Planar Point Location

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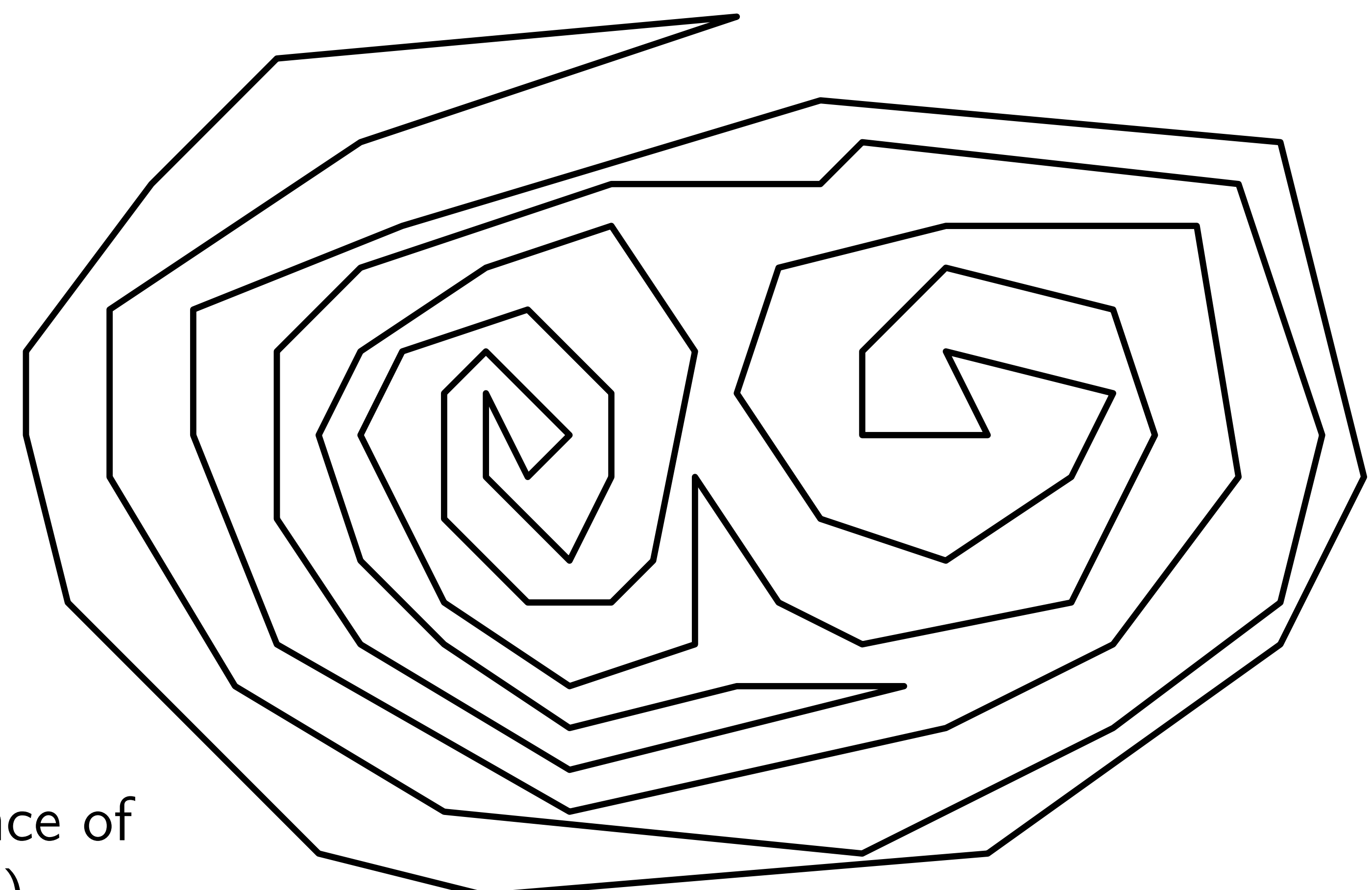

q given by its coordinates

P given by circular sequence of its corners (by coordinates)

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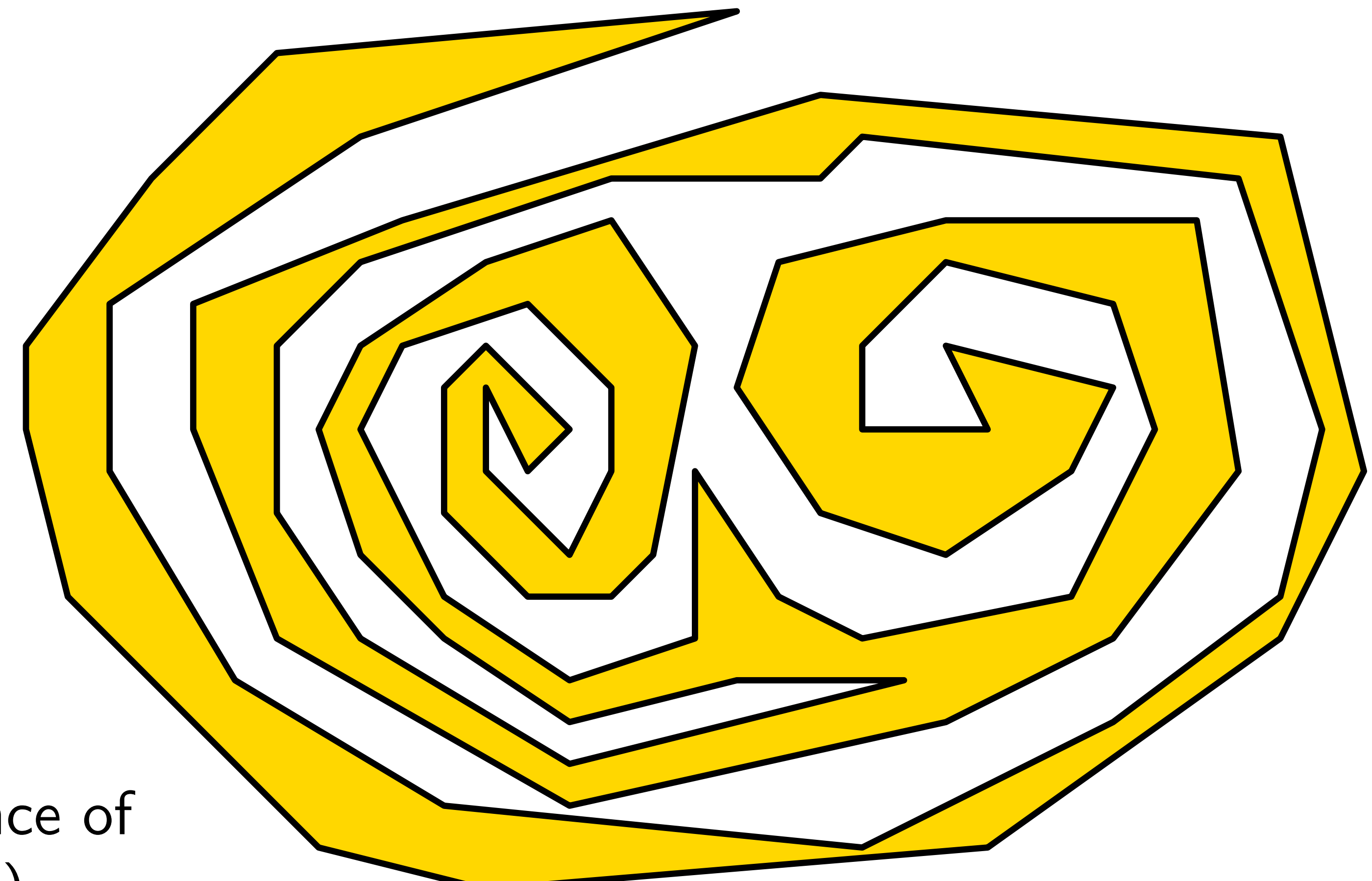

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Point in Polygon Test

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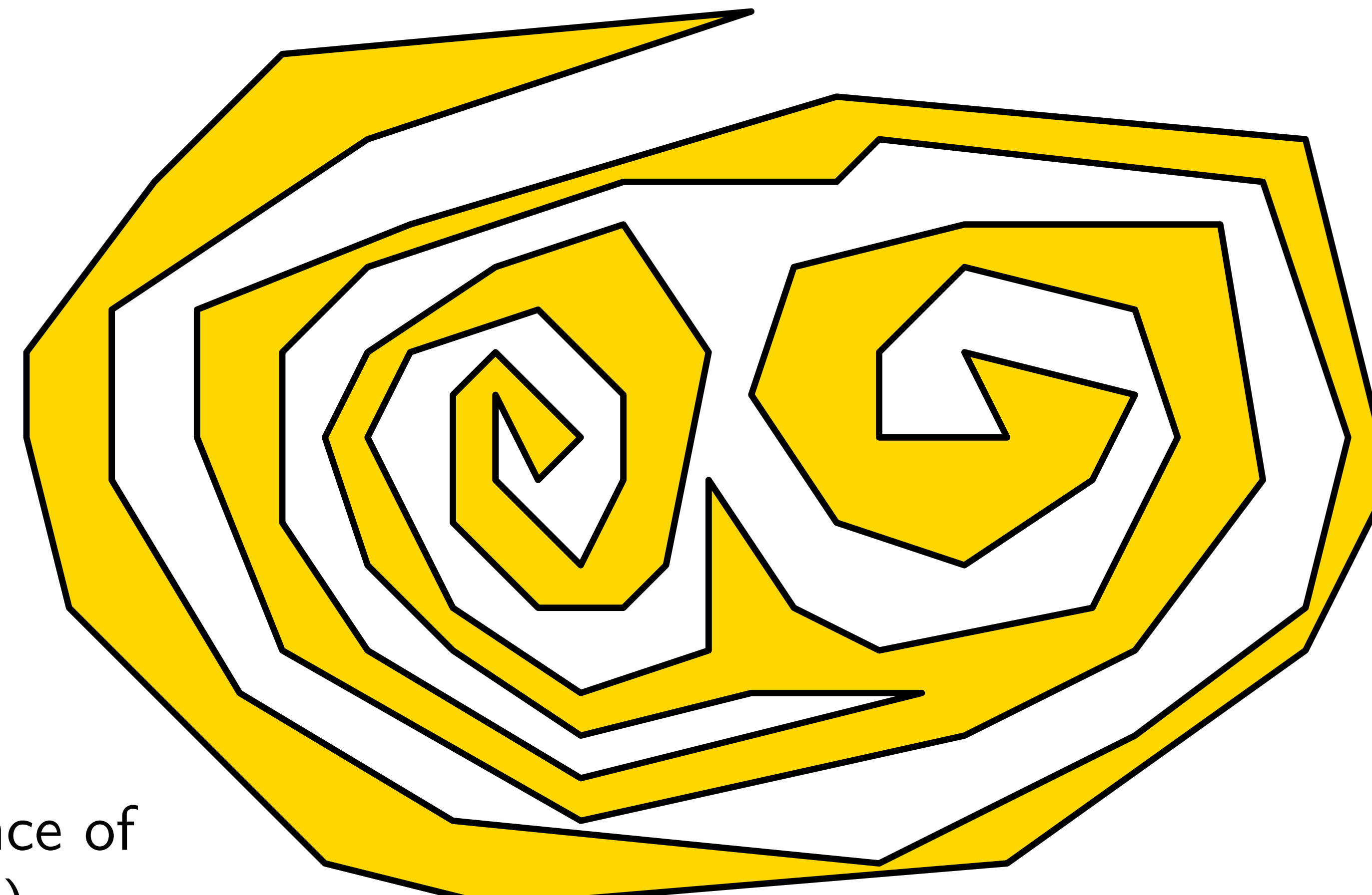

q given by its coordinates

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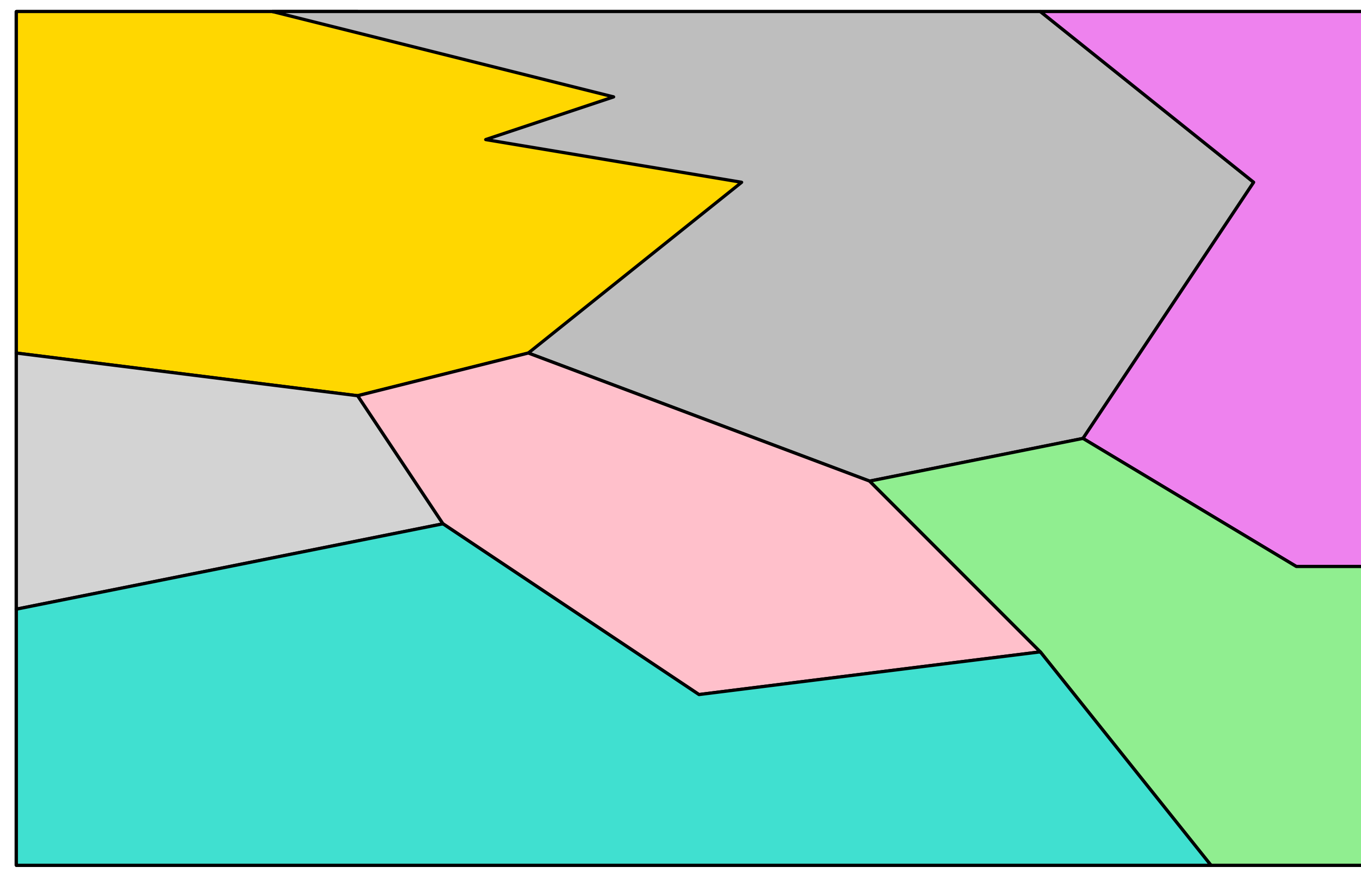
Planar Point Location

Preprocess a given partition of the plane (or a bounding box) so that for every query point q it can be determined quickly which region of the partition contains q.



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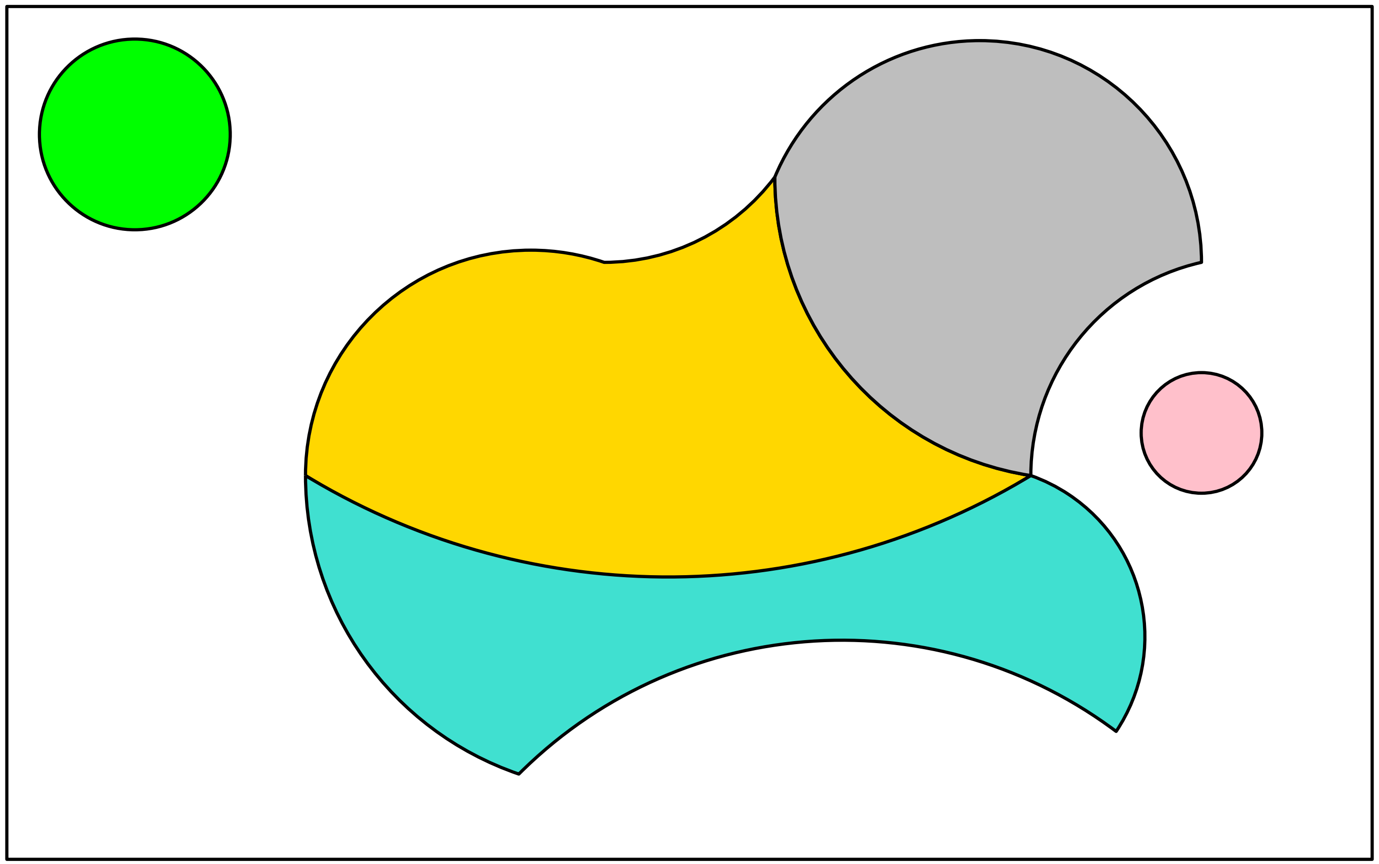
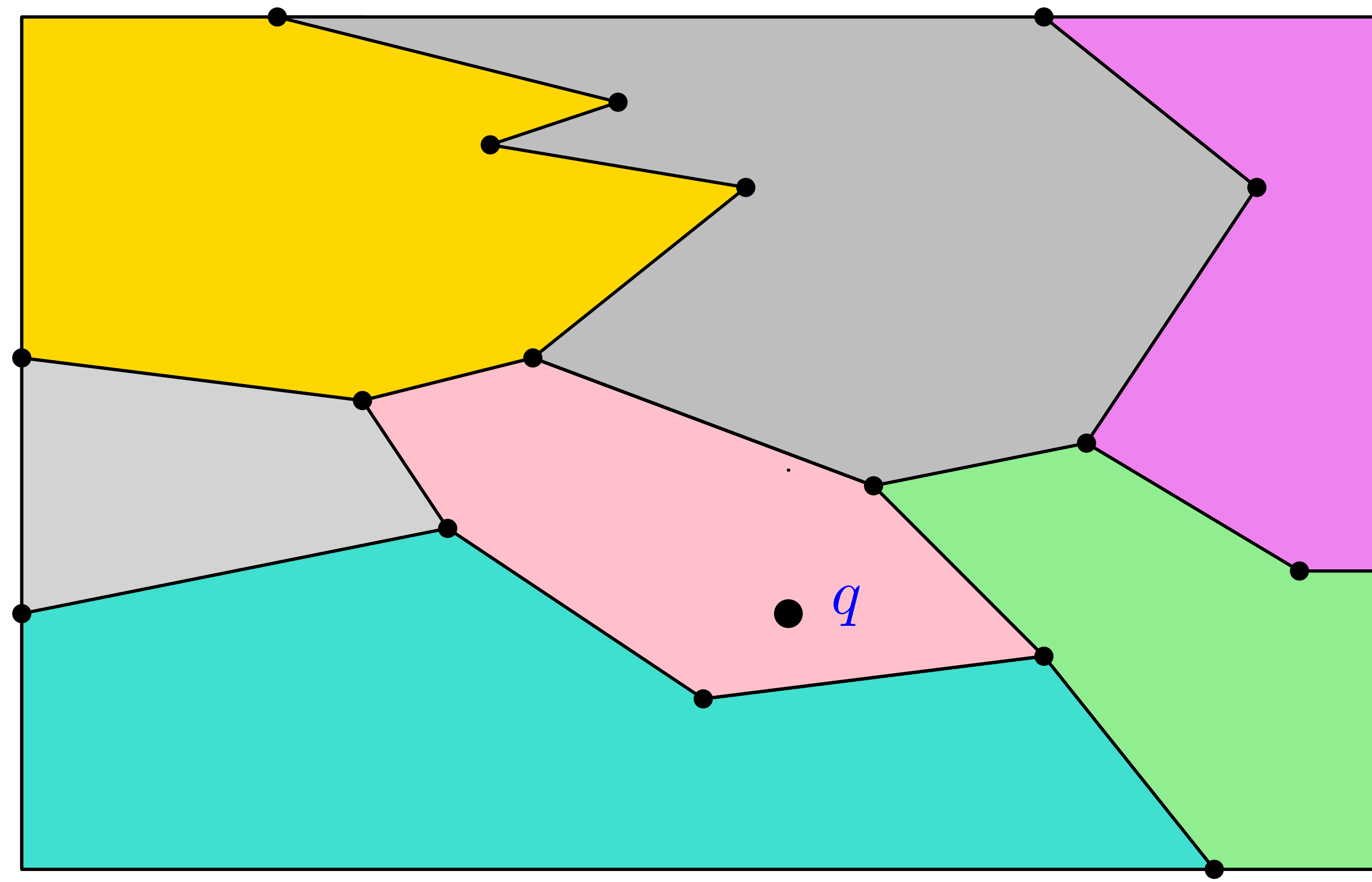


