Karl Bringmann and Marvin Künnemann Summer 2019

Fine-Grained Complexity Theory, Exercise Sheet 4

www.mpi-inf.mpg.de/departments/algorithms-complexity/teaching/summer19/fine-complexity/

Total Points: **40** + 2 bonus points Due: Tuesday, **June 11**, 2019

You are allowed to collaborate on the exercise sheets, but you have to write down a solution on your own, ***using your own words****. Please indicate the names of your collaborators for each exercise you solve. Further, cite all external sources that you use (books, websites, research papers, etc.).*

You need to collect at least 50% of all points on exercise sheets to be admitted to the exam.

Exercise ◯◯ — 2 bonus points

Read the lecture notes (of the last three lectures), identify as many typos and other mistakes as you can, and add them as a list to your solutions. You get one bonus point for at least one typo/mistake and 2 bonus points for at least five typos/mistakes.

Exercise 1 — **10** points

Recall the k–**Clique** problem from the lecture:

k–**Clique**: Given an undirected, unweighted graph G, determine whether G contains a k–clique (that is, a set of k vertices which are pairwise adjacent).

Show that if $3 \mid k$, then k–**Clique** can be solved in time $O(n^{\frac{\omega k}{3}})$.
What running time can you obtain when $3 \nmid k$?

Note: This is the best running time known for this problem.

Exercise 2 — **10** points

In the lecture we proved a subcubic reduction from **All–Pairs Negative Triangle** to **Negative Triangle**. Adapt this reduction to obtain a "combinatorial" subcubic reduction from **All–Pairs Triangle** to **Triangle**, that is, prove

$$(\textbf{All–Pairs Triangle}, n^3) \leq_{fgr} (\textbf{Triangle}, n^3).$$

Why does this reduction not prove an $N^{\omega-o(1)}$ lower bound for **Triangle** under the BMM hypothesis?

Exercise 3 — **5** points

Consider the following geometric problem:

Segment Visibility: Given a set S of N line segments and two distinguished line segments a and b. Determine whether there are a point p on a and a point q on b such that the line through the points p and q does not intersect any line segment in S (that is, we want to check whether a is "visible" from b).

Prove the following fine-grained reduction from **3SUM** to **Segment Visibility**:

$$(\textbf{3SUM}, n^2) \leq_{fgr} (\textbf{Segment Visibility}, N^2).$$

Exercise 4 — 15 points

Consider the following problem on directed acyclic graphs (DAGs):

All–Pairs Lowest Common Ancestor in DAGs (DAG–AP–LCA): Given a DAG $G = (V, E)$, determine for all vertices $i, j \in V$ *any* lowest common ancestor $v \in V$.
Here, a vertex $u \in V$ is a *common ancestor* of the vertices i and j if there is a path in G from u to i and from u to j (that is, i and j are descendants of u). The vertex u is a *lowest common ancestor* of the vertices i and j if no descendant of u is a common ancestor of i and j.

Note that in DAGs (as opposed to trees), the vertices i and j might have more than one lowest common ancestor. (In this case, the problem just asks for *any* lowest common ancestor.)

a) *(3 points)* Show that **DAG–AP–LCA** has no algorithm running in time $O(|V|^{\omega-\varepsilon})$ (and no *"combinatorial"* algorithm running in time $O(|V|^{3-\varepsilon})$) for any $\varepsilon > 0$, unless the BMM hypothesis fails.

This lower bound suggests that in order to obtain strongly subcubic algorithms, we should use "non-combinatorial" tools, that is, fast matrix multiplication. Indeed, we will see how to do this in the remaining parts of this exercise.

We will use the following intermediate problem:

BMM–MaxWitness: Given Boolean matrices $A = (a_{ij}), B = (b_{ij}) \in \{0, 1\}^{n \times n}$, determine for all $1 \leq i, j \leq n$ the maximum index k such that $a_{ik} = b_{kj} = 1$ if such a k exists (and $\perp$ otherwise).

b) *(3 points)* Show that the transitive closure of a graph $G = (V, E)$ can be computed using $O(\log(|V|))$ Boolean matrix multiplications.

c) *(4 points)* Prove that there is a (subcubic) fine-grained reduction from **DAG–AP–LCA** to **BMM–MaxWitness**, that is, prove

$$(\textbf{DAG–AP–LCA}, |V|^3) \leq_{fgr} (\textbf{BMM–MaxWitness}, n^3).$$

(Hint: Use a topological sort and part b).)

d) *(5 points)* Solve **BMM–MaxWitness** in subcubic time.

(Hint: Use (rectangular) matrix multiplication on suitably-sized subproblems.)

In total, this yields a strongly subcubic time algorithm for **DAG–AP–LCA**.