Computational Geometry

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Sessions: Tu, Th 10–12 on Zoom (roughly every 4th session will be a tutorial)

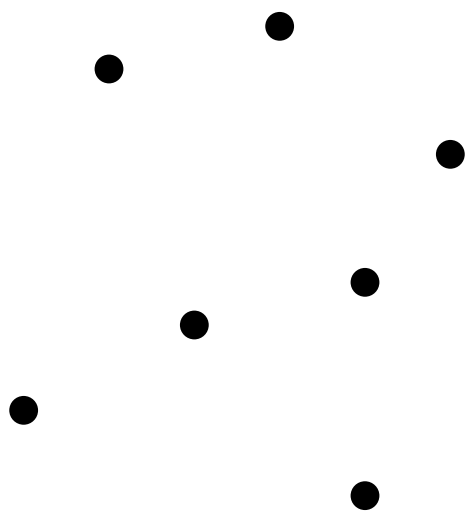
Homeworks: about every other week; half of the homework points necessary to qualify for the final exam

Exam: take-home exam; date to be determined

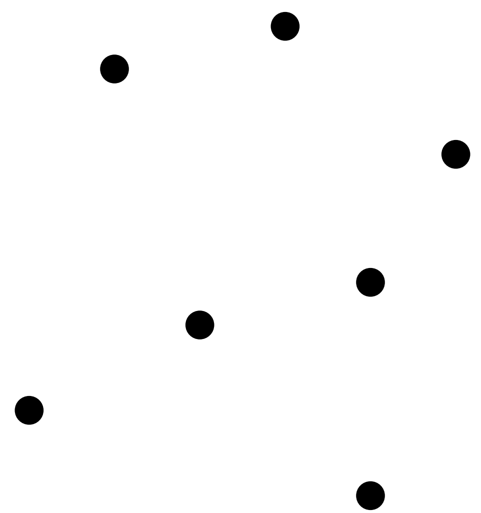
Coursepage: https://www.mpi-inf.mpg.de/departments/algorithms-complexity/teaching/summer20/computational-geometry

Problem 1: Let S be a set of n points in the plane:

Are all points distinct?



Problem 2: Let S be a set of n points in the plane:



Is S degenerate, i.e. are there 3 points of S on a common (straight) line?

Computational Model and Geometric Primitives

Real RAM: RAM that also has cells that can hold real numbers.

- arithmetic operations and comparisons of reals exactly and in constant time;
- possibly other operations such as squareroot, logarithm, etc. exactly and in constant time as well;

This model is convenient for concentrating on the geometric issues in the problems at hand. It can be unrealistic, i.e. difficult to realize. A lot of "abuse" is possible.

Computational Model and Geometric Primitives

Possible solution: Encapsulate all arithmetic on input reals into *geometric primitives:*

For example: how does point p lie with respect to the oriented line through the points q and r?

$$\text{sidedness}(p; q, r) = \begin{cases} 1 & \text{if } p \text{ left of } \overrightarrow{qr} \\ 0 & \text{if } p \text{ on } \overrightarrow{qr} \\ -1 & \text{if } p \text{ right of } \overrightarrow{qr} \end{cases}$$

This can be arithmetically realized as the sign of the determinant

$$\begin{vmatrix} 1 & p_1 & p_2 \\ 1 & q_1 & q_2 \\ 1 & r_1 & r_2 \end{vmatrix}$$

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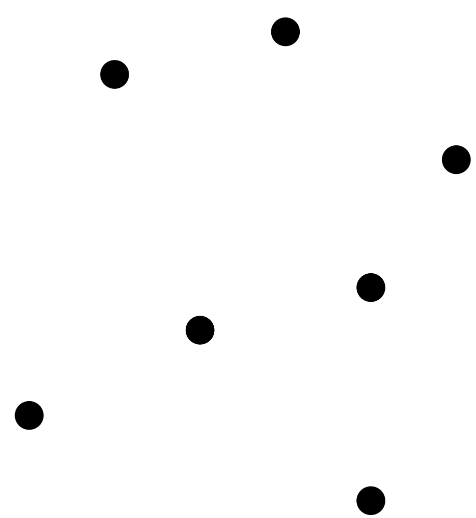

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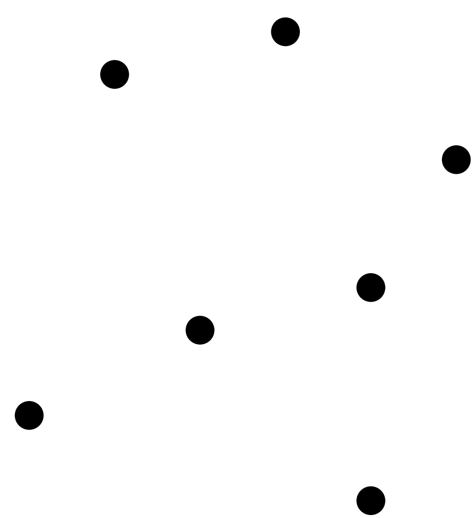

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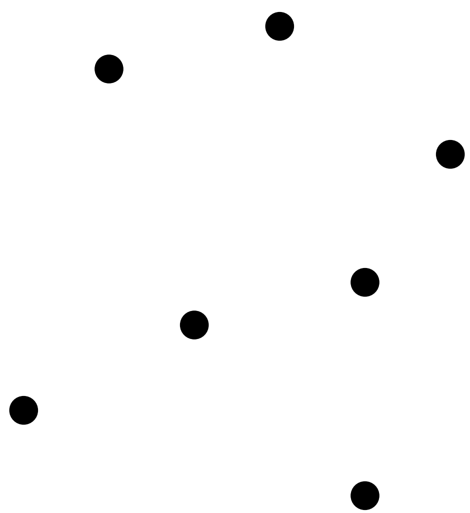
The area of the triangle spanned by p, q, r is given the absolute value of

$$\frac{1}{2}\begin{vmatrix} 1 & p_1 & p_2 \\ 1 & q_1 & q_2 \\ 1 & r_1 & r_2 \end{vmatrix}$$

Problem 4: Let S be a set of n points in the plane:

What is the smallest circle that contains S?