Vertical Ray Shooting

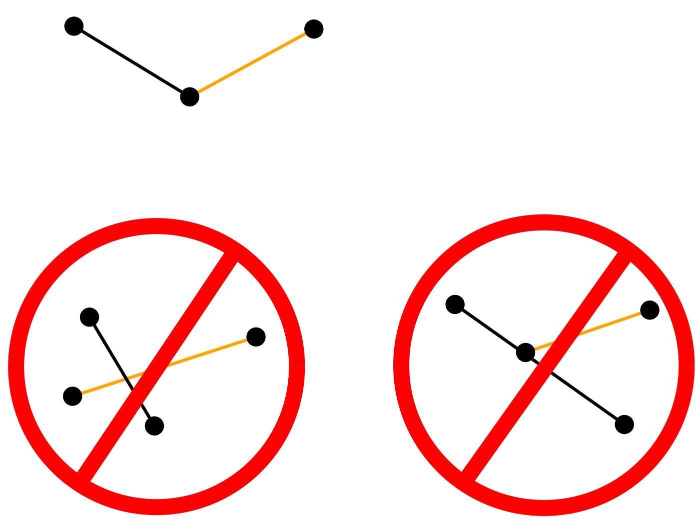
Preprocess a given set S of non-crossing segments in the plane (or a bounding box) so that for every query point q it can be determined quickly which segment of S lies immediately above(below) q.

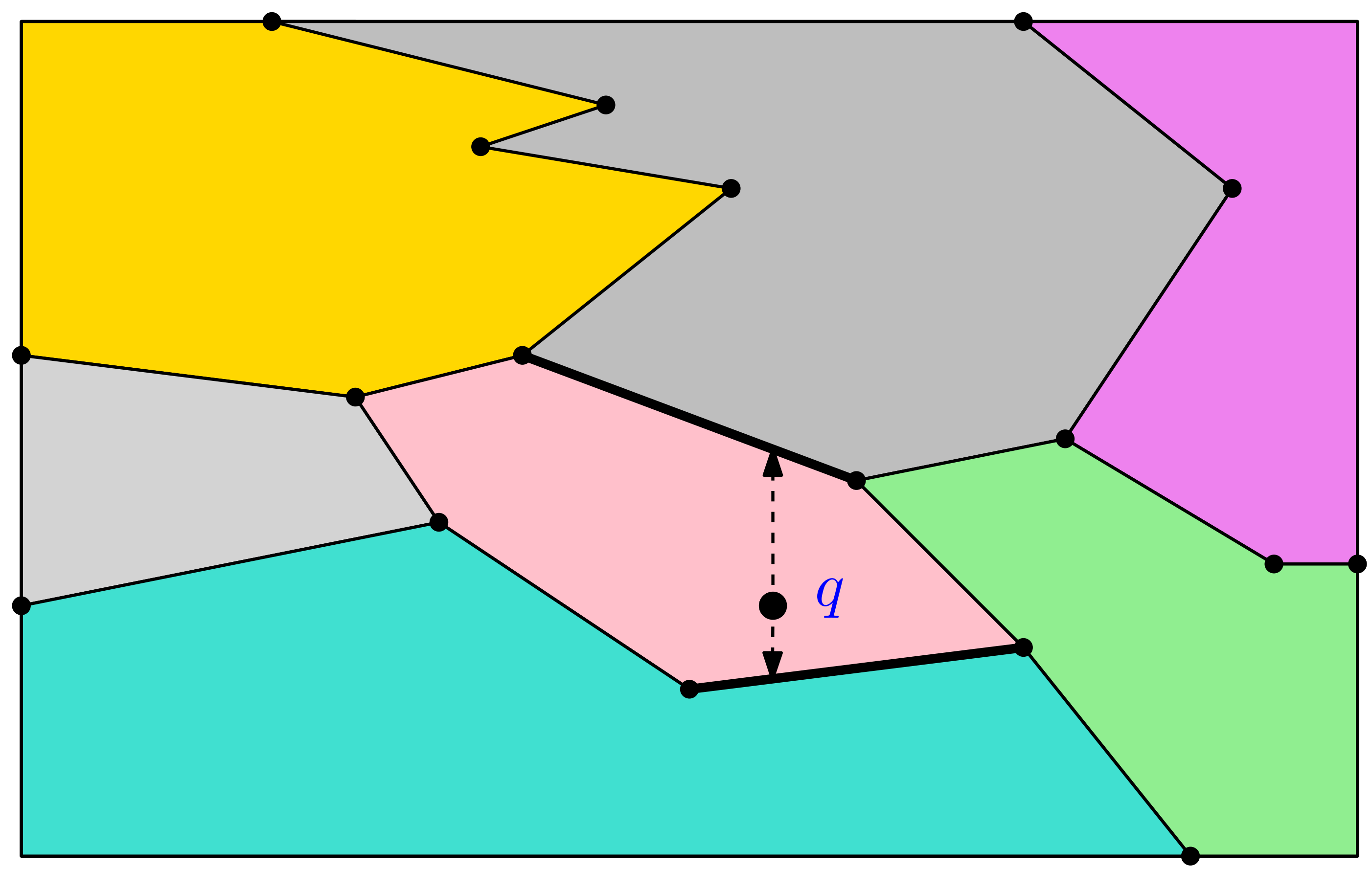


Vertical Ray Shooting

Preprocess a given set S of non-crossing segments in the plane (or a bounding box) so that for every query point q it can be determined quickly which segment of S lies immediately above(below) q.

if they intersect, then they intersect in a common endpoint





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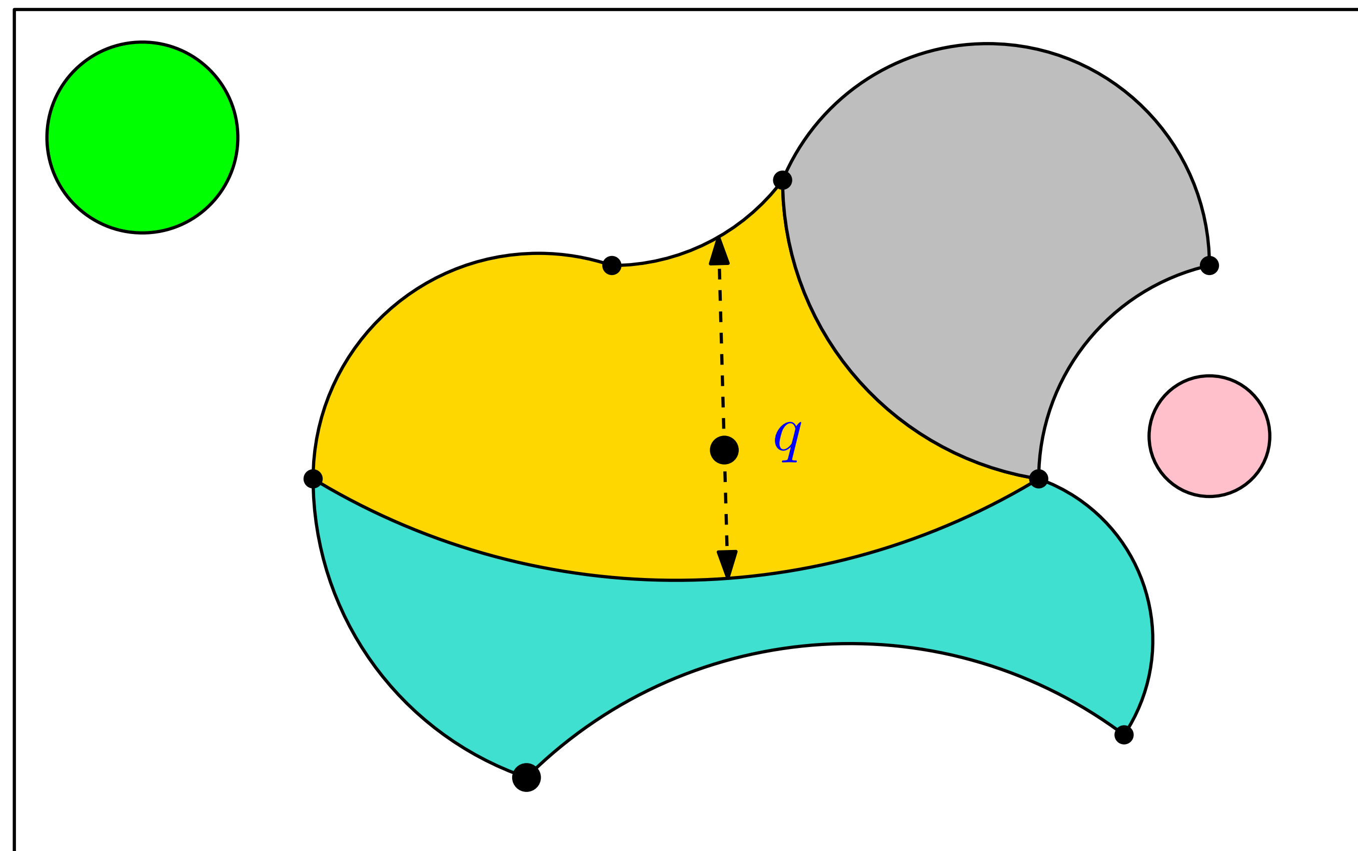
Vertical Ray Shooting

Preprocess a given set S of non-crossing curves in the plane (or a bounding box) so that for every query point q it can be determined quickly which curve of S lies immediately above(below) q.



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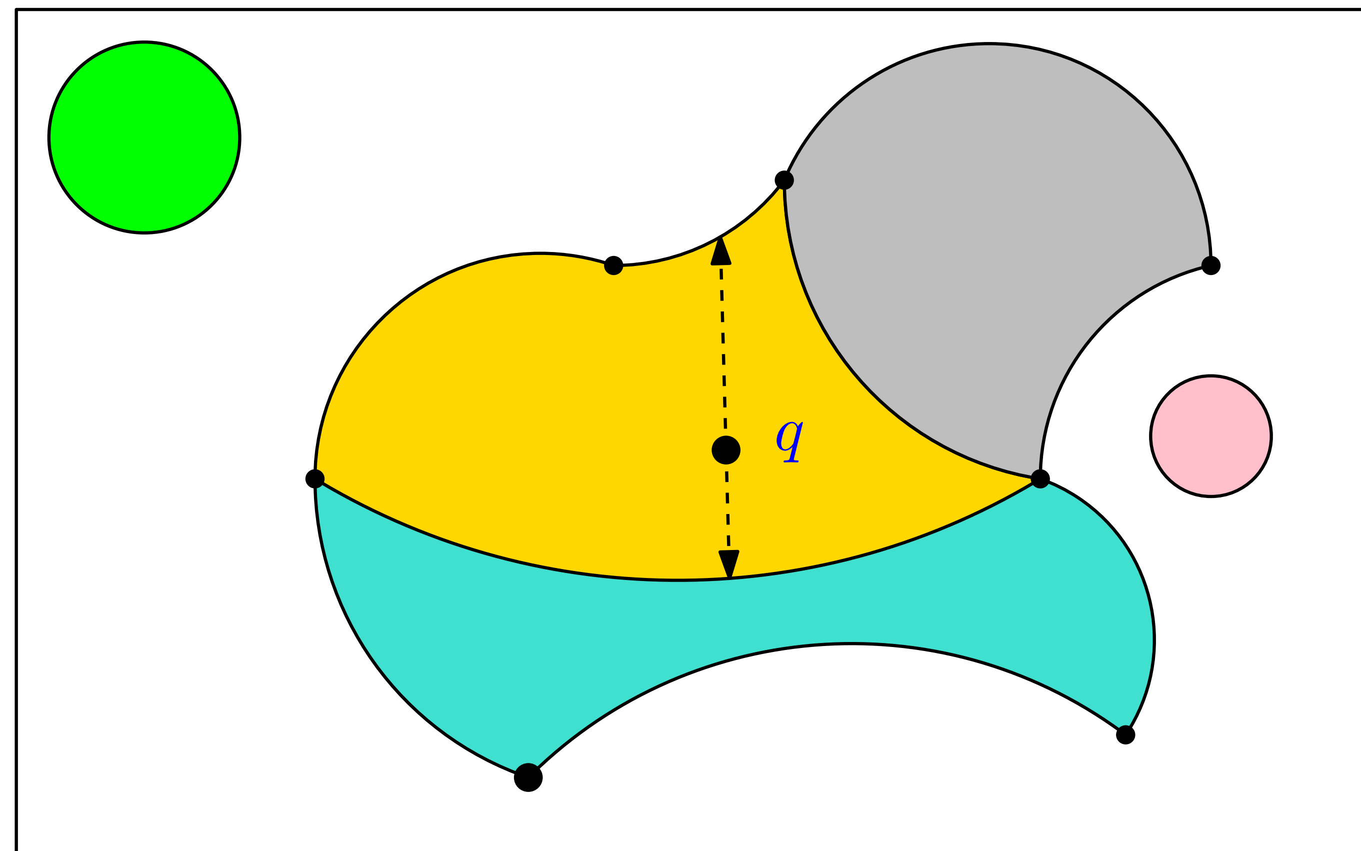
Preprocess a given set S of non-crossing x-monotone curves in the plane (or a bounding box) so that for every query point q it can be determined quickly which curve of S lies immediately above(below) q.



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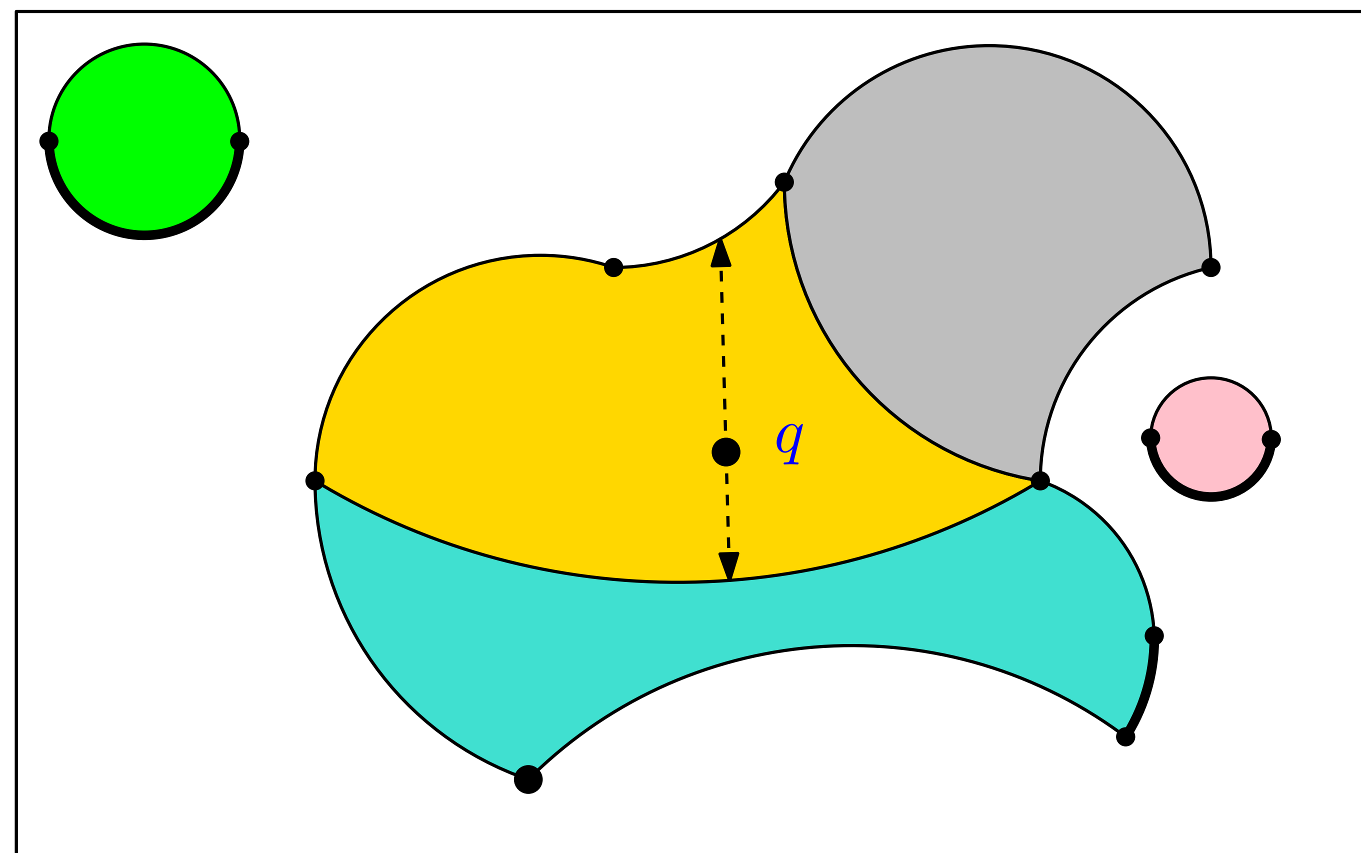

