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Fine-Grained Complexity Theory, Exercise Sheet 6

www.mpi-inf.mpg.de/departments/algorithms-complexity/teaching/summer19/fine-complexity/

Total Points: **40** + 7 bonus points — Due: Tuesday, **July 9**, 2019

You are allowed to collaborate on the exercise sheets, but you have to write down a solution on your own, ***using your own words****. Please indicate the names of your collaborators for each exercise you solve. Further, cite all external sources that you use (books, websites, research papers, etc.).*

You need to collect at least 50% of all points on exercise sheets to be admitted to the exam.

Recall the definition of (co-)nondeterministic algorithms from the lecture:

A decision problem P is in *nondeterministic* time $t(n)$ if there is a deterministic algorithm V_{yes}, that runs on an input of size n in time $t(n)$ such that

- for any "YES"-instance I of P there is a string π for which the algorithm $V_{yes}(I, \pi)$ accepts; and
- for any "NO"-instance I of P and for every string π, the algorithm $V_{yes}(I, \pi)$ *rejects*.

In contrast, a decision problem P is in *co-nondeterministic* time $t(n)$ if there is a deterministic algorithm V_{no}, that runs on an input of size n in time $t(n)$ such that

- for any "YES"-instance I of P and for every string π, the algorithm $V_{no}(I, \pi)$ *rejects*; and
- for any "NO"-instance I of P there is a string π for which the algorithm $V_{no}(I, \pi)$ accepts.

Exercise ◯◯ — 2 bonus points

Read the lecture notes (of the last three lectures), identify as many typos and other mistakes as you can, and add them as a list to your solutions. You get one bonus point for at least one typo/mistake and 2 bonus points for at least five typos/mistakes.

Exercise 1 — **6 + 8** points

Recall the following problem from the lecture:

Maximum Submatrix: Let A be an $n \times n$ matrix with entries in $\{-\lfloor n^c \rfloor, -\lfloor n^c \rfloor + 1, \dots, \lfloor n^c \rfloor\}$ for some constant $c > 0$. Find the maximum sum of all entries of any submatrix of A, where a submatrix is a choice of some consecutive rows and some consecutive columns.

a) Demonstrate an $O(n^3)$ time algorithm for **Maximum Submatrix** (solve it directly without any reductions).

b) A submatrix of A is called **centered** if it contains the center (entry $A_{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor}$) of matrix A. Now the problem is to find a **Maximum Centered Submatrix**. Show that if APSP has subcubic algorithm then so does this problem, that is, show that there is subcubic reduction from this problem to APSP.

Exercise 2 — 6 + 5 + 4 points

a) Given a weighted graph $G = (V, E, w)$ and a prime p, demonstrate an algorithm running in time $O(p^2 \cdot |V|^\omega)$ that computes the number of triples $(i, j, k) \in V^3$ such that

$$w(i, j) + w(j, k) + w(k, i) \equiv 0 \pmod{p}.$$

b) Demonstrate nondeterministic and co-nondeterministic algorithms running in time $O(|V|^{3-\varepsilon})$ for **ZeroTriangle** (for some $\varepsilon > 0$).

Hint: Use part a); the (co-nondeterministic) ***3SUM*** *algorithm from the lecture is similar.*

c) Consider the following decision problem:

$(\min, +)$–**Verification**: Given $n \times n$ matrices A, B, C, determine whether $A \odot B = C$, where $\odot$ denotes the $(\min, +)$–product of two matrices.

Demonstrate nondeterministic and co-nondeterministic algorithms running in time $O(n^{3-\varepsilon'})$ for $(\min, +)$–**Verification** (for some $\varepsilon' > 0$).

Hint: Use known reductions.

Exercise 3 — 3 + 6 + 1 + 6 points

Consider the following graph problem:

Negative Matching Walk: Given a weighted graph $G = (V, E, w)$, an alphabet Σ, vertex labels $l : V \to \Sigma$, and a string $s = s_1 s_2 \ldots s_T \in \Sigma^T$, determine whether there is a walk $v_1, \ldots, v_T \in V$ in G, such that

- the labels $l(v_1), l(v_2), \ldots, l(v_T)$ form the string s, that is

$$l(v_1)\, l(v_2)\, \ldots\, l(v_T) = s,$$

- the total edge weight of the walk is negative, that is

$$\sum_{1 \le i < T} w(v_i, v_{i+1}) < 0.$$

a) Construct a deterministic algorithm for **Negative Matching Walk** that runs in time $O(T \cdot |V|^2)$.

b) Show the following fine-grained reduction for the case $T = |V|$:

$$(\textbf{APSP}, n^3) \le_{fgr} (\textbf{Negative Matching Walk}, |V|^3).$$

We will now proceed to show that **NSETH** implies that there are no tight **SETH**-based lower bounds for this problem.

c) Demonstrate a nondeterministic algorithm for **Negative Matching Walk** that runs in time $O(|V|^{3-\varepsilon})$ for some $\varepsilon > 0$ and $T = |V|$.

d) Demonstrate a co-nondeterministic algorithm for **Negative Matching Walk** that runs in time $O(|V|^{3-\varepsilon'})$ for some $\varepsilon' > 0$ and $T = |V|$.

Hint: Let $u_j^{(i)}$ denote the minimum weight of a walk ending in vertex j that forms the string $s_1 \ldots s_i$. Guess $u^{(i)}$ for all i and verify them using one $(\min, +)$ product verification (and other suitable checks).