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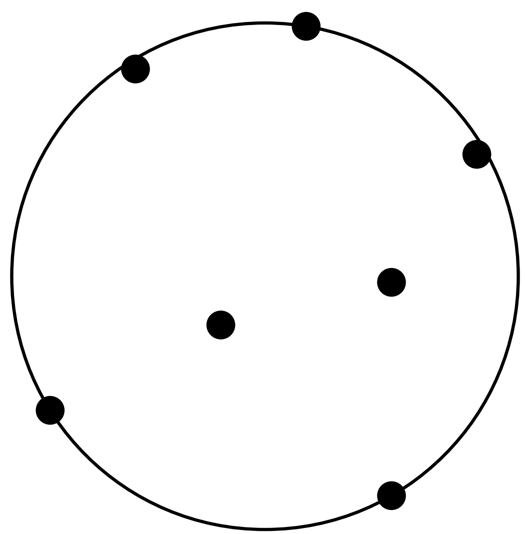

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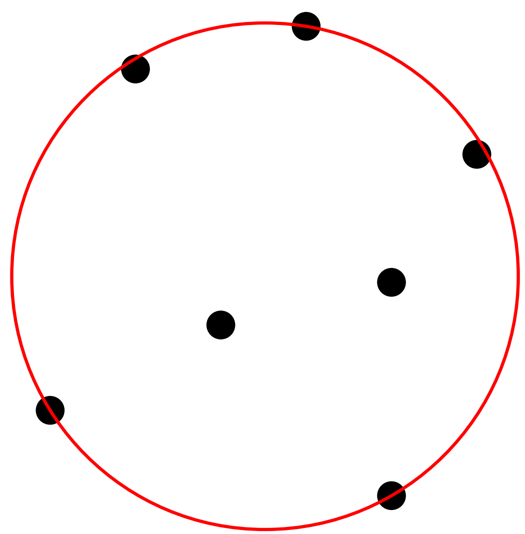

What is the smallest circle that contains S ?

Geometric primitive:

The location of point p with respect to the circle through r, s, t is determined by the sign of the determinant

$$\begin{vmatrix} 1 & p_1 & p_2 & p_1^2 + p_2^2 \\ 1 & r_1 & r_2 & r_1^2 + r_2^2 \\ 1 & s_1 & s_2 & s_1^2 + s_2^2 \\ 1 & t_1 & t_2 & q_1^2 + t_2^2 \end{vmatrix}$$

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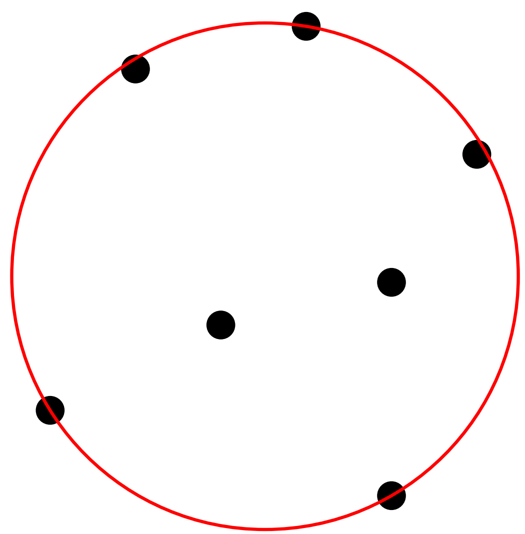

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in relation to the sign of the determinant

$$\begin{vmatrix} 1 & r_1 & r_2 \\ 1 & s_1 & s_2 \\ 1 & t_1 & t_2 \end{vmatrix}$$

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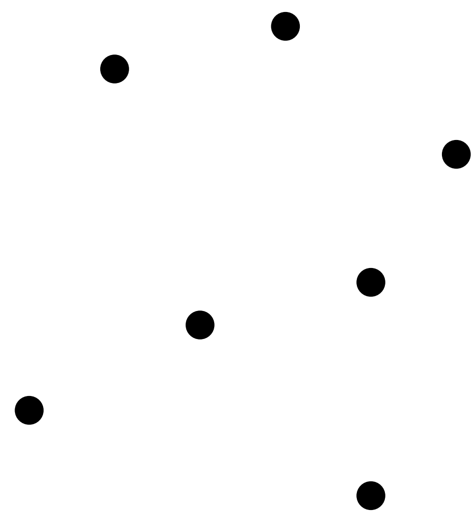

What is the smallest convex polygon that contains S?

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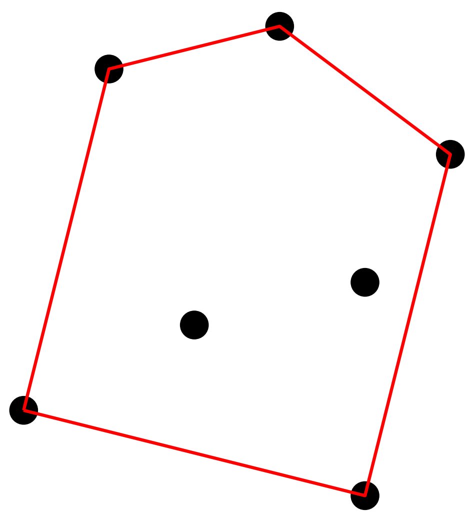
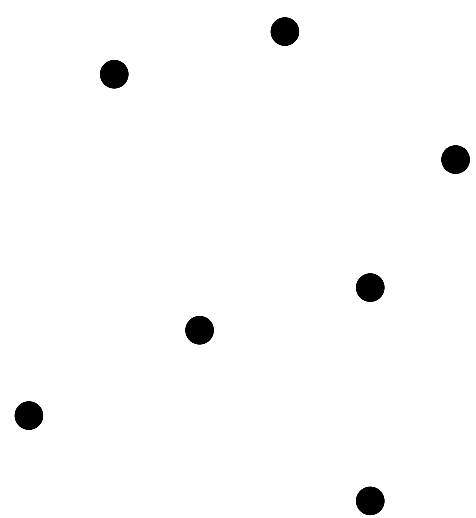


What is the smallest convex polygon that contains S?

convex hull of S

Problem 6: Let S be a set of n points in the plane:



Compute a triangulation of S ?