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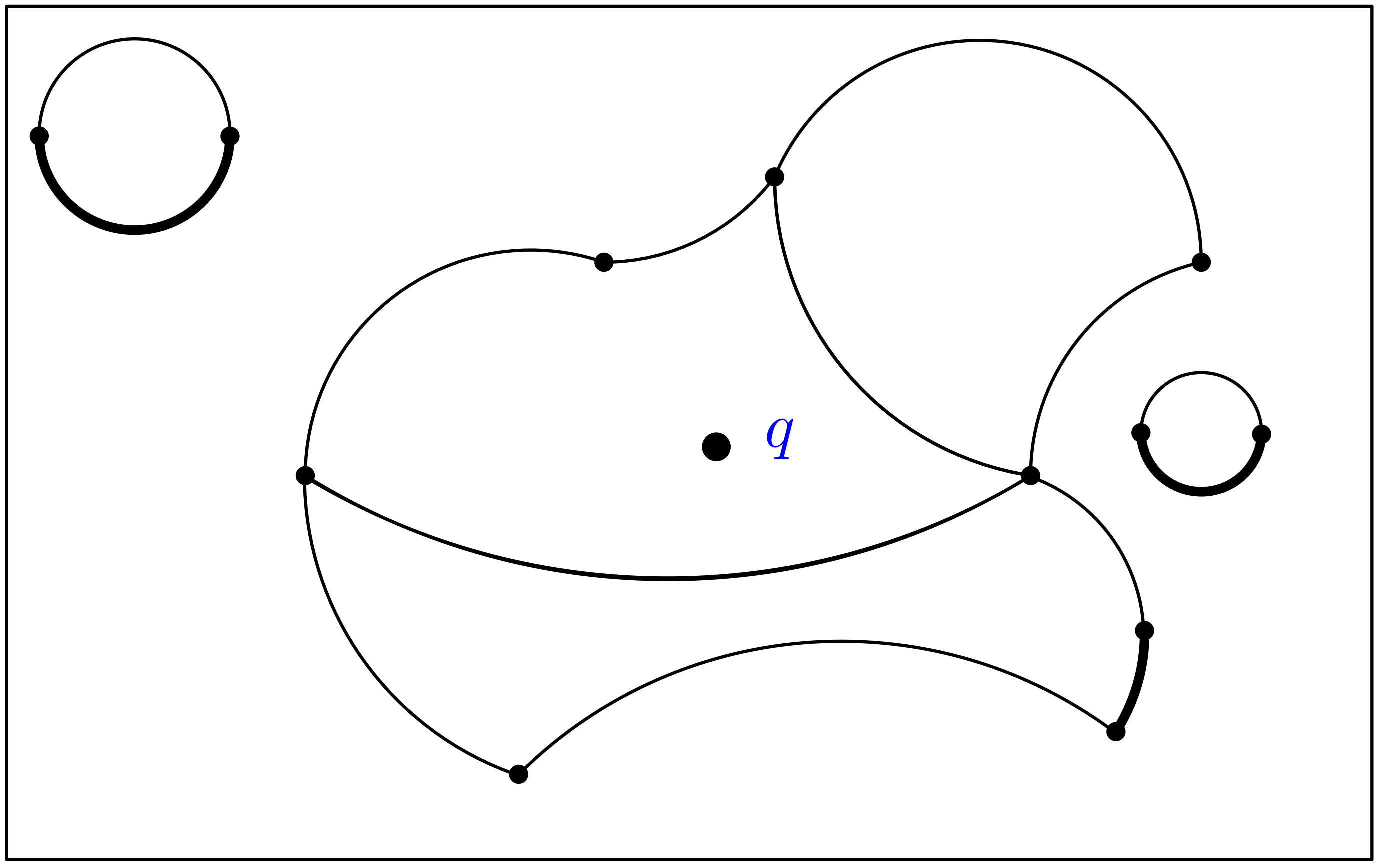
Preprocess a given set S of non-crossing x-monotone curves in the plane (or a bounding box) so that for every query point q it can be determined quickly which curve of S lies immediately above(below) q.

Computational assumption
If the vertical line through a point q intersects an x-monotone segment s then it can be determined in constant time whether q lies above, on, or below s.



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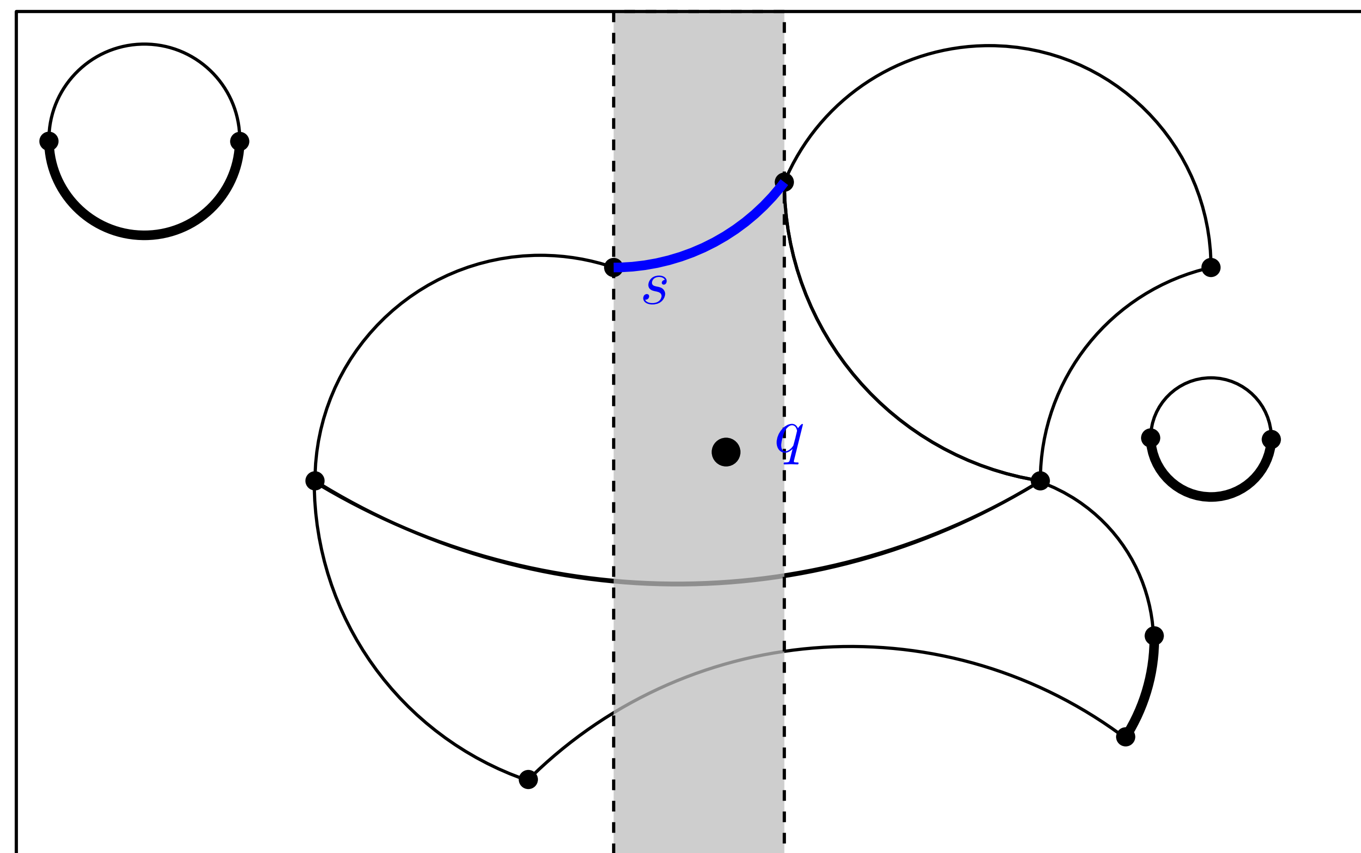
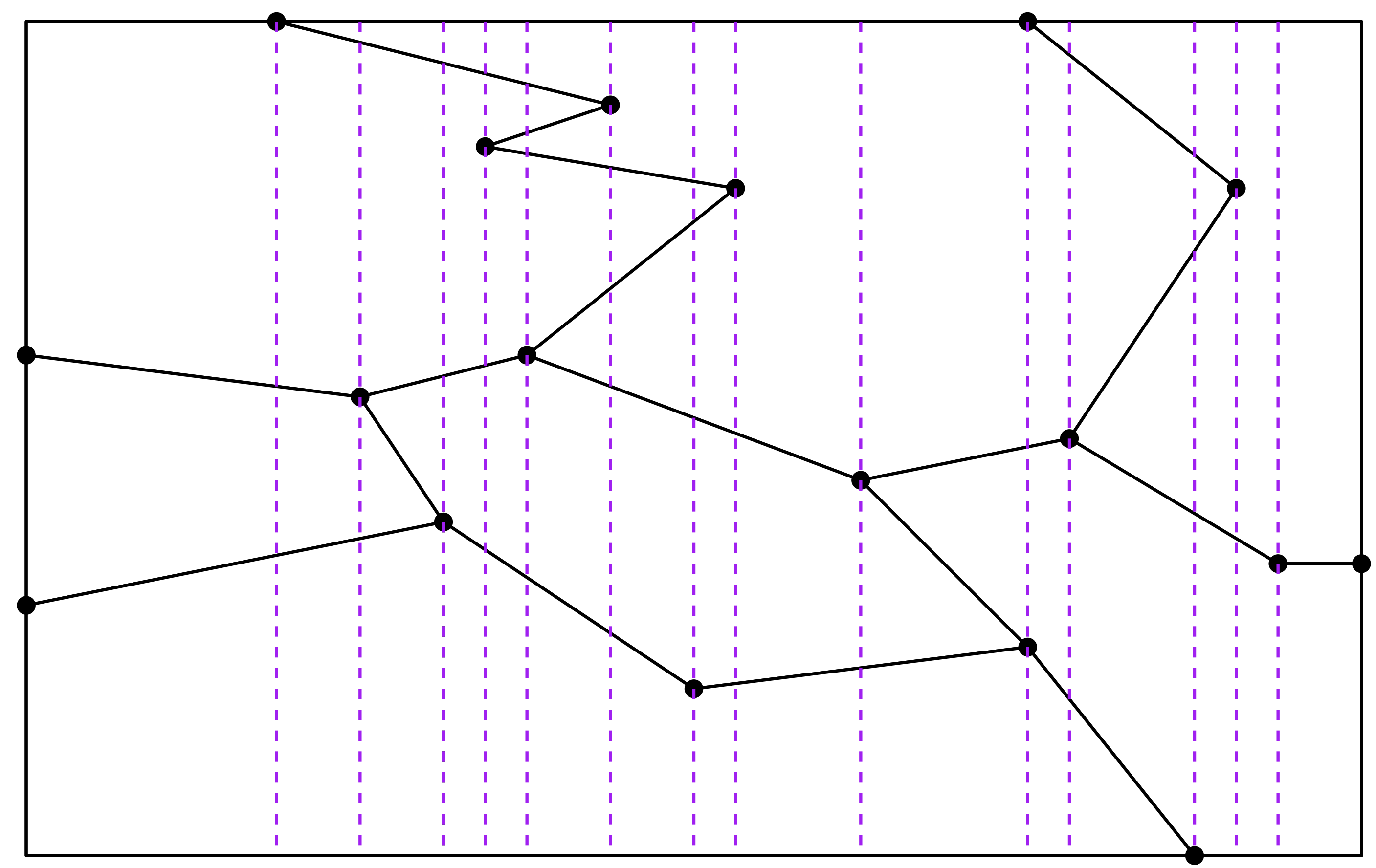


The Slab Method of Dobkin & Lipton

1. Draw a vertical line through each segment endpoint, which partitions the bounding box into slabs.
   Build a binary search structure (x-structure) that allows to determine the slab containing a query point in logarithmic time.



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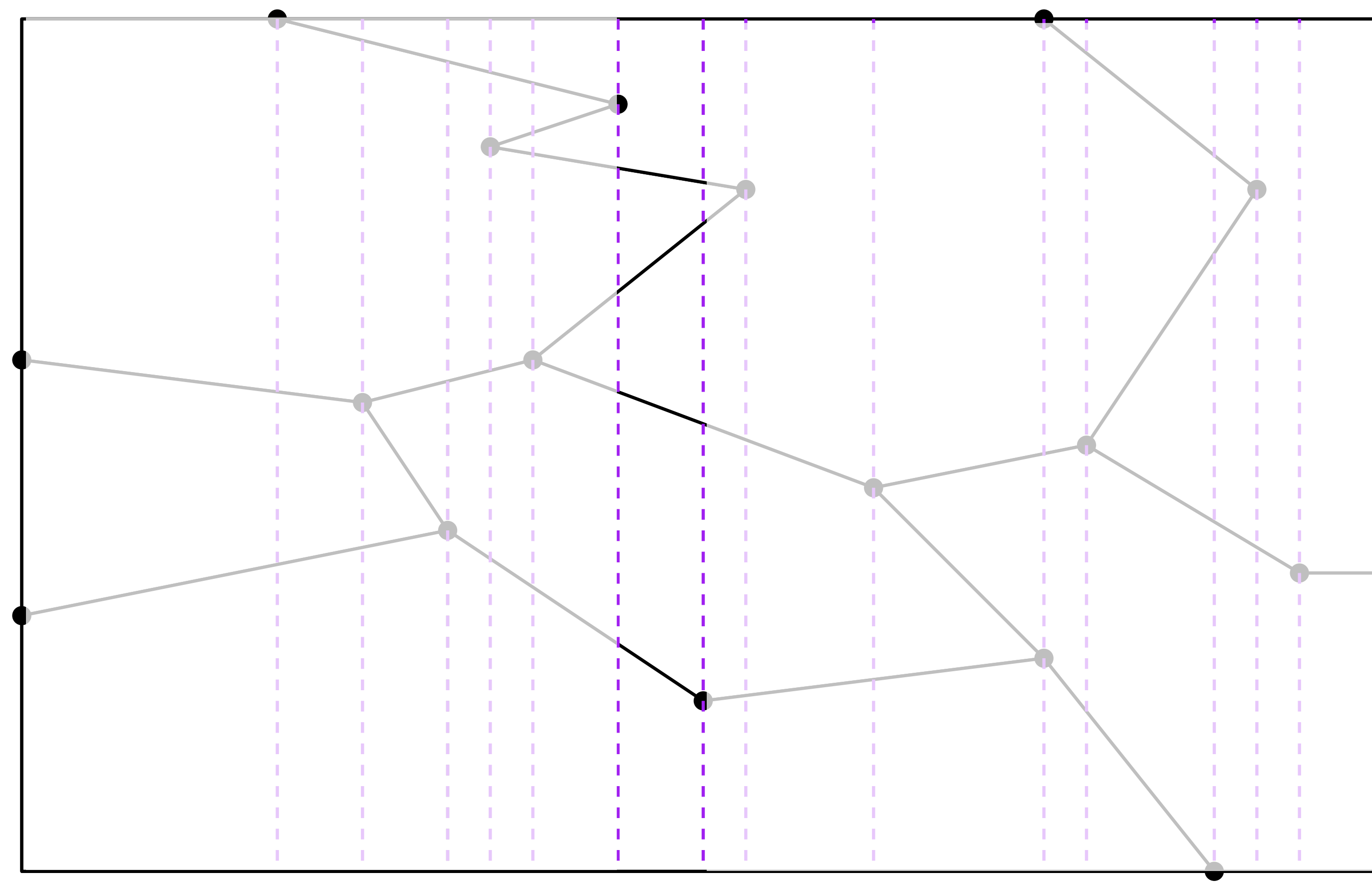

2. In each slab the segments crossing the slab are totally ordered vertically.

   For each slab build a binary search structure (y-structure) to determine the segments immediately above and below the query point.

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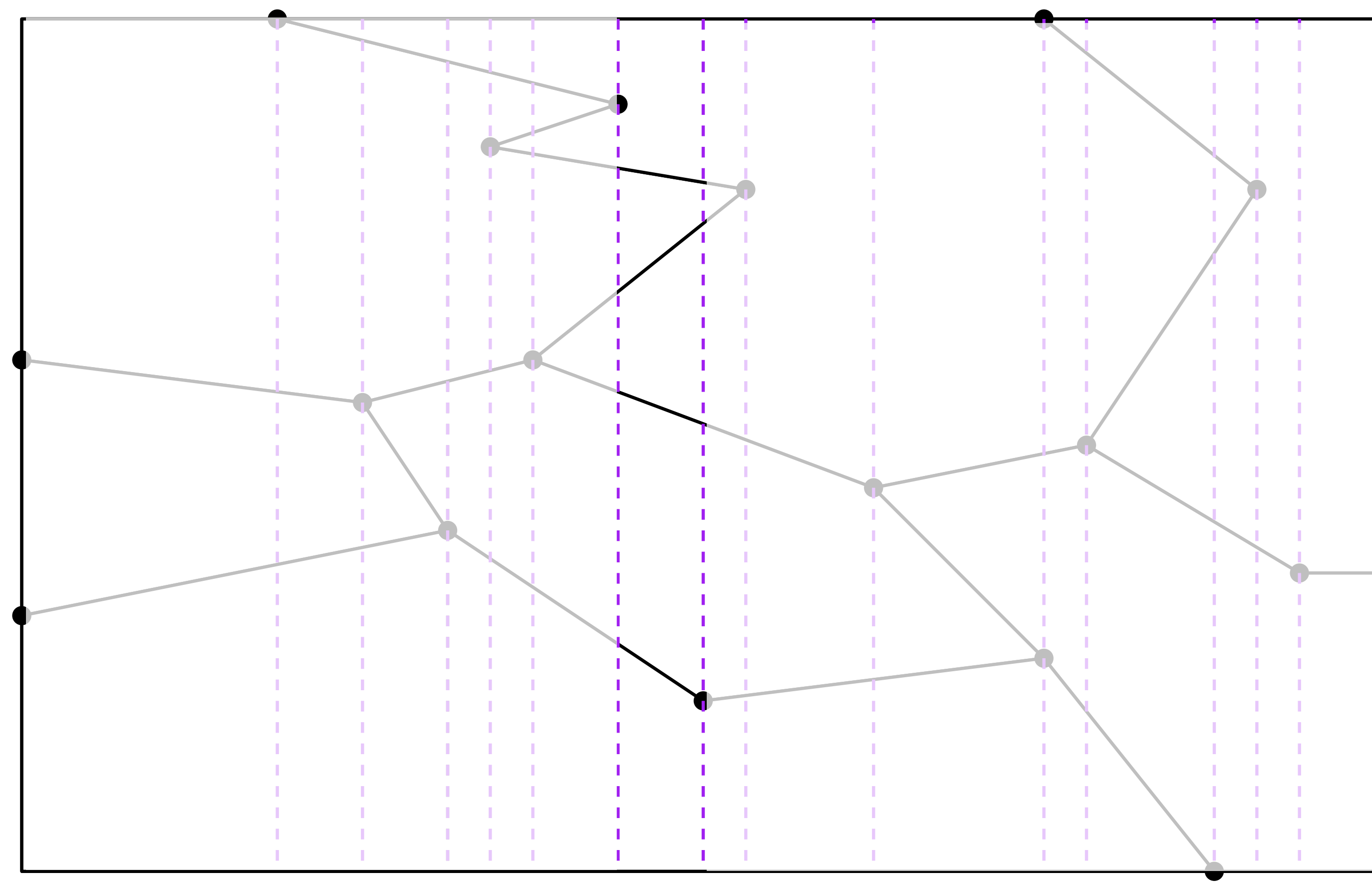


2. In each slab the segments crossing the slab are totally ordered vertically.

   For each slab build a binary search structure (y-structure) to determine the segments immediately above and below the query point.

