Homework Sheet 9: Graph Connectivity: Sequential Algorithms

Due Date: AA July 2019

1. Recall the notion of a (global) mincut in a graph. Let $G$ be a graph such that mincut of this graph has at least $k$ edges. Consider any subset of edges $X = \{e_1, e_2, \ldots, e_t\}$ and let $G'$ be the graph obtained by contracting all edges of $X$ in $G$. Show that the minimum degree of $G'$ is at least $k$.

2. (i) Let $G$ be a undirected graph. Let $D_G$ be the digraph obtained from $G$ by converting each edge $(u, v) \in E(G)$ into two directed edges $(u, v)$ and $(v, u)$. Show that $G$ is $k$-edge-connected if and only if $D_G$ is $k$-edge-connected.

(ii) Let $D$ be a directed graph. Let $G_D$ be the graph obtained from $G$ by adding an edge $(u, v)$ whenever one of the two directed edges $(u, v), (v, u)$ exists in $D$. Show that $D$ is $k$-edge-connected if and only if $G_D$ is $k$-edge-connected.

3. (i) Let $D$ be a directed graph. Let $r \in V(D)$ be a vertex. Let $T^O_1, T^O_2, \ldots, T^O_k$ be a collection of edge disjoint spanning out-branchings in $D$. Then show that, for any vertex $v \in V(D) - r$, there are at least $k$ edge disjoint paths in $D$ from $r$ to $v$.

(ii) Similarly, let $T^I_1, T^I_2, \ldots, T^I_k$ be a collection of edge disjoint spanning in-branchings in $D$. Then show that, for any vertex $v \in V(D) - r$, there are at least $k$ edge disjoint paths in $D$ from $v$ to $r$.

(iii) From the above, conclude that if $D$ is a digraph and $r$ is a vertex such that,

- there is a collection of $k$-edge disjoint spanning out-branchings in $D$,
- and there is a collection of $k$-edge disjoint spanning in-branchings in $D$,

then $D$ is $k$-edge-connected.