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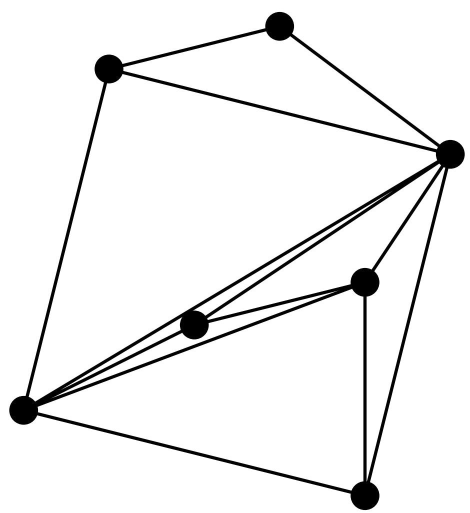

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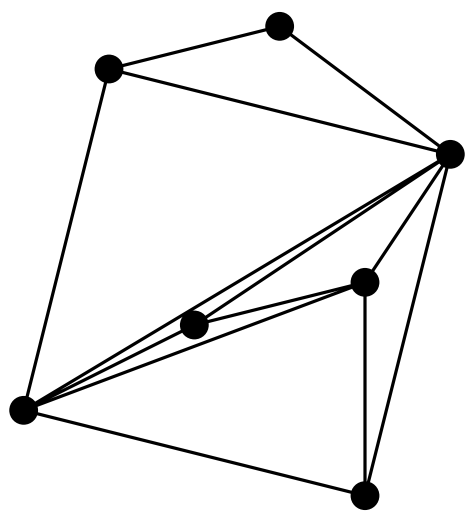
Problem 7: Let S be a set of n points in the plane:



Compute a **"good"** triangulation of S?

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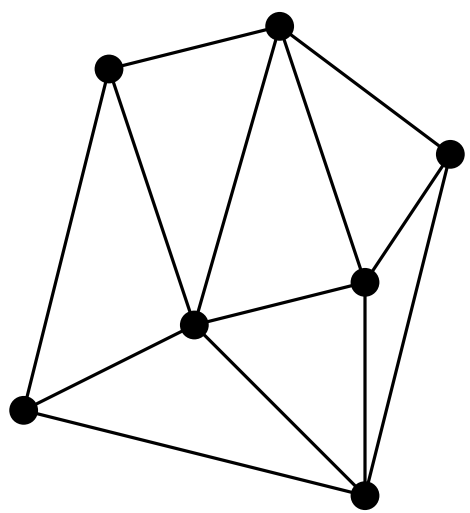
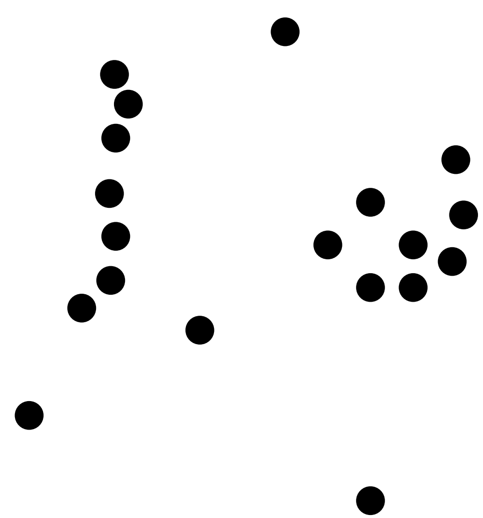


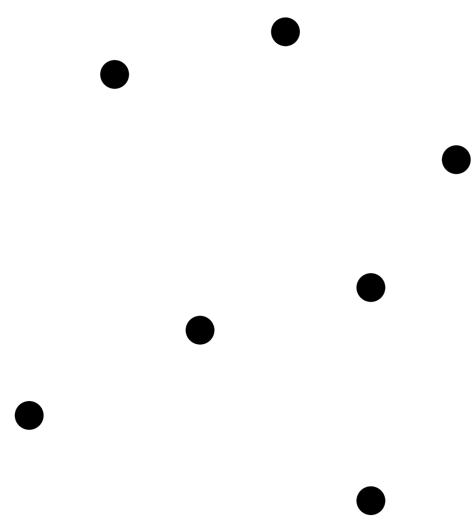
"Delaunay triangulation"

Problem 7: Let S be a set of n points in the plane:



Determine the "clusters of S" ?

Problem 8: Let S be a set of n points in the plane:



For a given integer k, compute or count the k-sets of S.

Definition: A k-set of S is a subset B of S with k elements for which there is a line that separates B from $S \setminus B$.

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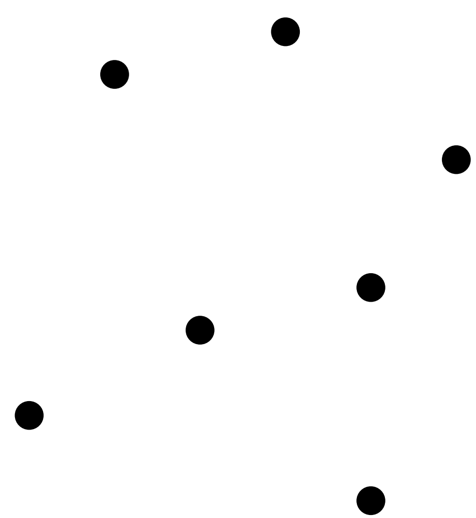


For a given integer k, compute or count the k-sets of S.

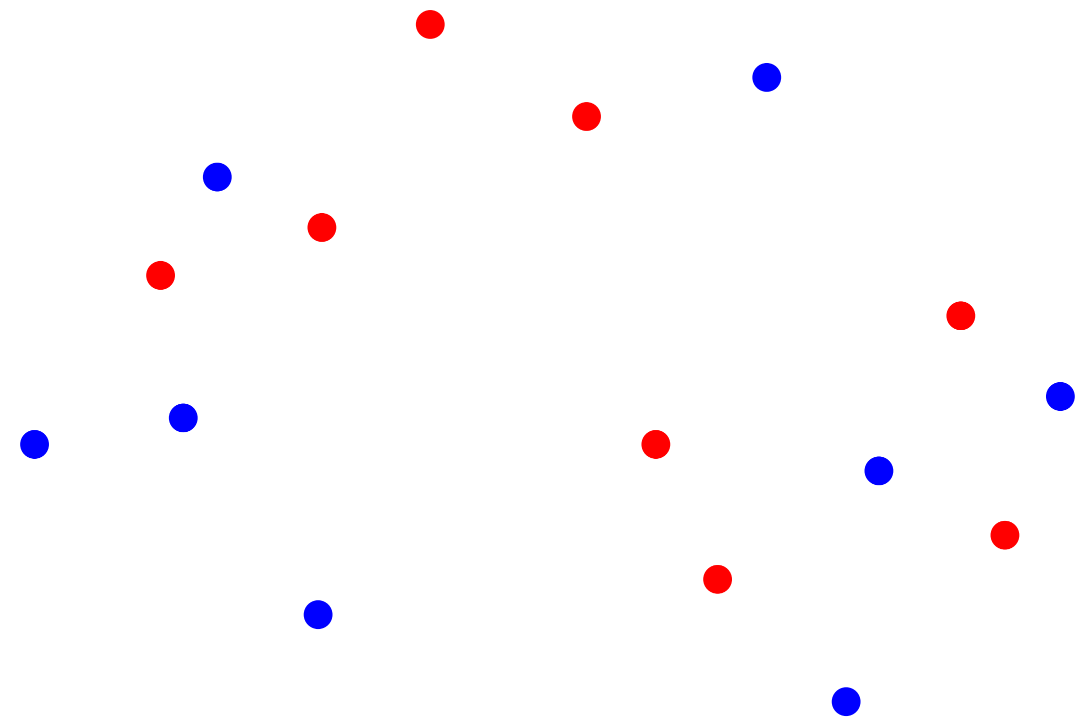
Definition: A k-set of S is a subset B of S with k elements for which there is a line that separates B from $S \setminus B$.