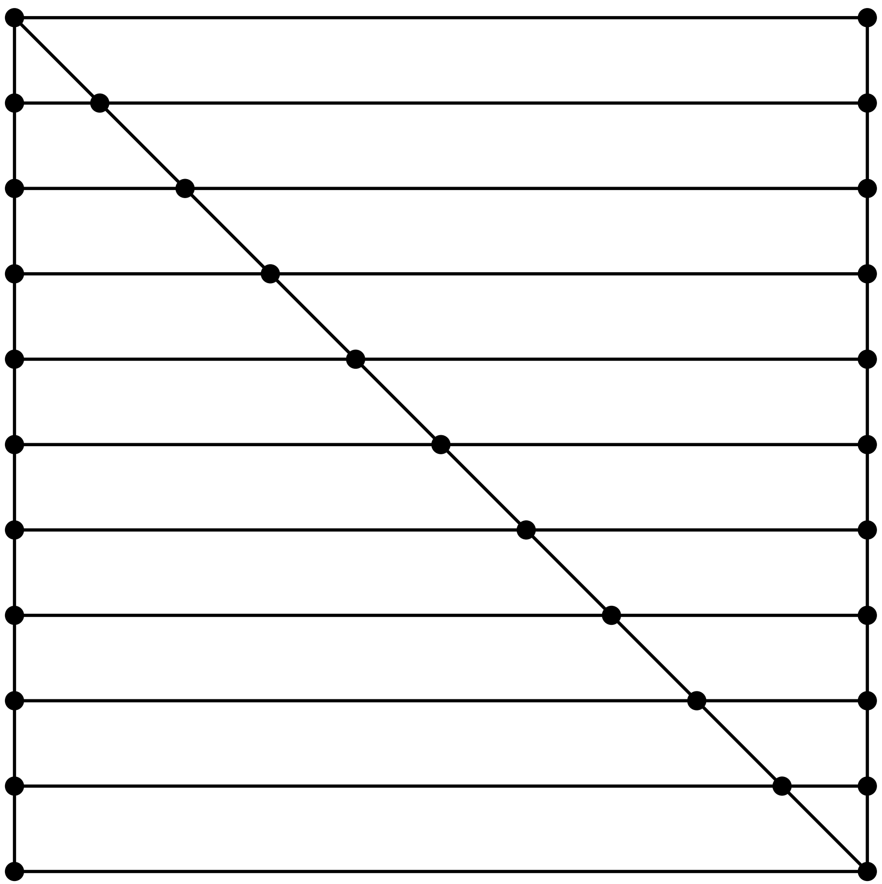
Query time is logarithmic:

$$Q(n) = 2\log_2 n + O(1)$$

The Slab Method of Dobkin & Lipton

Query time: $Q(n) = O(\log n)$

Space usage: $S(n) = O(n^2)$ in the worst case.



Inverse Range Searching Based Methods

Idea for processing query point q:

1. Identify the set $S(q)$, the set of segments in S that intersect the vertical line through q.
2. Find the correct answer within $S(q)$.

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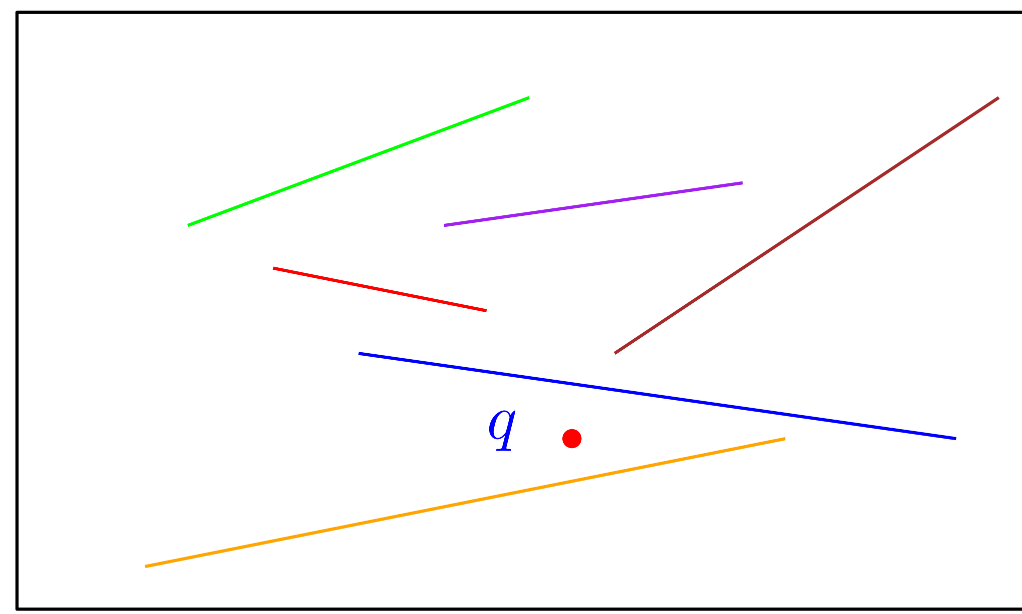

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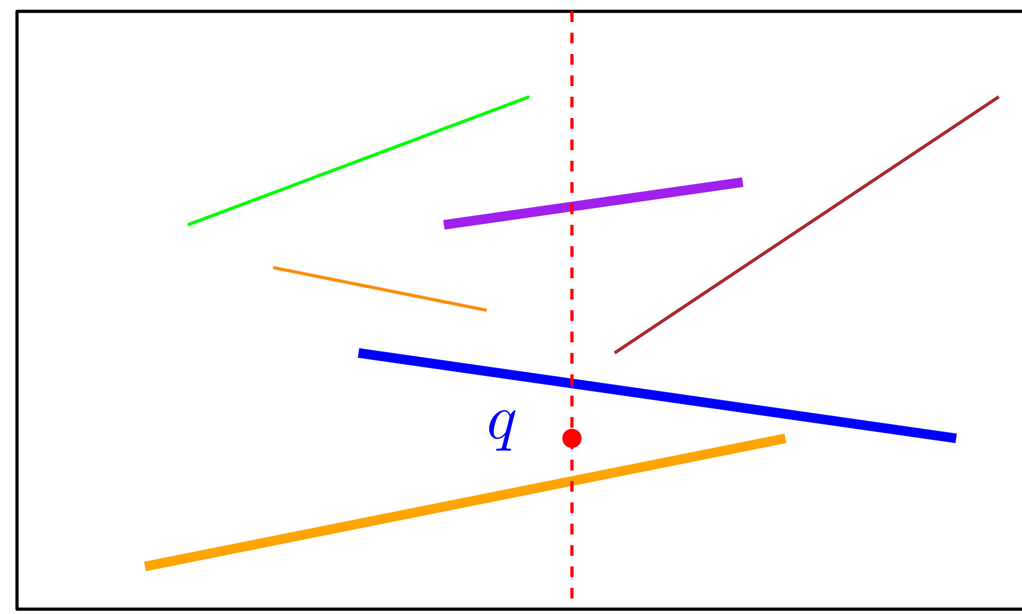

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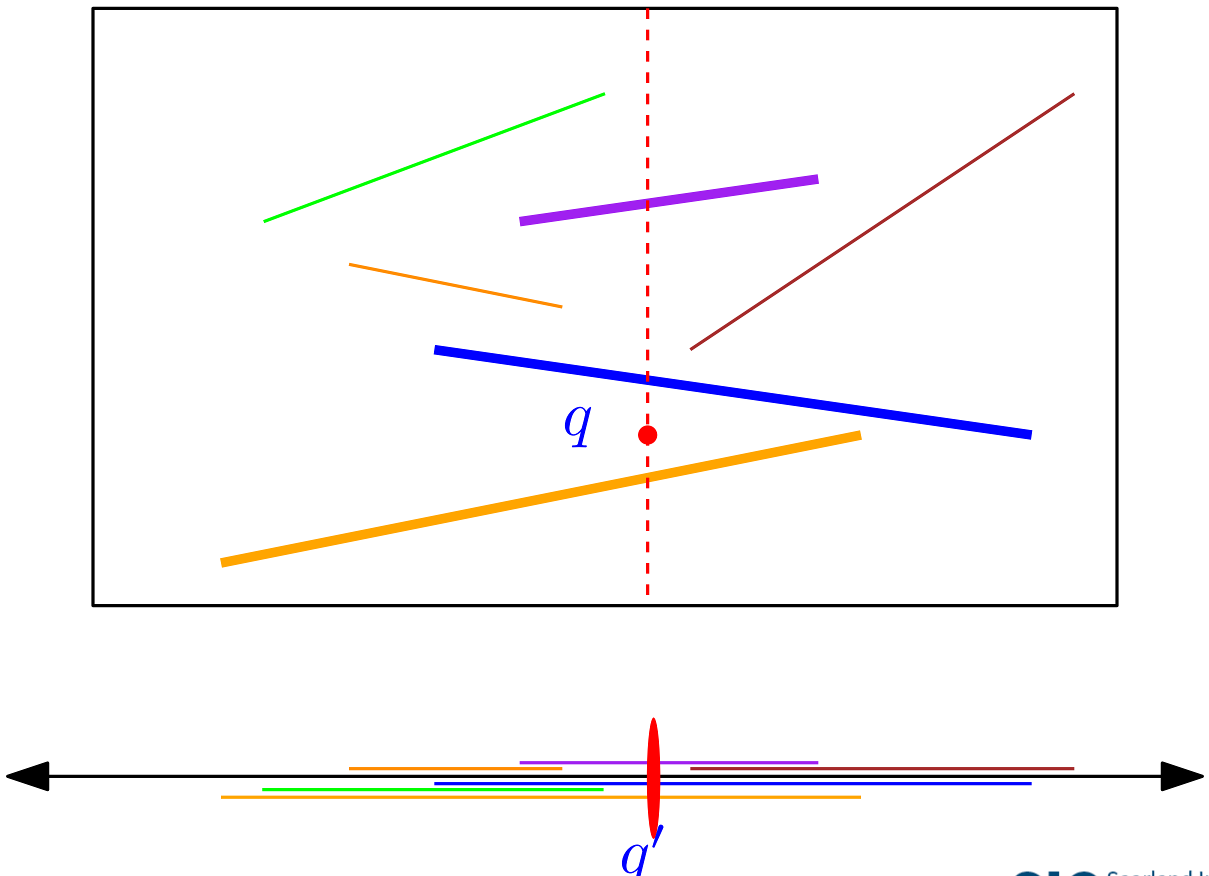
q' projection of q onto horizontal axis;
s' projection of s onto horizontal axis;

Step 1 corresponds to 1-dimensional problem of finding the intervals s' that contain q' ("inverse range searching")



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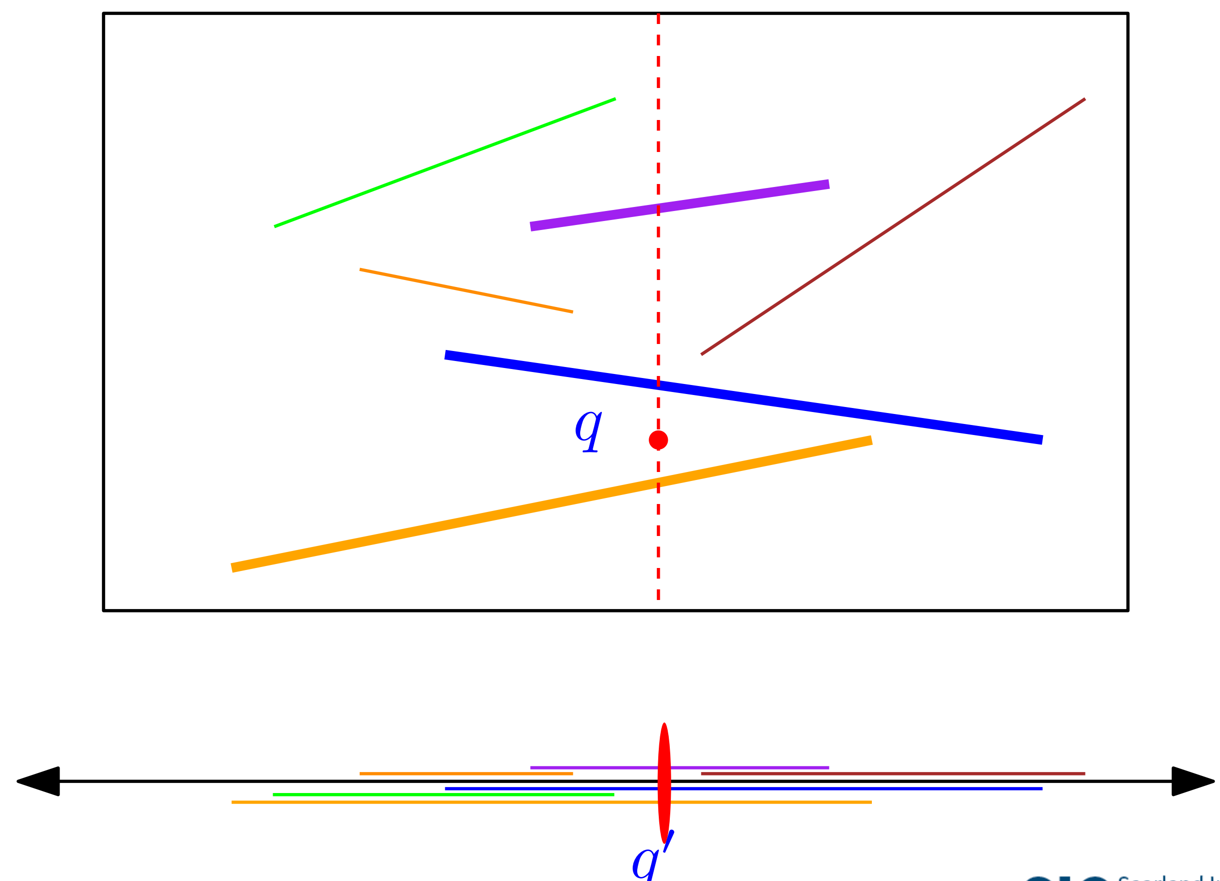
Possible solutions via
segment tree or
interval tree



Inverse Range Searching Based Methods: Segment Tree

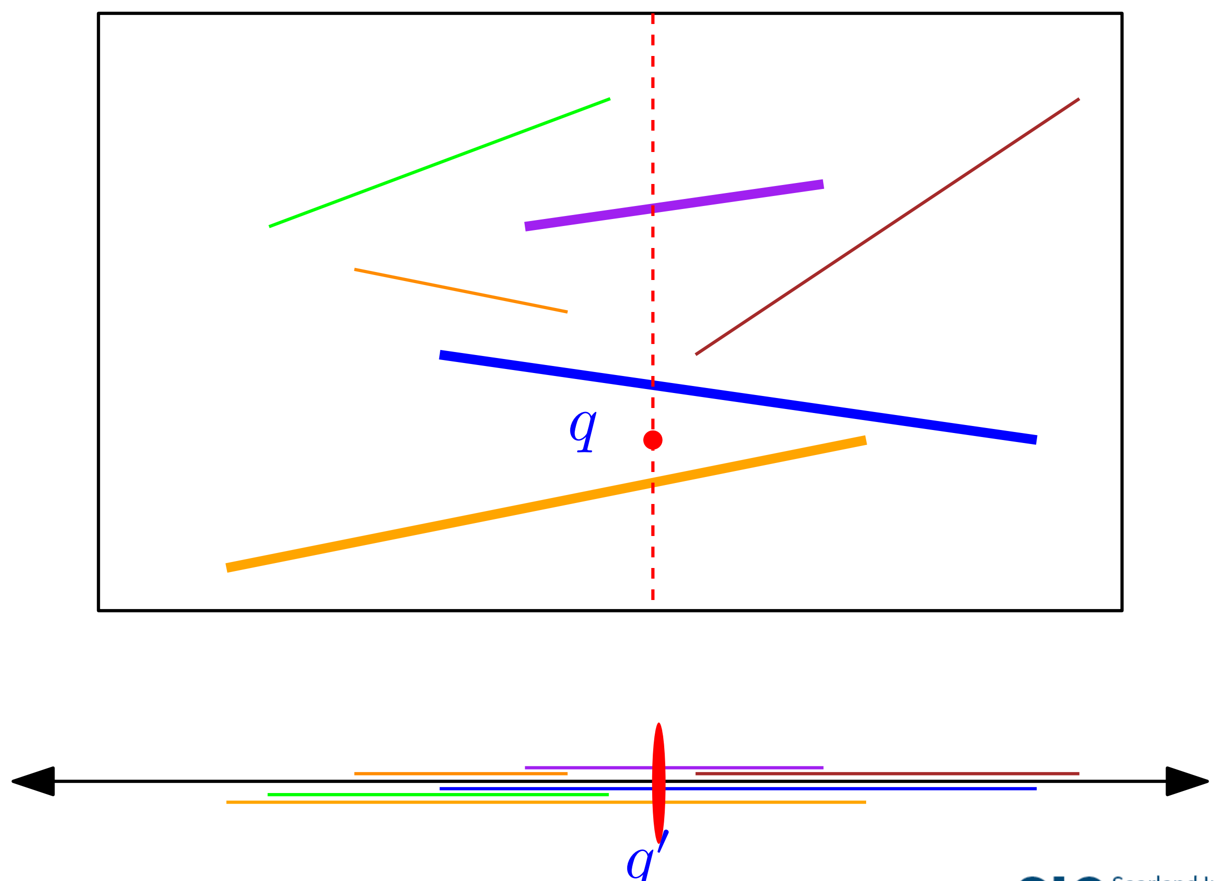
Idea for processing query point q:

1 Identify the set $S(q)$, the set of segments in S that intersect the vertical line through q.
2 Find the correct answer within $S(q)$.

Segment tree provides $S(q)$ as disjoint union of $O(\log n)$ canonical sets of segments (some S_v's from the segment tree)

Proprocess each canonical set S_v to allow vertical binary search for q

Search for q in each of the relevant canonical sets.



Query time $Q(n) = O(\log^2 n)$
Space usage $S(n) = O(n \log n)$

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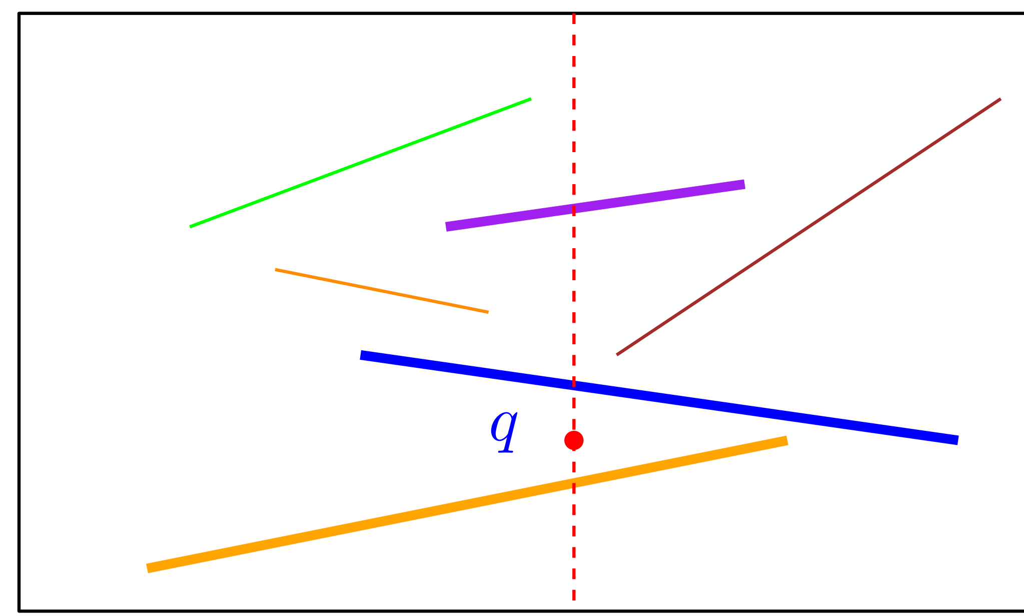

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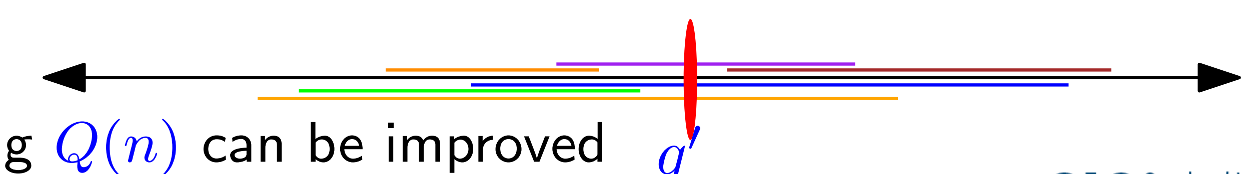


With appropriate fractional cascading $Q(n)$ can be improved to $O(\log n)$. (homework)

Inverse Range Searching Based Methods: Interval Tree

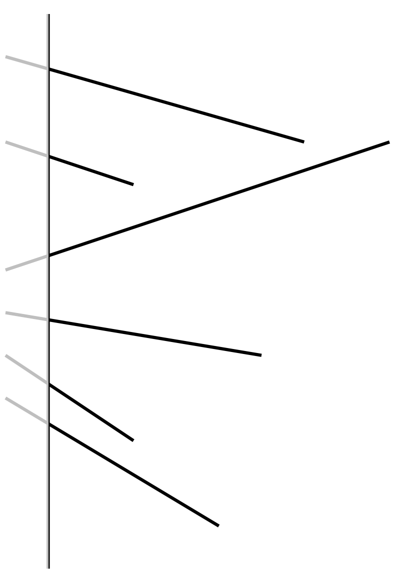
Idea for processing query point q:

1 Identify the set $S(q)$, the set of segments in S that intersect the vertical line through q.

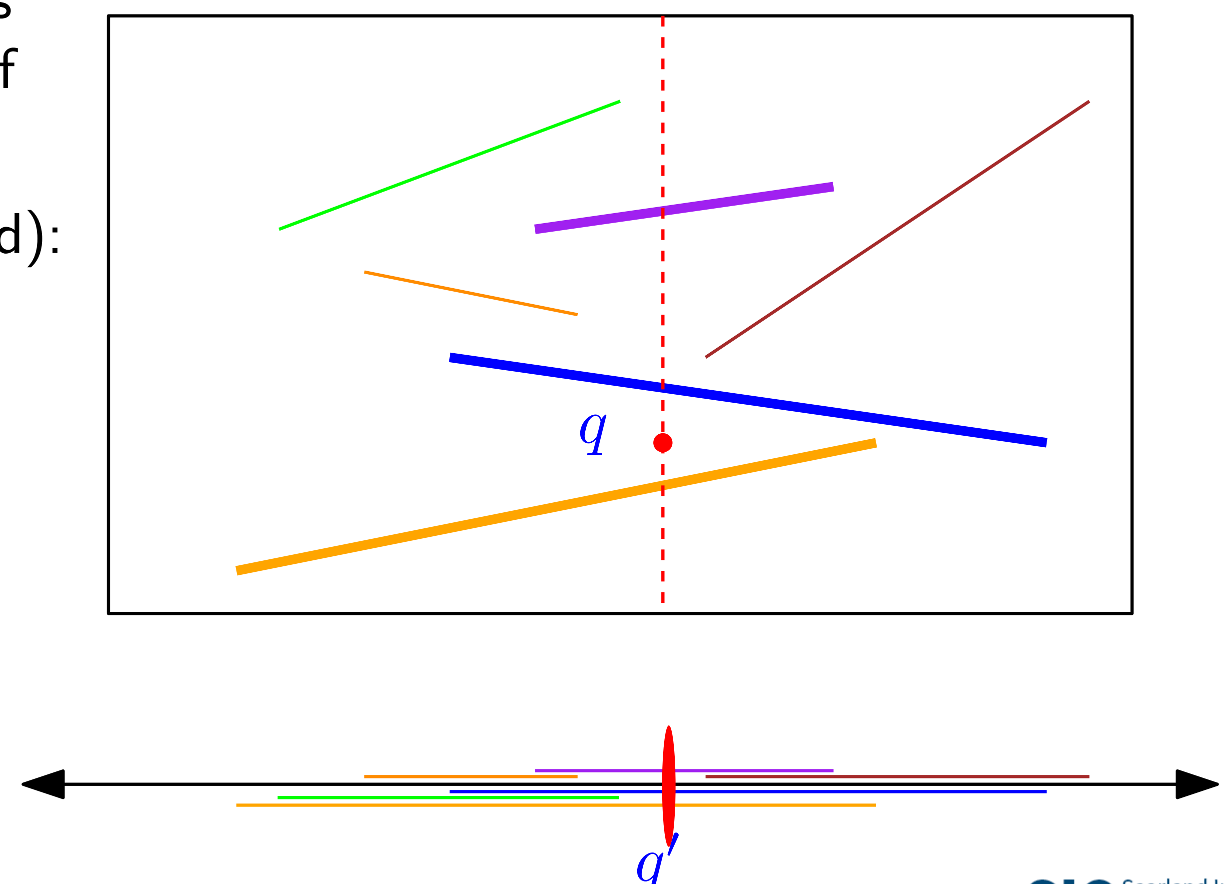
2 Find the correct answer within $S(q)$.

Interval tree provides a superset of $S(q)$ as disjoint union of $O(\log n)$ canonical sets of segments.

They have the following form (left attached):



or mirror image (right attached)



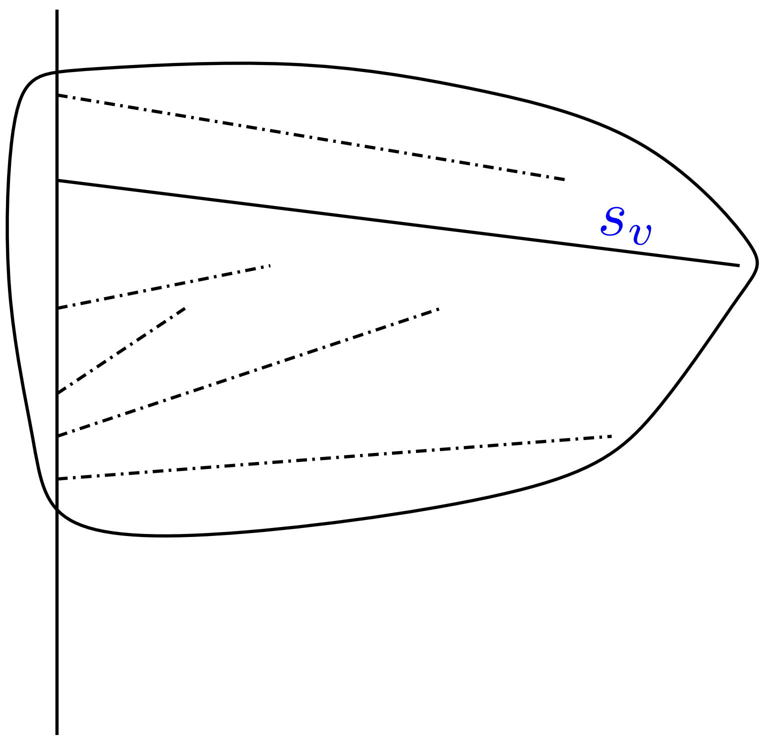
Inverse Range Searching Based Methods: Interval Tree

Want to do fast vertical ray shooting in left attached segments.

segments are vertically ordered according to their attachment point;

build binary tree T whose leaves are the segments in this vertical ordering;

for each node v in the tree store the segment s_v from T_v that extends furthest away from the attachment line;



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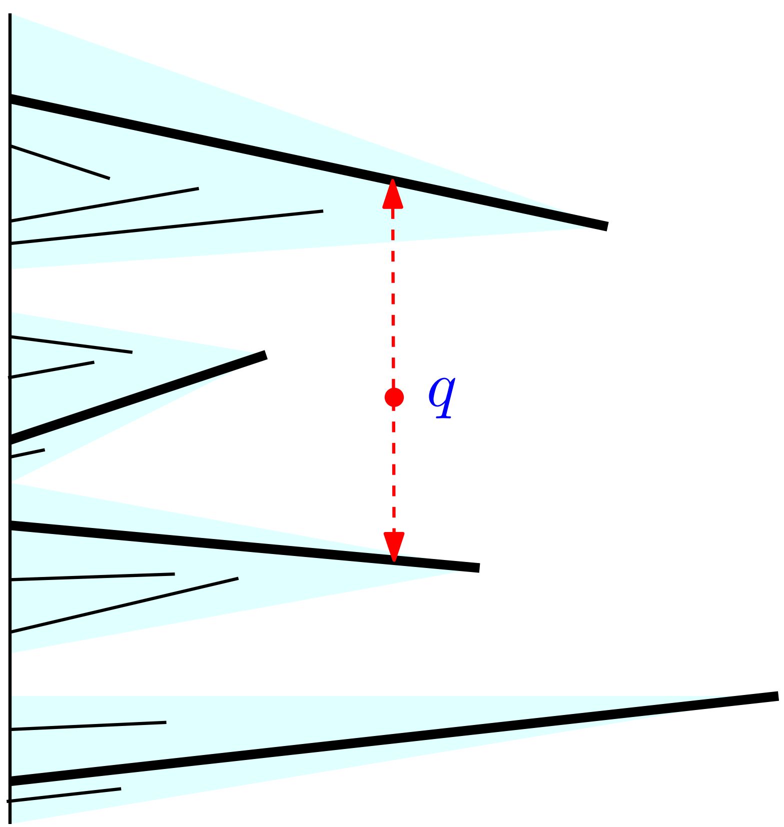
Inverse Range Searching Based Methods: Interval Tree

Want to do fast vertical ray shooting in left attached segments.

Vertical ray shooting among the s_v's from 4 nodes of T on the same level allows to eliminate at least two subtrees from consideration.

Recurse in the remaining trees.



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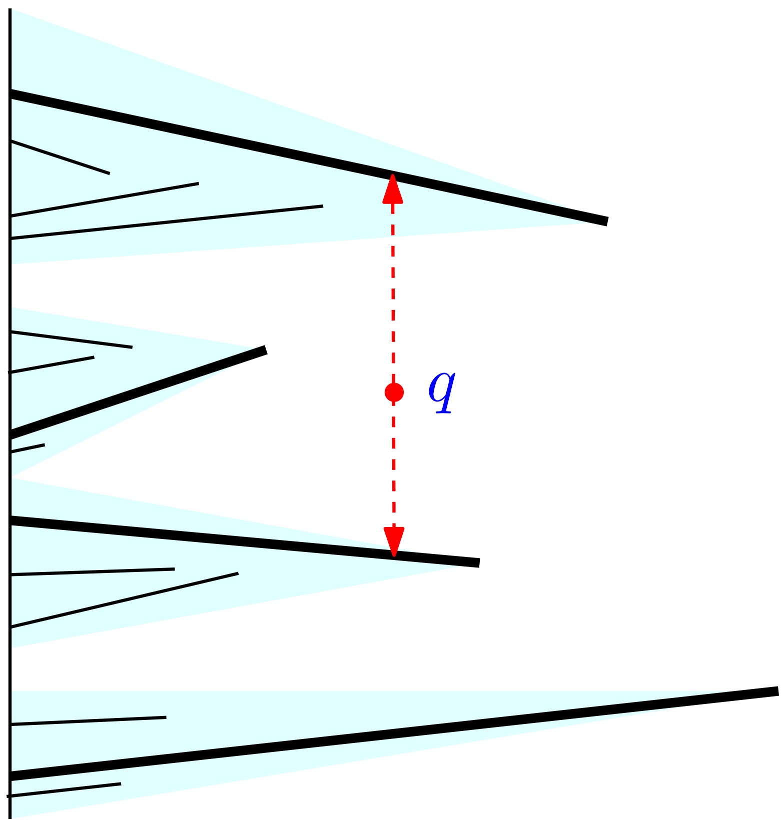

constant number of comparisons necessary to descend down one level in the tree

Therefore logarithmis search time within one set of attached segments

$Q(n) = O(\log^2 n)$ since $O(\log n)$ attached sets need to be searched

$S(n) = O(n)$ since every segment occurs in only two attachment sets

Cheng and Janardan 1992

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Optimal Planar Point Location ?

Segment tree + fractional cascading: $Q(n) = O(\log n)$ $S(n) = O(n \log n)$

Interval trees: $Q(n) = O(\log^2 n)$ $S(n) = O(n)$

Is optimal query time $Q(n) = O(\log n)$ with space $S(n) = O(n)$ possible?

Optimal Planar Point Location ?

Segment tree + fractional cascading: $Q(n) = O(\log n)$ $S(n) = O(n \log n)$

Interval trees: $Q(n) = O(\log^2 n)$ $S(n) = O(n)$

Is optimal query time $Q(n) = O(\log n)$ with space $S(n) = O(n)$ possible?

YES

1978 Lipton and Tarjan using the new planar separator theorem (very complicated, horrible constants)

1979 Kirkpatrick (simple, moderate consants, but specialized)

1984 Edelsbrunner, Guibas, and Stolfi ($Q(n) \leq 3 \cdot \log_2 n$)

1986 Sarnak and Tarjan using persistent search trees

1986 Cole based on searching similar lists

1997 Goodrich, Orletsky, and Ramaiyer ($Q(n) \leq 2 \cdot \log_2 n$)

1998 Adamy and Seidel $Q(n) \leq 1 \cdot \log_2 n + 2\sqrt{\log_2 n} + O(\sqrt[4]{\log} n)$

1990 Mulmuley / Seidel randomized methods

• Planar point location

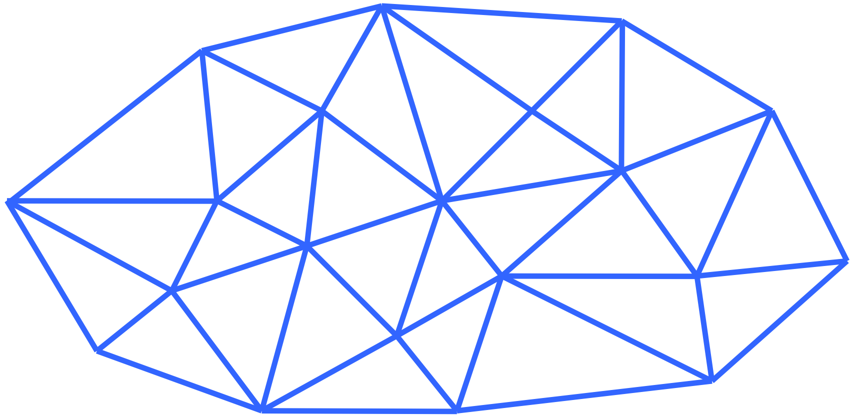
Optimal methods:

- Lipton – Tarjan
- Kirkpatrick
- Edelsbrunner – Guibas - Stolfi
- Cole
- Sarnak – Tarjan
- randomized

Other methods:

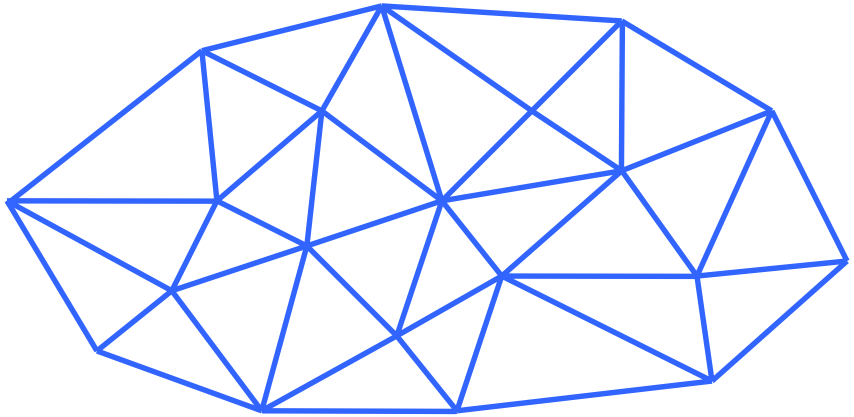
- via segment trees / via interval trees
- trapezoidal search trees
- constant optimal methods
- via cuttings
- distribution adaptive methods
- ...

Kirkpatrick's hierarchy for straight edge, triangulated subdivisions



subdivision G

Kirkpatrick's hierarchy for straight edge, triangulated subdivisions



subdivision G

to obtain **smaller** G'

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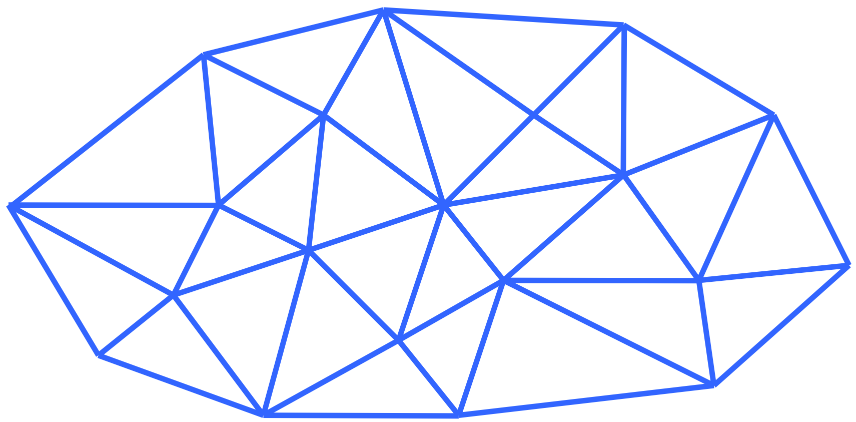

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remove low degree vertex
and retriangulate hole

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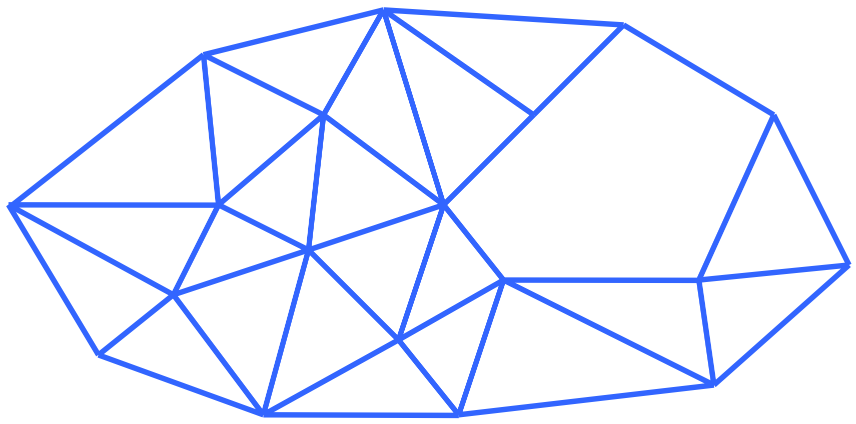

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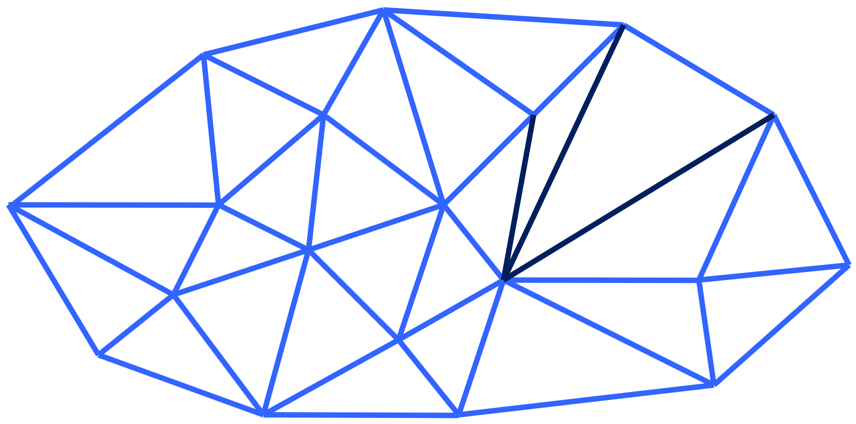