The geometric-combinatorial problem of giving good bounds for $f_k(S)$, the number of k-sets of S is still open:

$$n \cdot e^{\Omega(\sqrt{\log k})} \leq f_k(S) \leq O(n\sqrt[3]{k})$$

Problem 9: Ham-Sandwich cuts for planar point sets:

Let R be a set of n red points and B be a set of n blue points, compute a line that simultaneously halves R and B.



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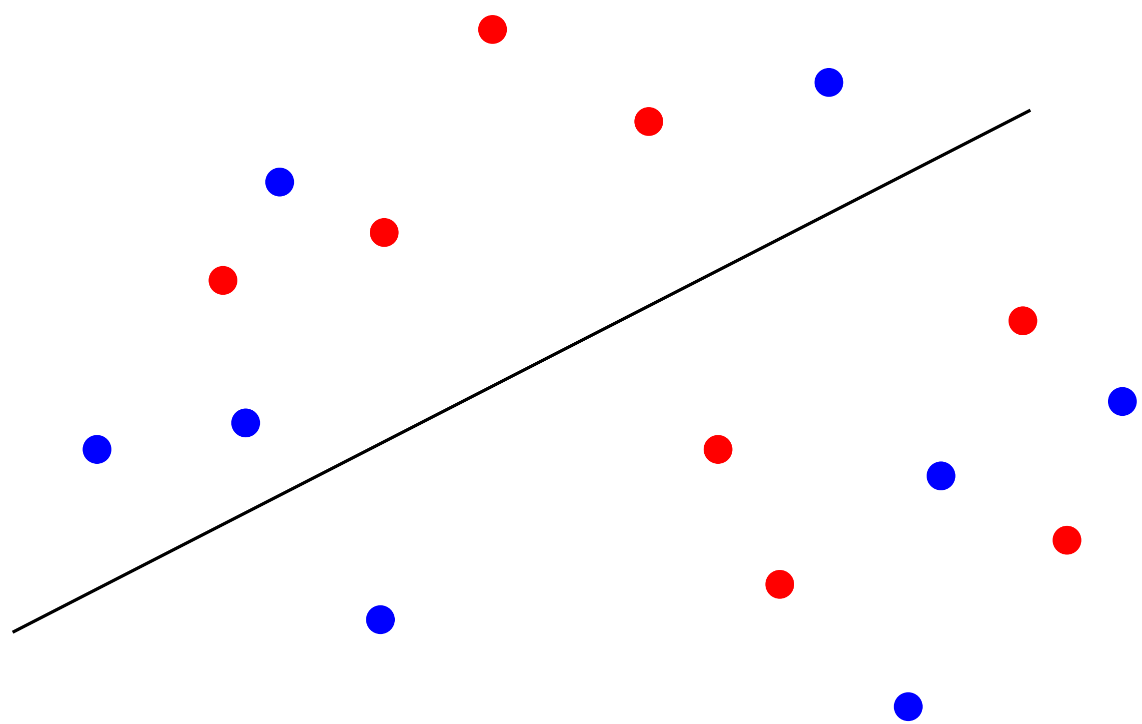

Problem 10: Discrepancy

How well does a set S of n points in a unit disc D reflect the area of halfplanes intersecting D?

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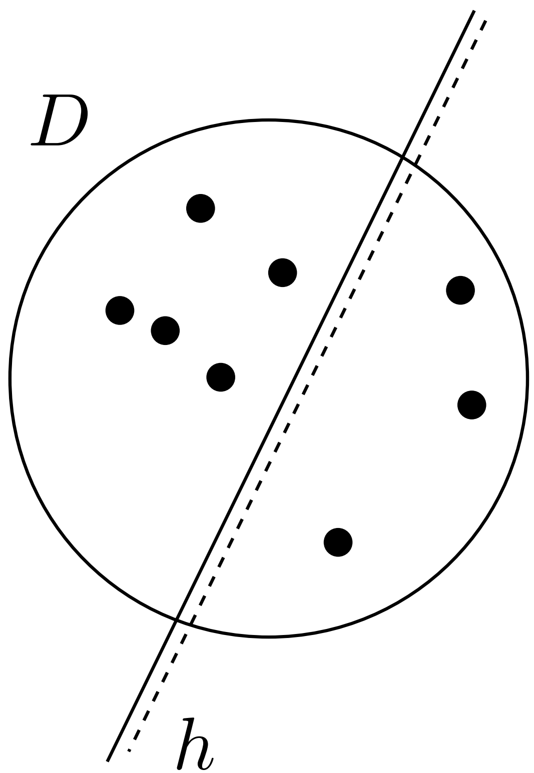


h halfplane

$$a_D(h) = \frac{\text{area}(h \cap D)}{\text{area}(D)}$$

$$a_S(h) = \frac{|h \cap S|}{|S|}$$

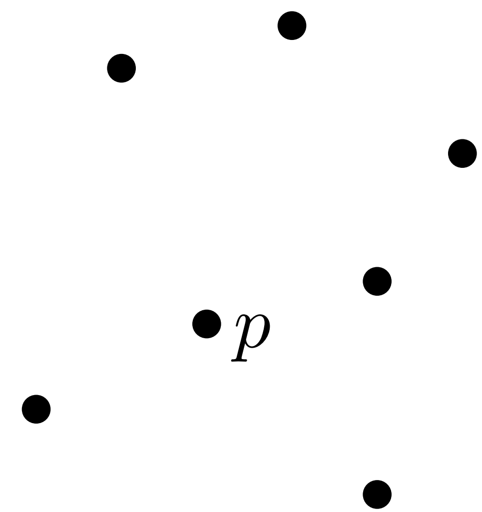
$$\text{discrepancy}(S) = \inf_h |a_D(h) - a_S(h)|$$

Problem 2: Let S be a set of n points in the plane:

Is S degenerate, i.e. are there 3 points of S on a common (straight) line?