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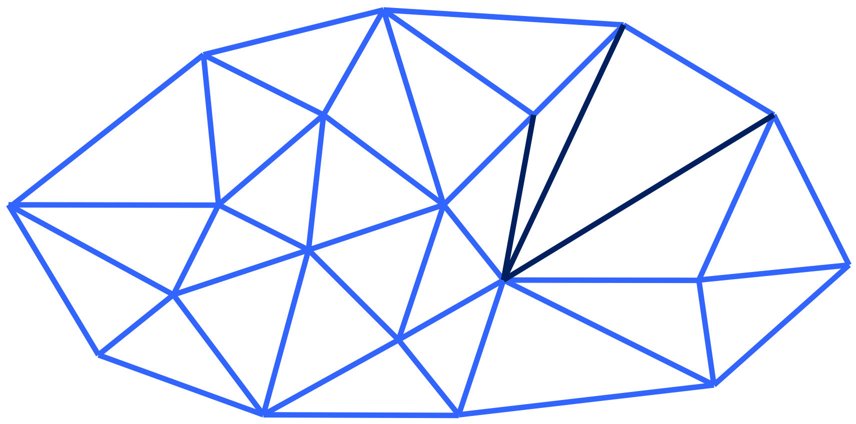


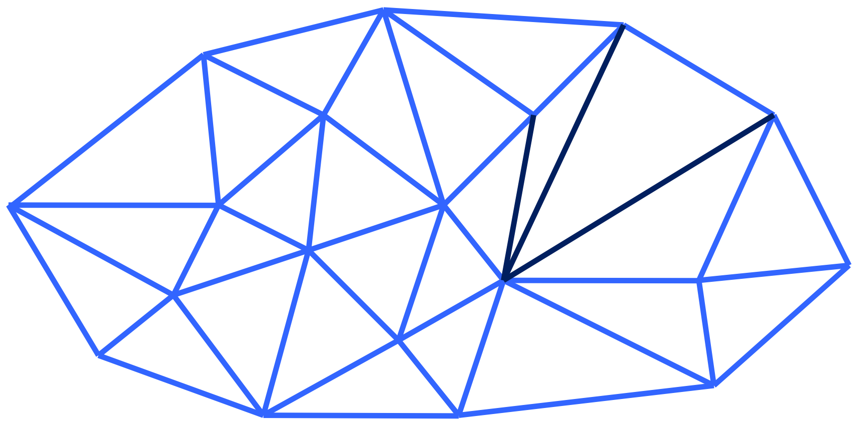
subdivision *G*

to obtain **smaller** *G'*

- remove low degree vertex
  and retriangulate hole

repeat recursively

Kirkpatrick's hierarchy for straight edge, triangulated subdivisions



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Query for point q :

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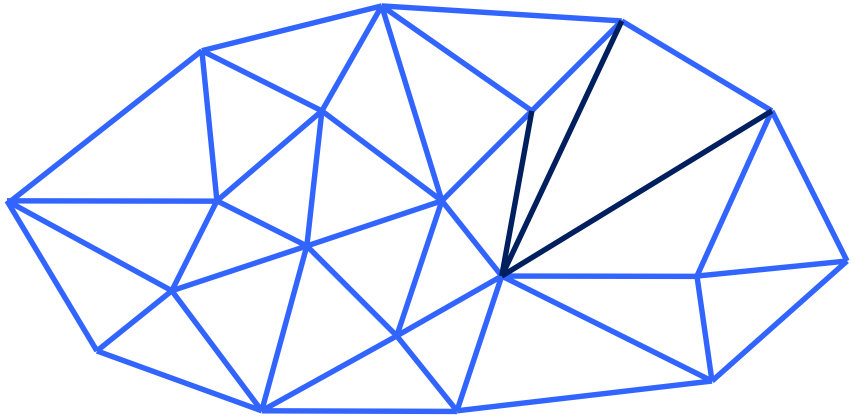

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Query for point q :

locate q in G'

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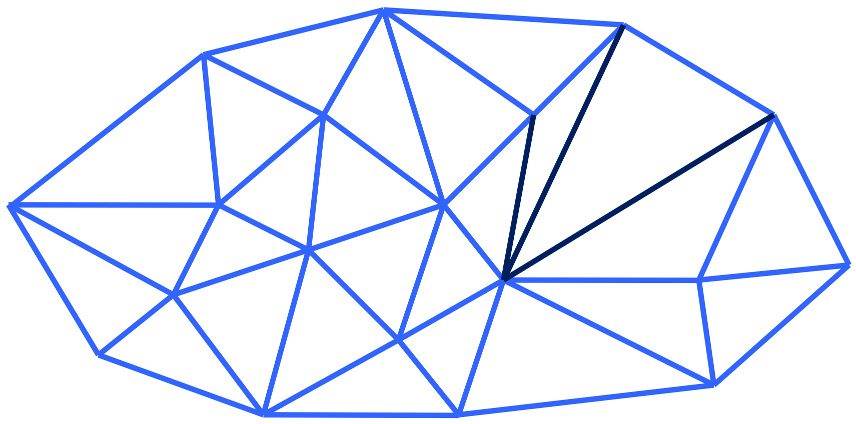

subdivision G

to obtain **smaller** G'

remove low degree vertex
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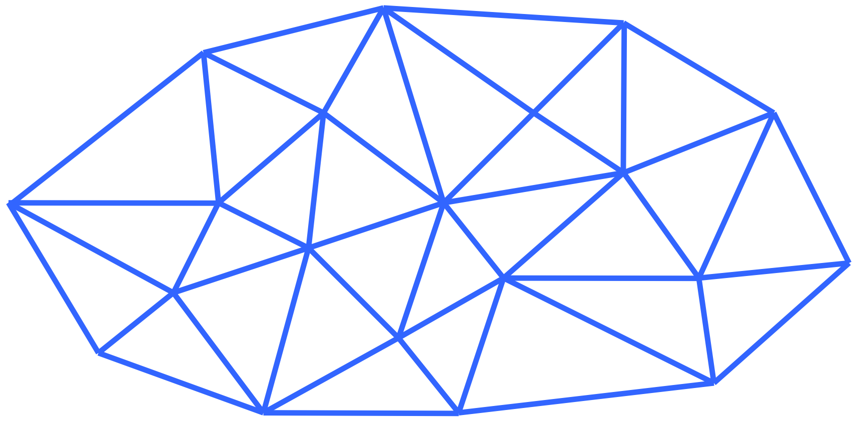
Query for point q :

locate q in G'

if q in "black" triangle then determine correct triangle of G
else triangle is correct answer already

Kirkpatrick's hierarchy for straight edge, triangulated subdivisions



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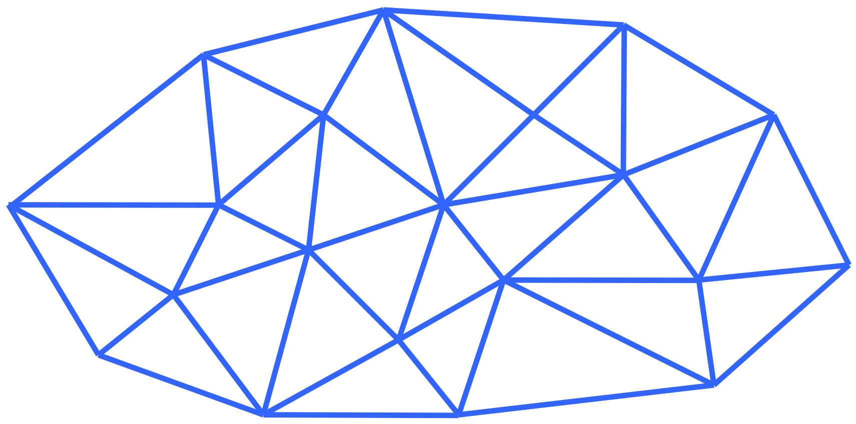
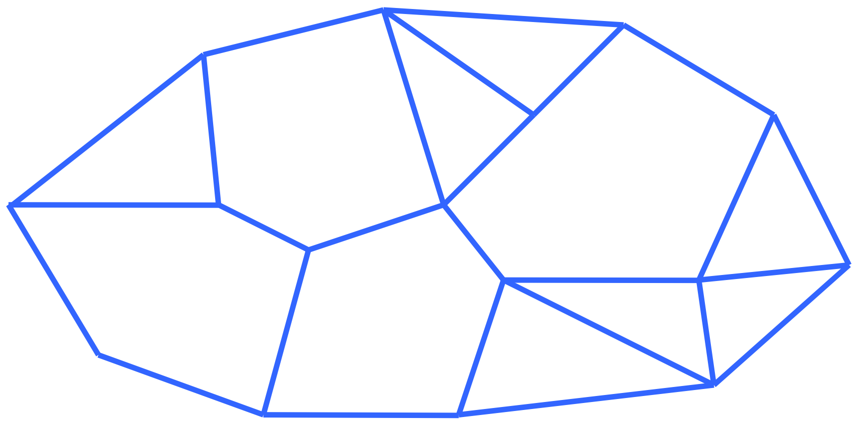


subdivision G

to obtain **smaller** G'

remove large independent set of low degree vertices and retriangulate holes

Kirkpatrick's hierarchy for straight edge, triangulated subdivisions



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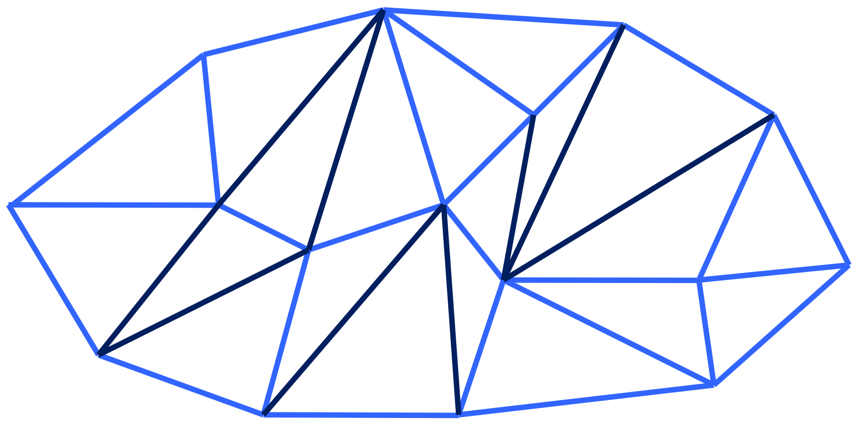

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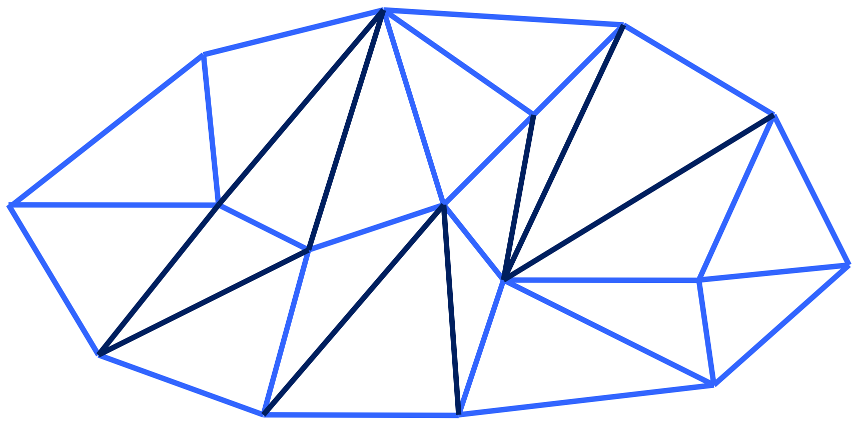

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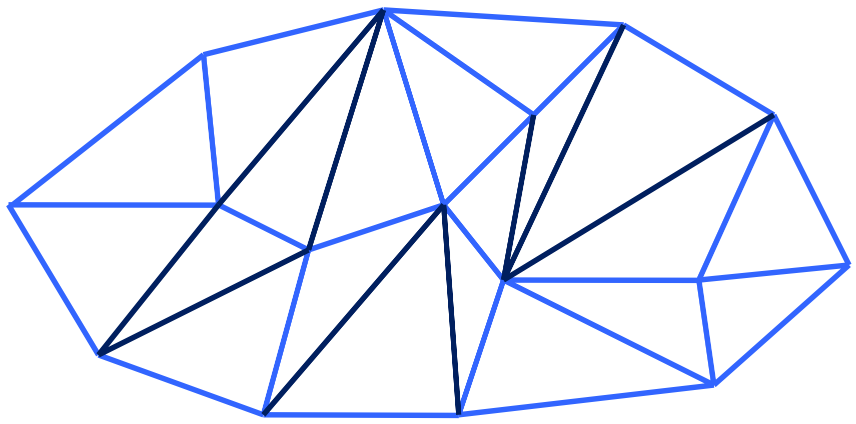

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Lemma: For every $d \geq 6$ there exists an $\alpha > 0$ such that every n-vertex planar graph has an independent set of at least αn vertices of degree $\leq d$.

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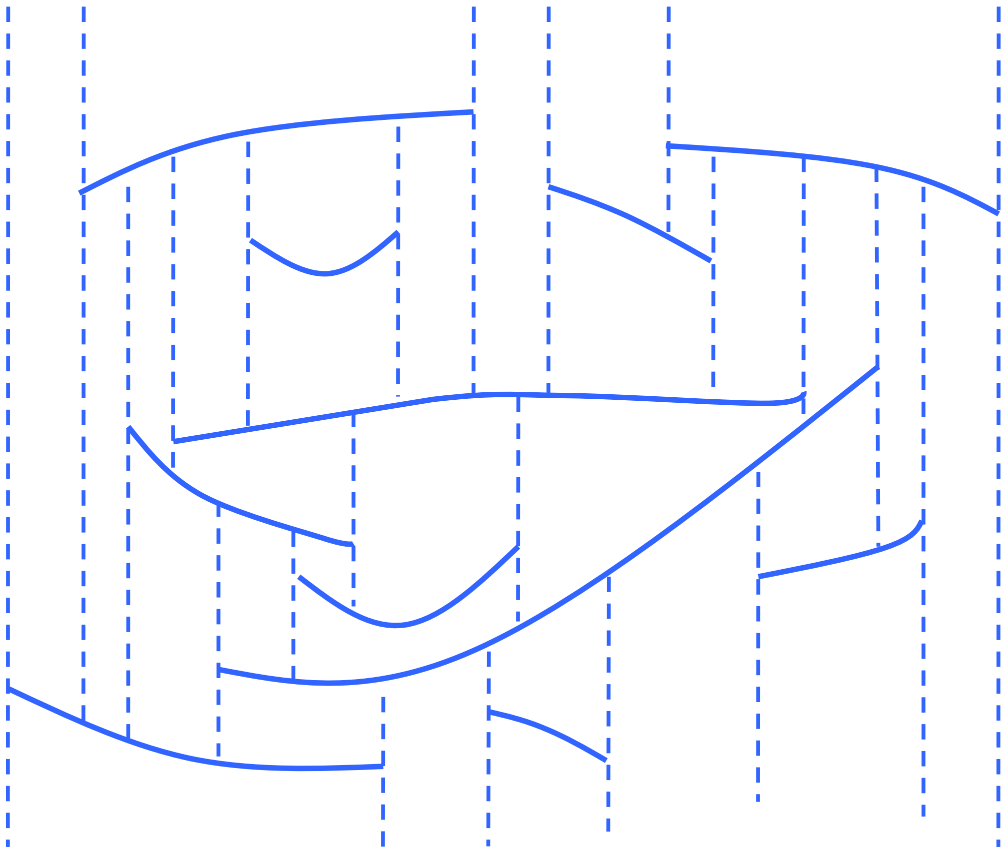
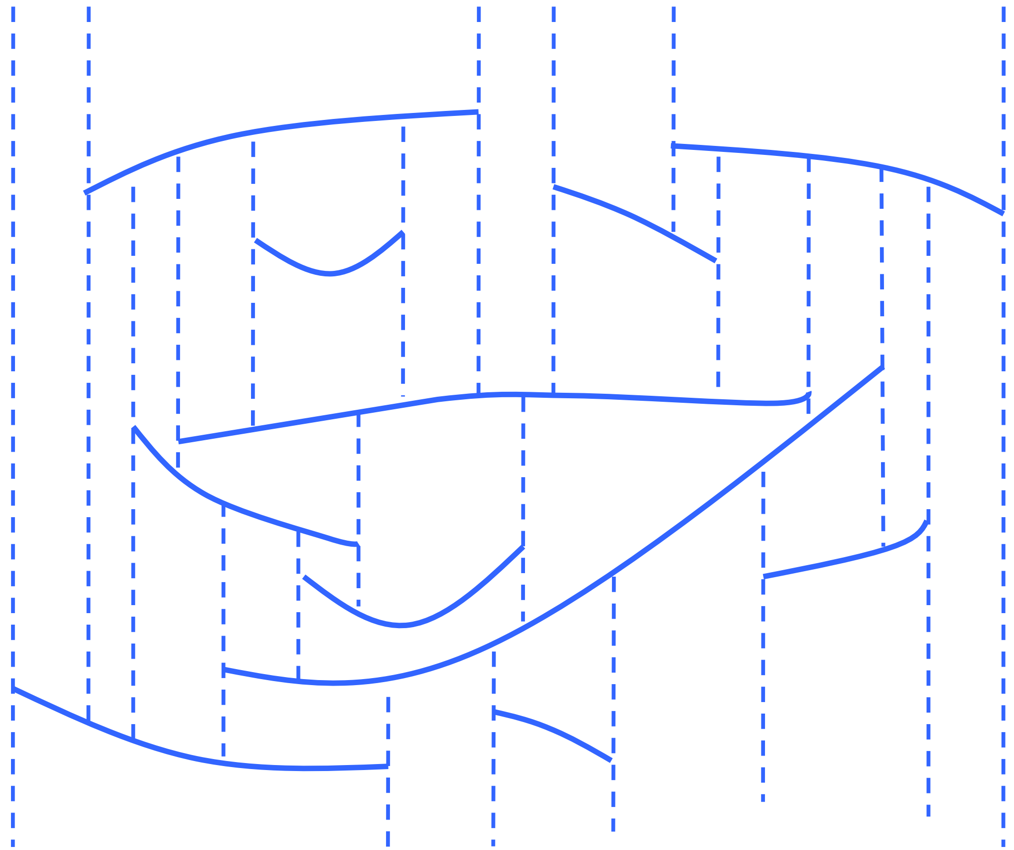
Idea: apply this hierarchical approach to trapezoidations and but remove segments instead of vertices.

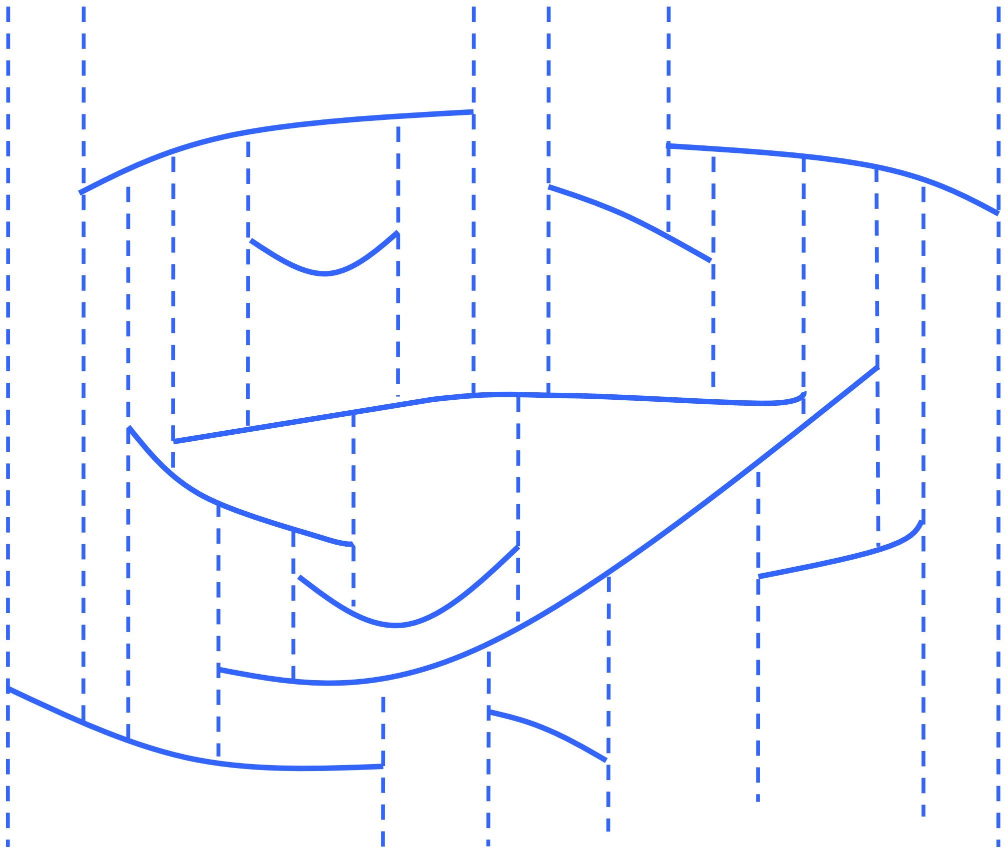


trapezoidation G



trapezoidation *G*

to obtain **smaller** *G'*



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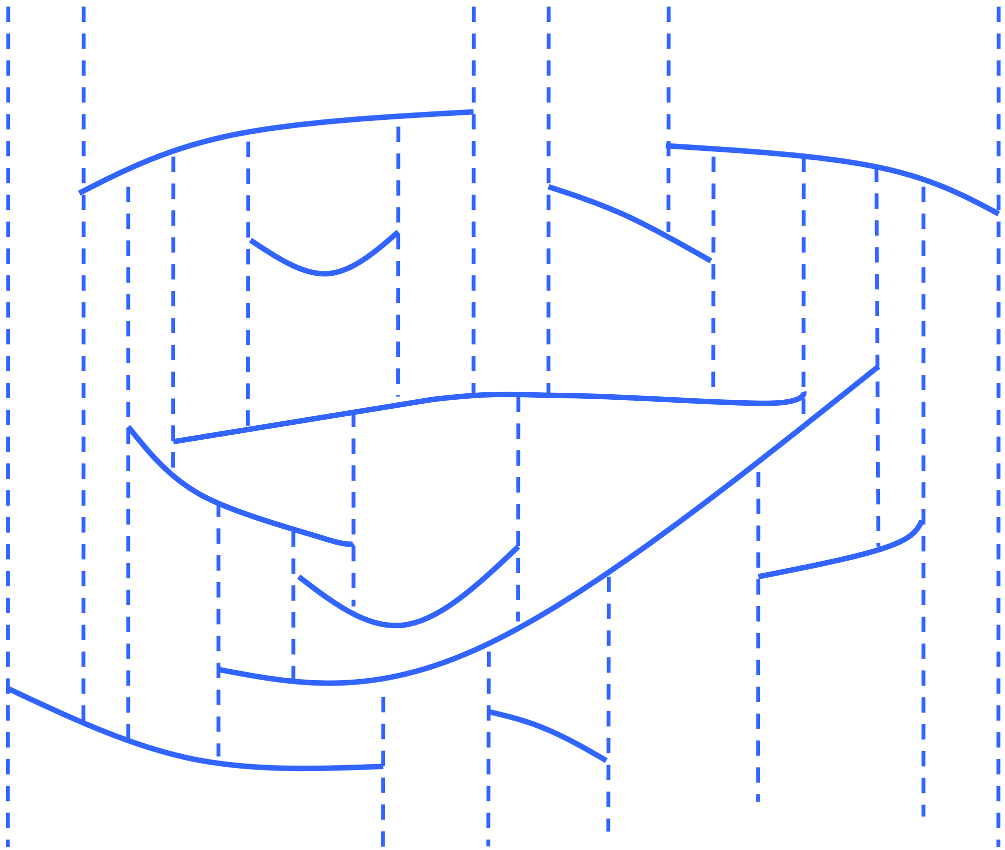
remove **some** segment



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remove **some** segment
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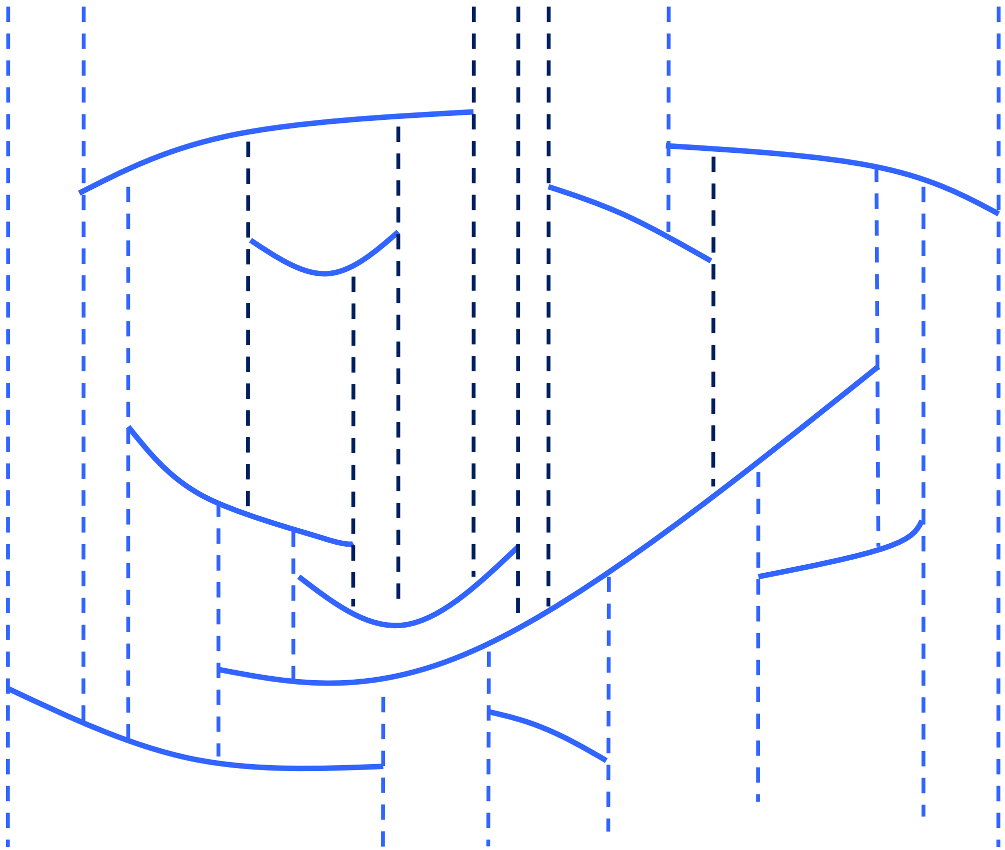

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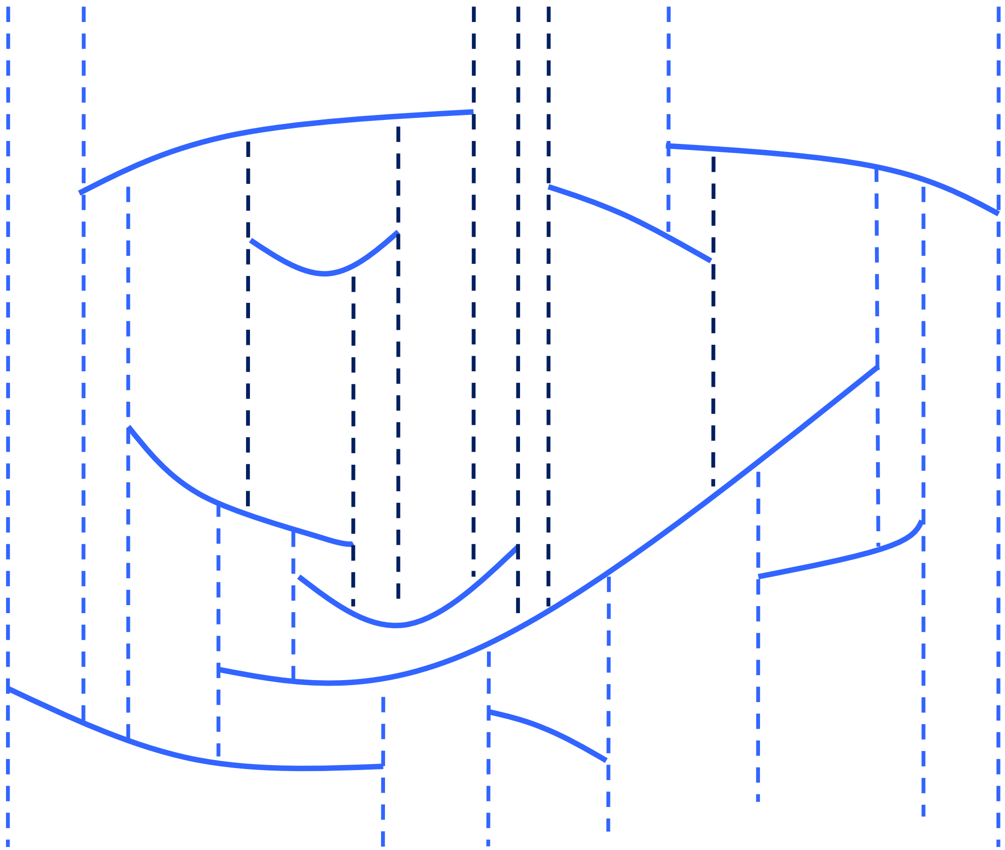

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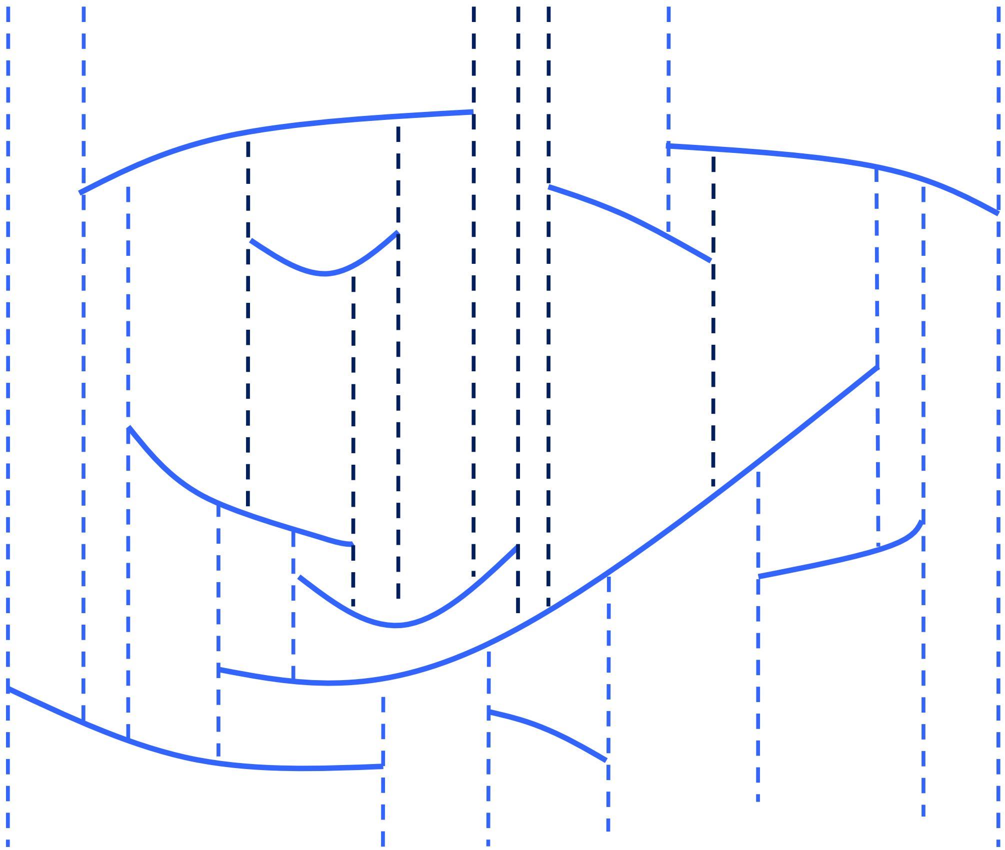

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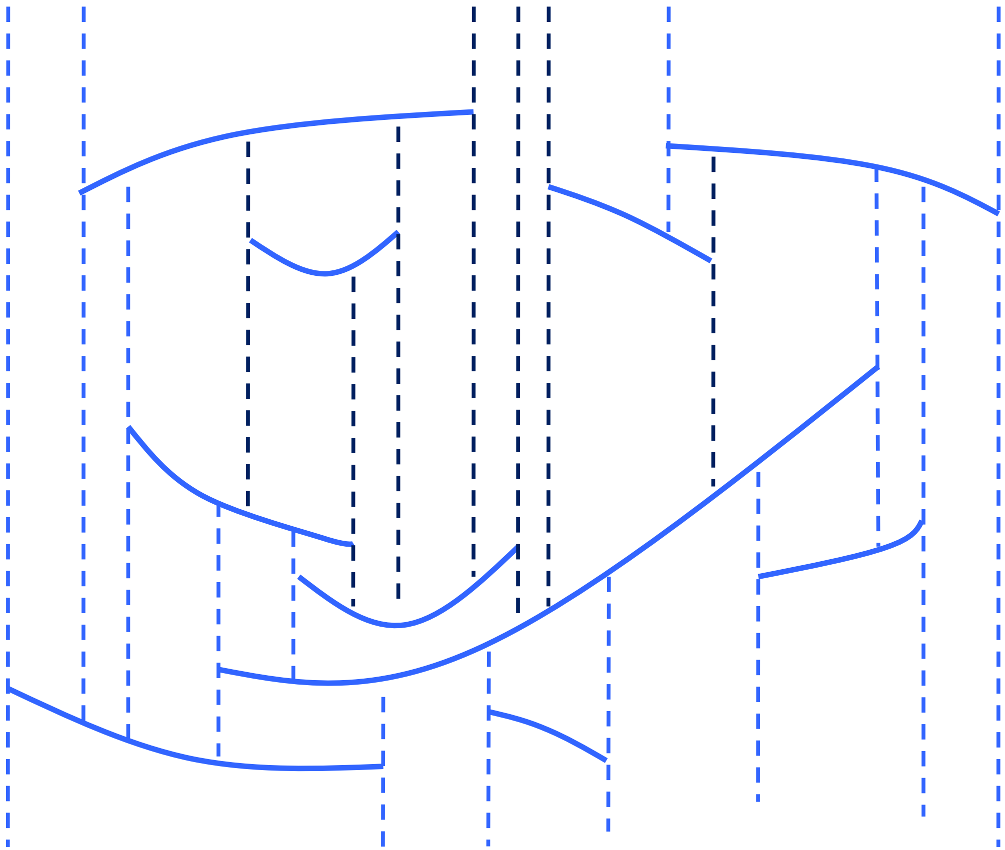

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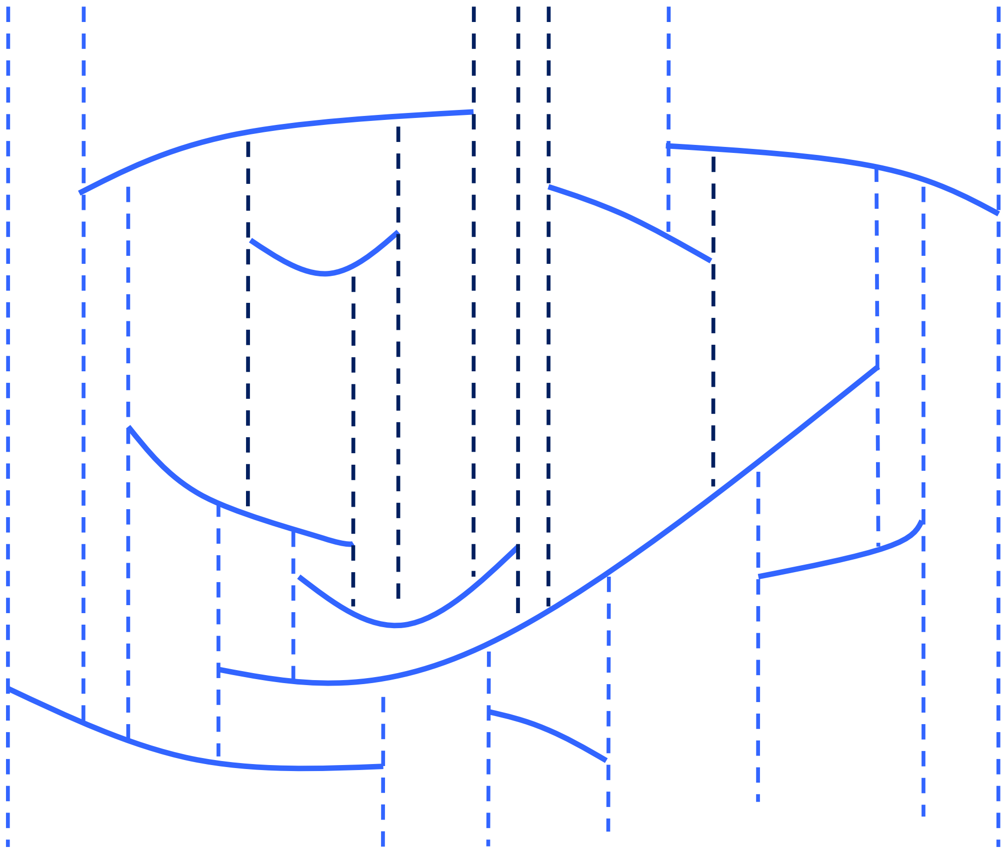

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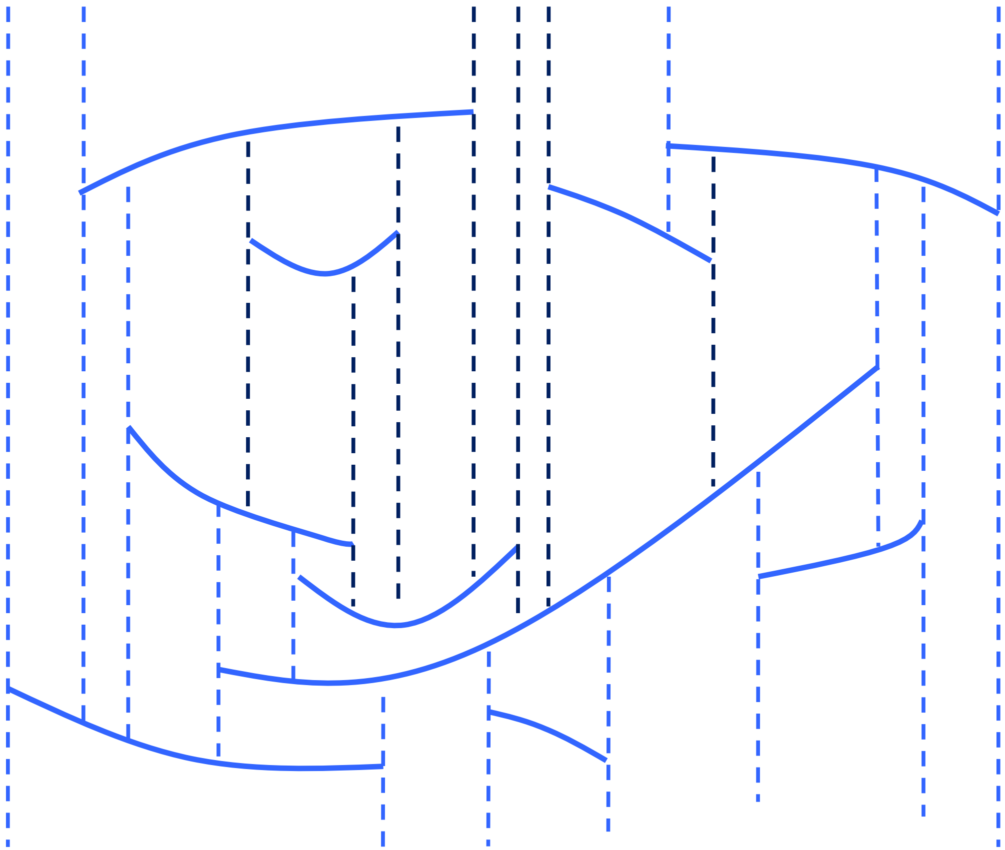