Solution 1: brute force, check every triple of points in S

$\Theta(n^3)$ time



Solution 2: For every $p \in S$ check whether among the $n-1$ lines spanned with the other points in S there are two with the same slope.

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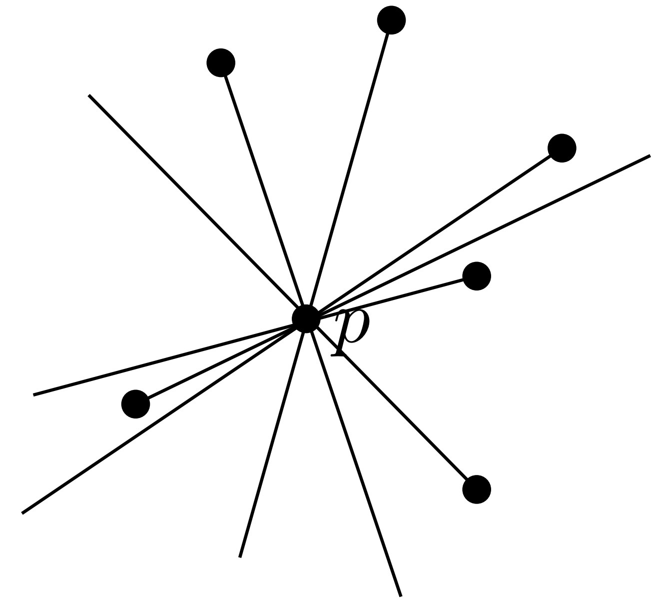

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Geometric point–line duality (polarity)

x-y–plane | ξ-η–plane

$$y + \eta = x \cdot \xi$$

point p given by (a, b) $\xrightarrow{\mathcal{D}}$ line λ given by $\eta = a \cdot \xi - b$

line ℓ given by $y = \alpha \cdot x - \beta$ $\xrightarrow{\mathcal{D}}$ point π given by (α, β)

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point p lies $\begin{smallmatrix}\text{above}\\ \text{on}\\ \text{below}\end{smallmatrix}$ line ℓ iff point $\pi = \mathcal{D}(\ell)$ lies $\begin{smallmatrix}\text{above}\\ \text{on}\\ \text{below}\end{smallmatrix}$ line $\lambda = \mathcal{D}(p)$

Signed vertical distance between p and ℓ is the same as the signed vertical distance between $\mathcal{D}(\ell)$ and $\mathcal{D}(p)$.

Geometric point–line duality (polarity), other version

x-y–plane

ξ-η–plane

$$x \cdot \xi + y \cdot \eta = 1$$

point p given by (a, b) $\xrightarrow{\mathcal{D}}$ line λ given by $a \cdot \xi + b \cdot \eta = 1$

line ℓ given by $\alpha \cdot x + \beta \cdot y = 1$ $\xrightarrow{\mathcal{D}}$ point π given by (α, β)

point p lies above / on / below line ℓ iff point $\pi = \mathcal{D}(\ell)$ lies above / on / below line $\lambda = \mathcal{D}(p)$

"above" means "on different side as the origin"
"below" means "on the same side as the origin"

Point-line duality — geometric interpretation

Embed x-y plane in 3-space as the $z = 1$ plane.
Embed ξ-η plane in 3-space as the $z = -1$ plane.

p **in** x-y-1 **plane:** $\mathcal{D}(p)$ is given by the line formed by the intersection of the ξ-η-(-1) plane with the plane through the origin o that is normal to $\overrightarrow{op}$

ℓ **in** x-y-1 **plane:** $\mathcal{D}(\ell)$ is given by the point formed by the intersection of the ξ-η-(-1) plane with the line through the origin o that is normal to the plane spanned by the o and ℓ.

Point-line duality — Degeneracy problems

Do 3 points in a set S of n point lie on a common line?

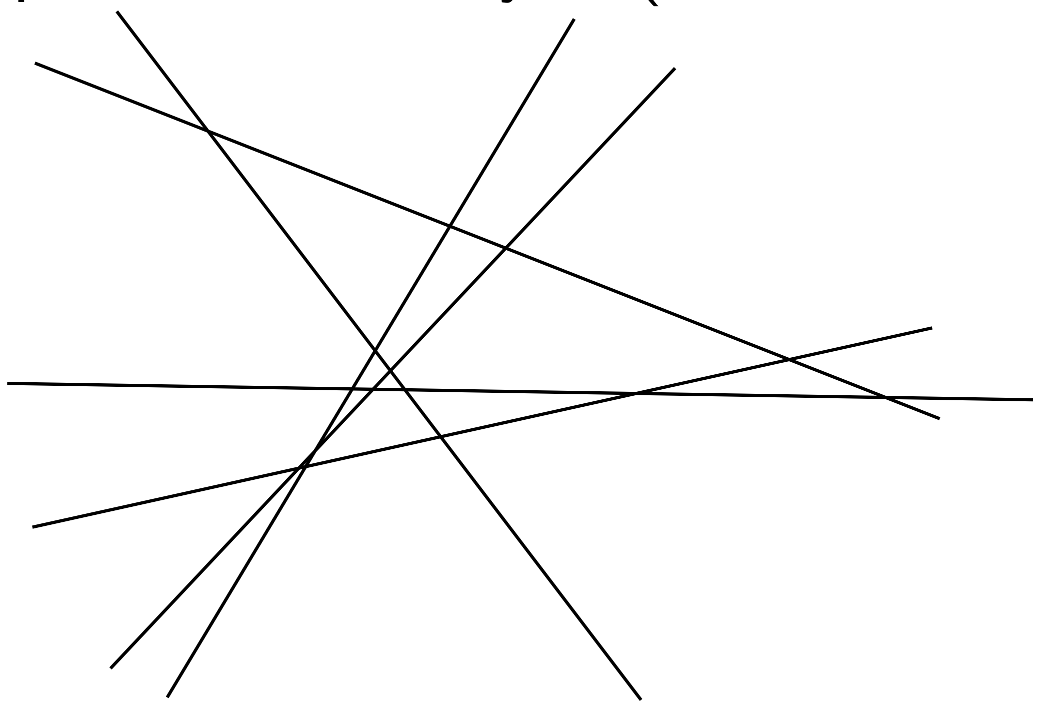
Under duality this become:

Do 3 lines in a set L of n lines contain a common point?

(i.e. do 3 lines intersect in a common point)

Arrangements of lines

For a set L of n lines $\mathcal{A}(L)$, the *arrangement of* L, is the partition of the plane induced by L (viewed as a planar graph).



$\binom{n}{2}$ vertices

n^2 edges

$\binom{n}{2} + \binom{n}{1} + \binom{n}{0}$ cells

This maximum is achieved when there is no degeneracy.

Incremental Construction of a line arrangement

Given a set L of n lines, we need to construct $\mathcal{A}(L)$ (as a plane graph).

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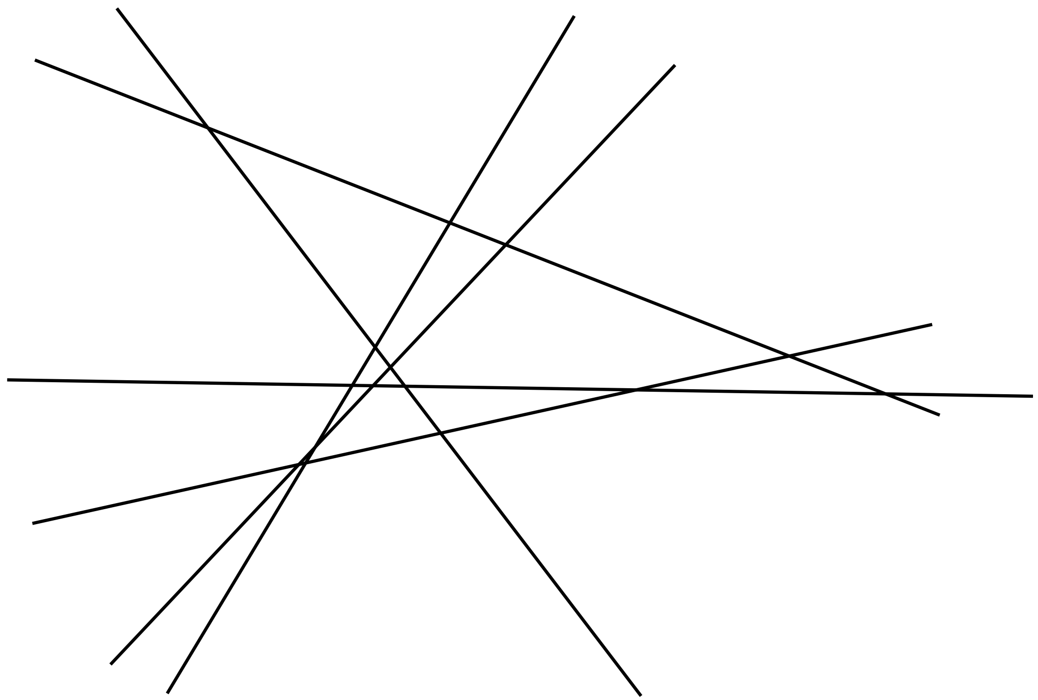

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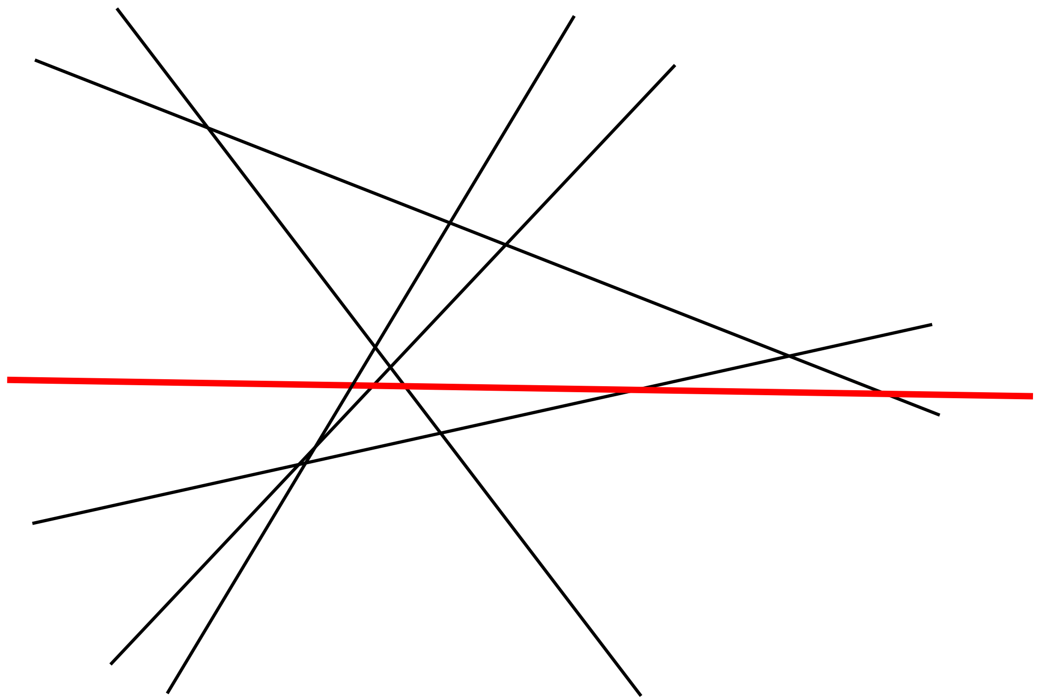

Incremental Construction

1. pick some $\ell \in L$

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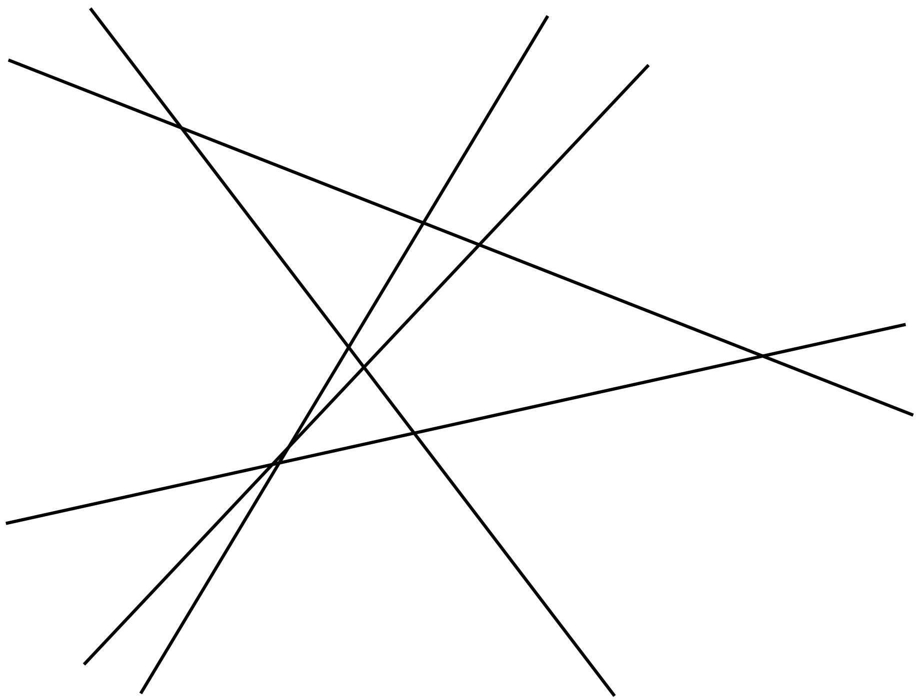

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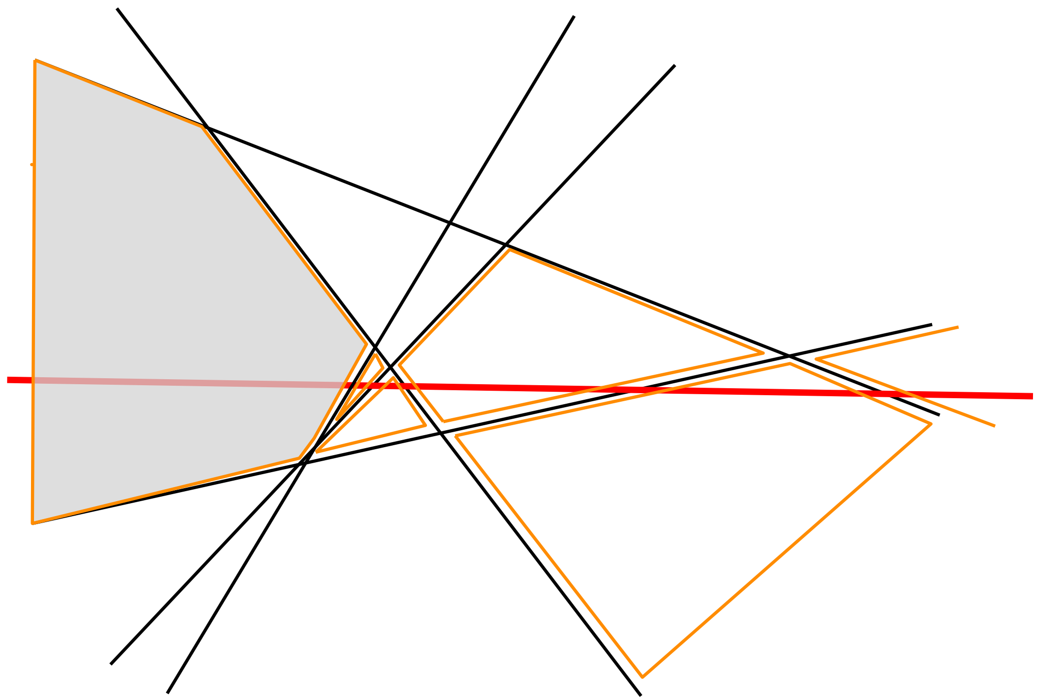
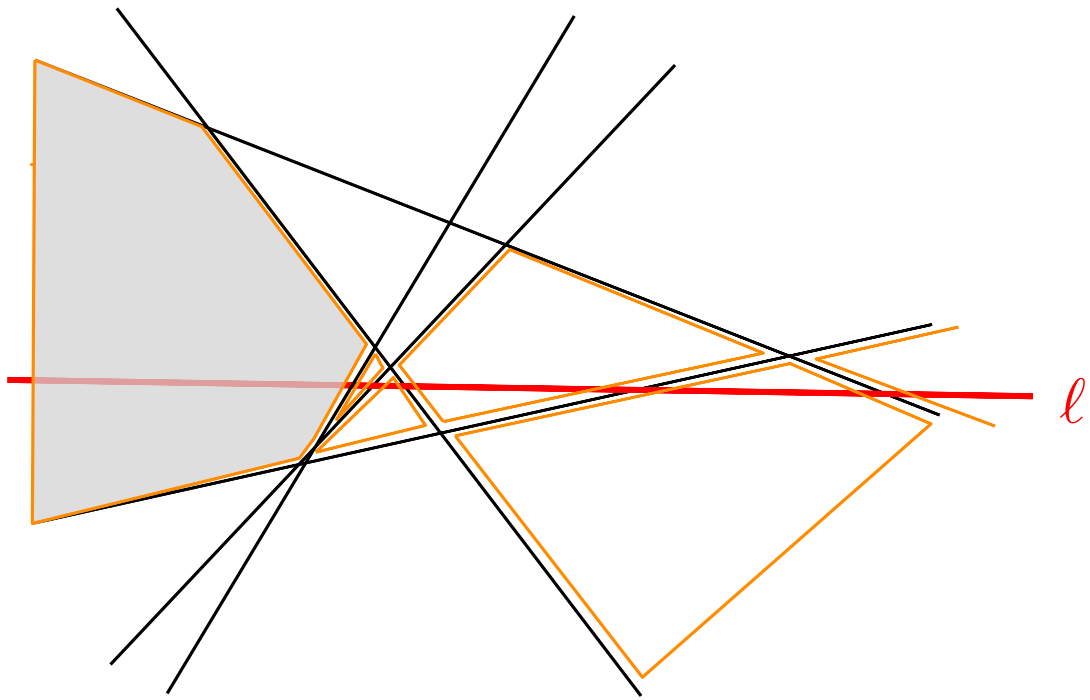


Incremental Construction

1. pick some $\ell \in L$
2. construct $\mathcal{A}(L')$, where $L' = L \setminus \{\ell\}$
3. construct $\mathcal{A}(L)$ from $\mathcal{A}(L')$ by "threading in" line ℓ

"Threading a line" into an arrangement



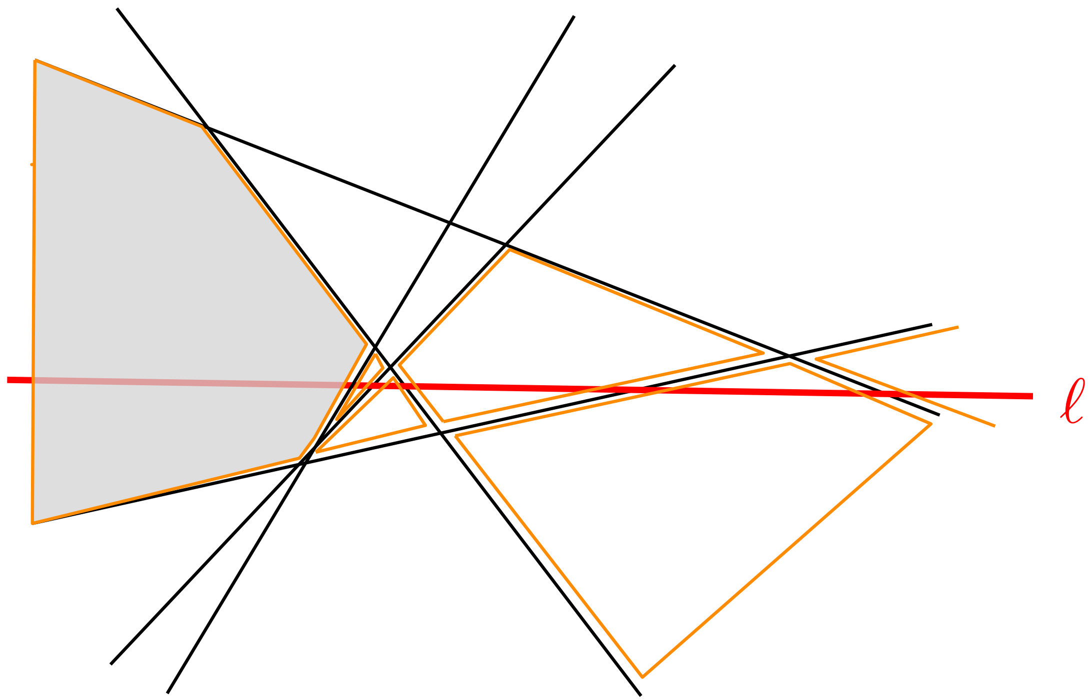
Cost of threading ℓ into $\mathcal{A}(L')$:

1. $O(n)$ for locating cell of $\mathcal{A}(L')$, where ℓ starts "from the left"
2. $O($sum of the sizes of the cells intersected by $\ell)$

zone(ℓ, L') = cells of $\mathcal{A}(L')$ that intersect ℓ

$z(\ell, L') = \sum_{c \in \text{zone}(\ell,\, L')} \#$ edges of c

"Threading a line" into an arrangement



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$$= O(z(\ell, L')) \leq 6n$$

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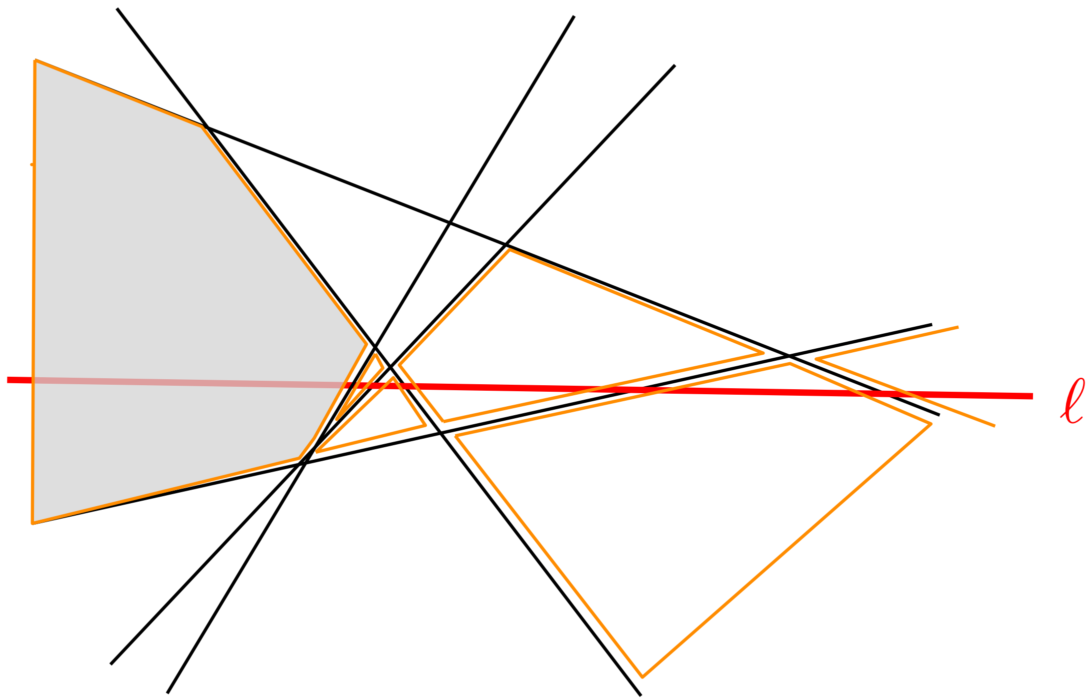
Zone Theorem



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Theorem: Let L be a set of n lines in the plane. For every $\ell \in L$ we have $z(\ell, L') \leq 6n$.

Line Arrangements

Theorem Given a set L of n lines its arrangement can be constructed in $O(n^2)$ time (and space).

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Consequences:

- The point degeneracy problem (and the line degeneracy problem) can be solve in $O(n^2)$ time.
- For a set S of n points the smallest (largest) area triangle spanned by 3 points of S can be found in $O(n^2)$ time.
- For a set R of n red points and a set B of n blue points a ham-sandwich line can be found in $O(n^2)$ time.
- The discrepancy problem can be solved in $O(n^2)$ time.