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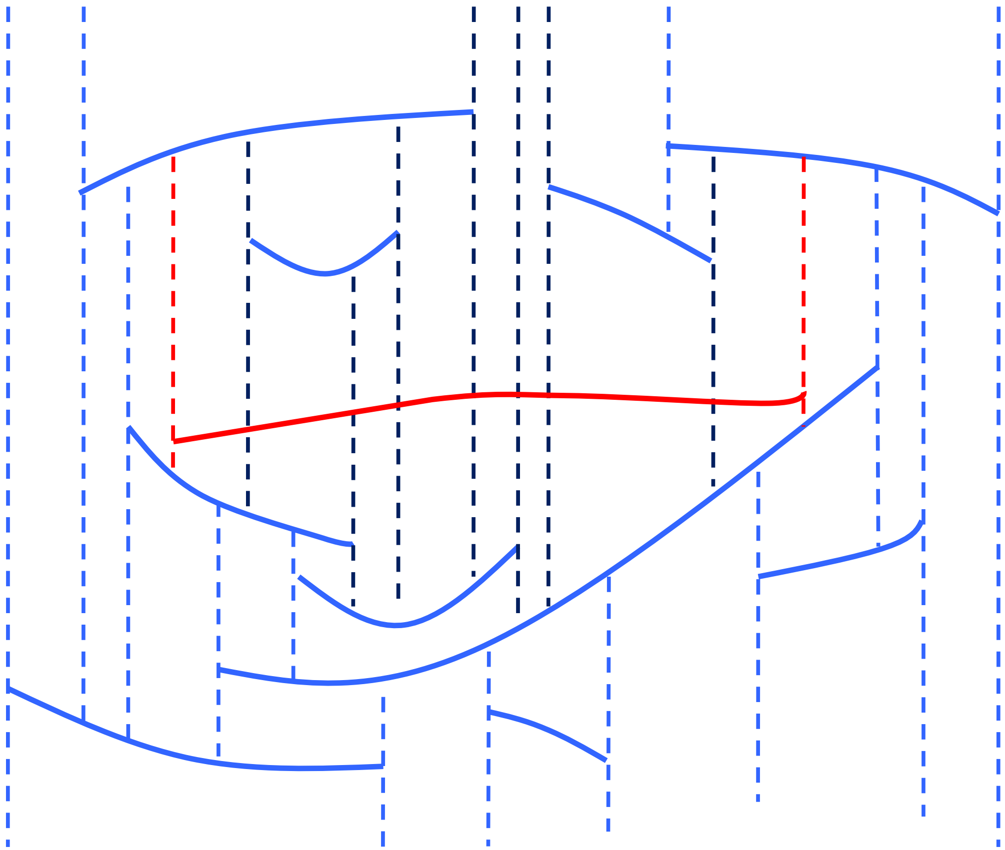

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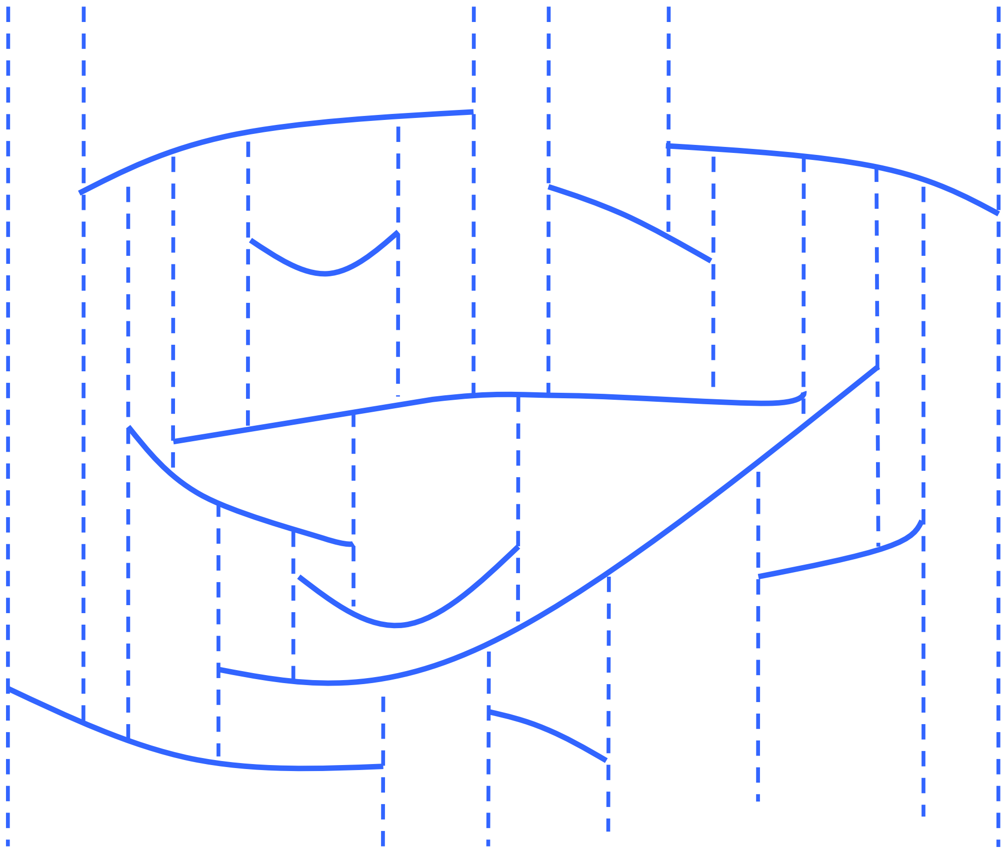

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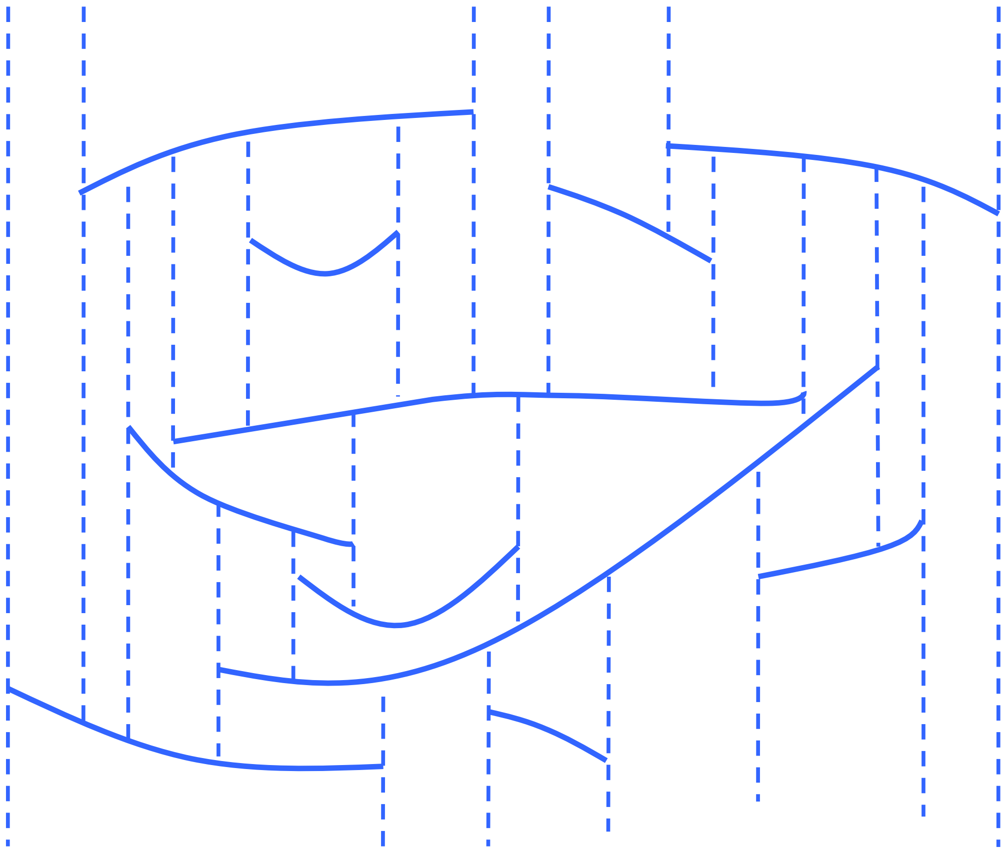

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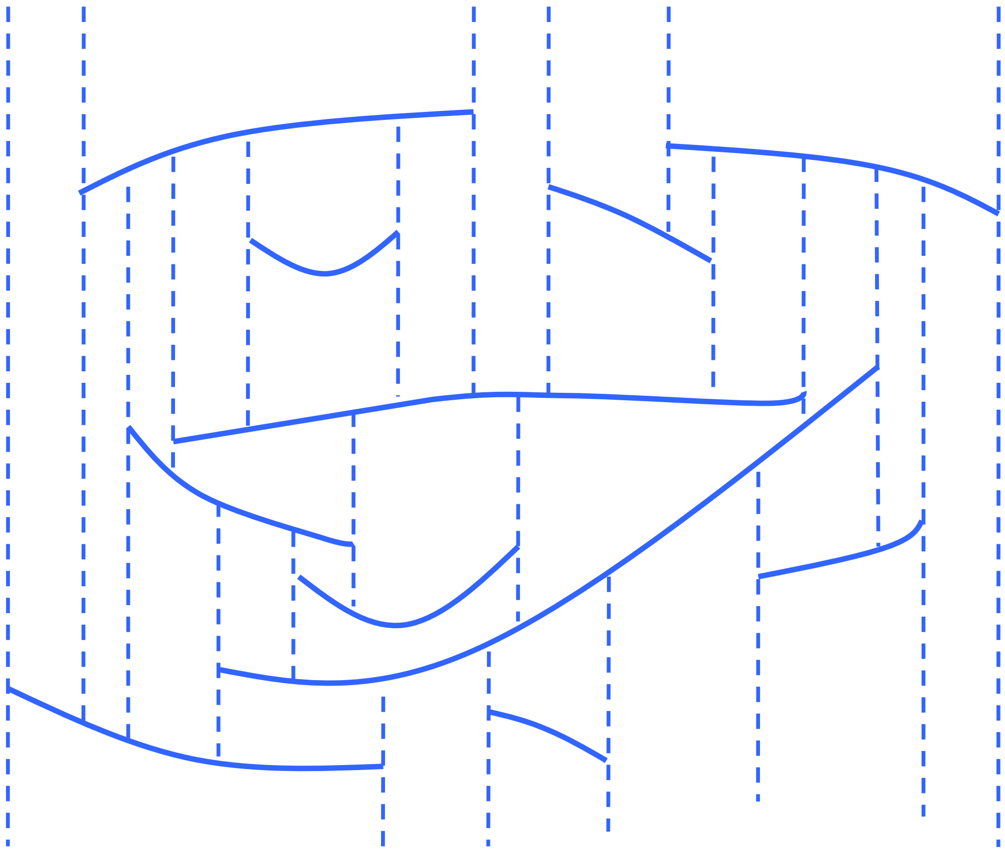

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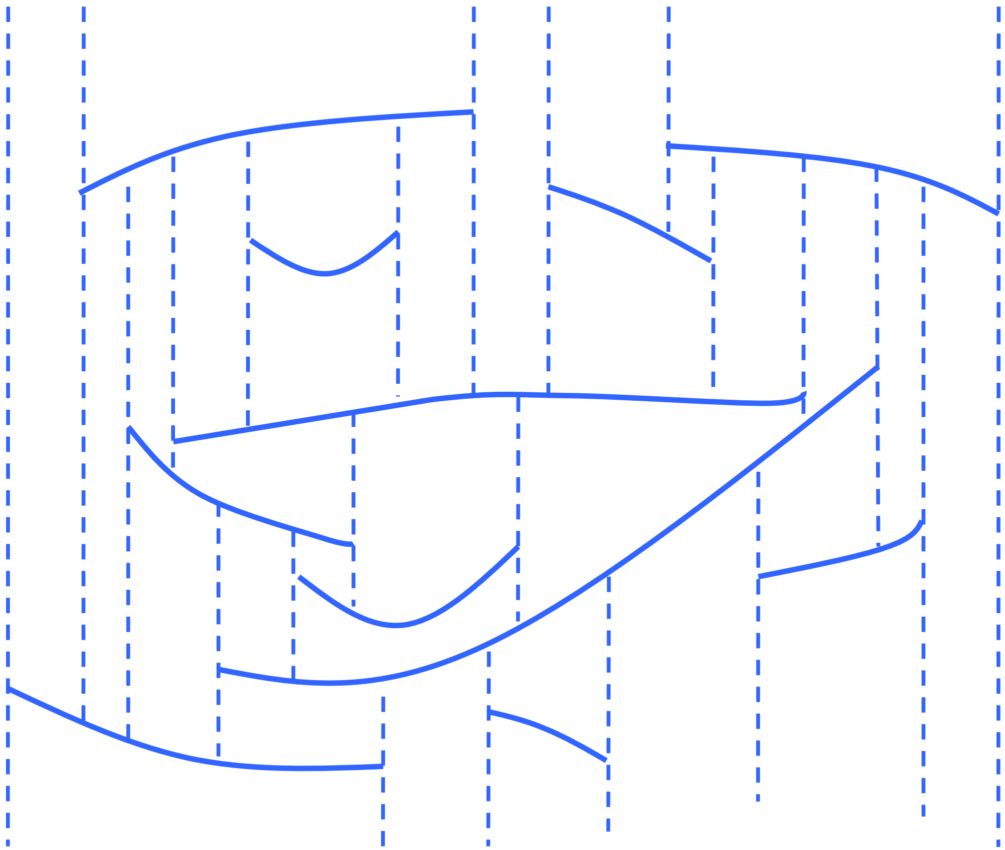

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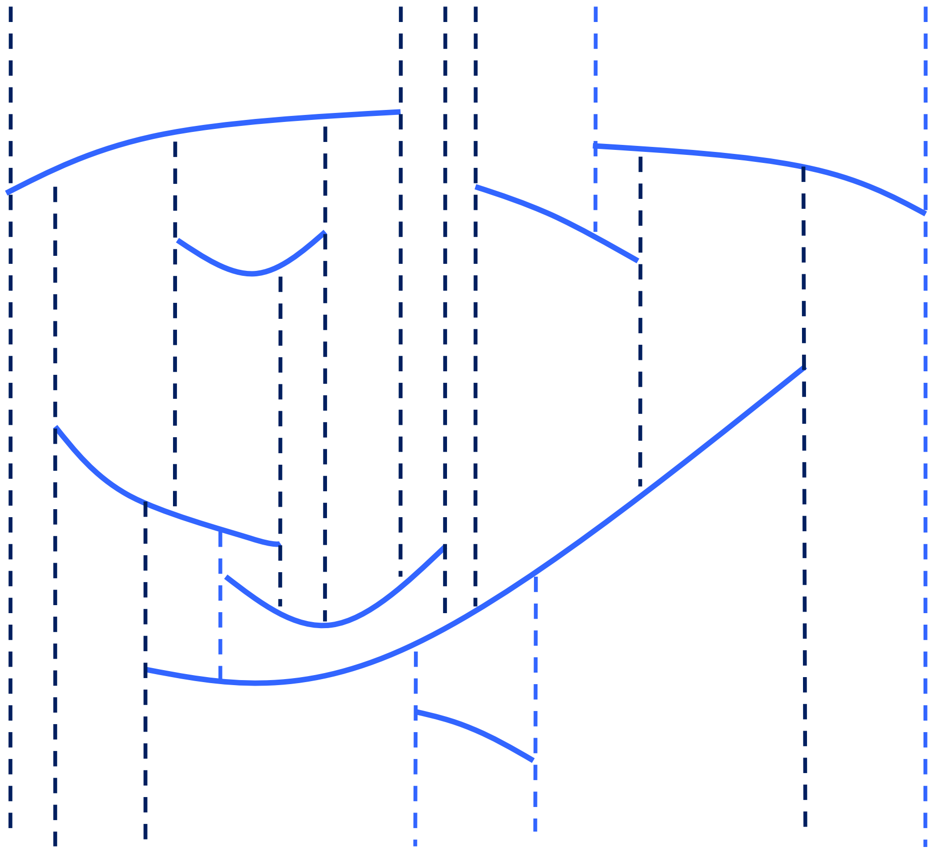

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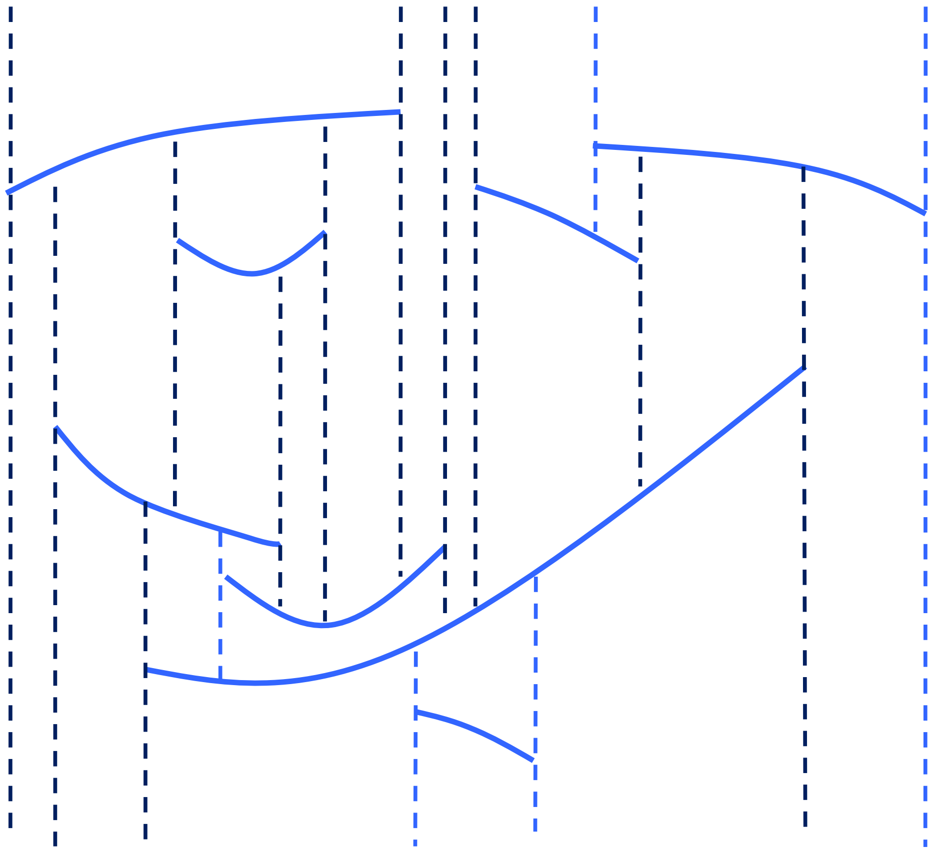

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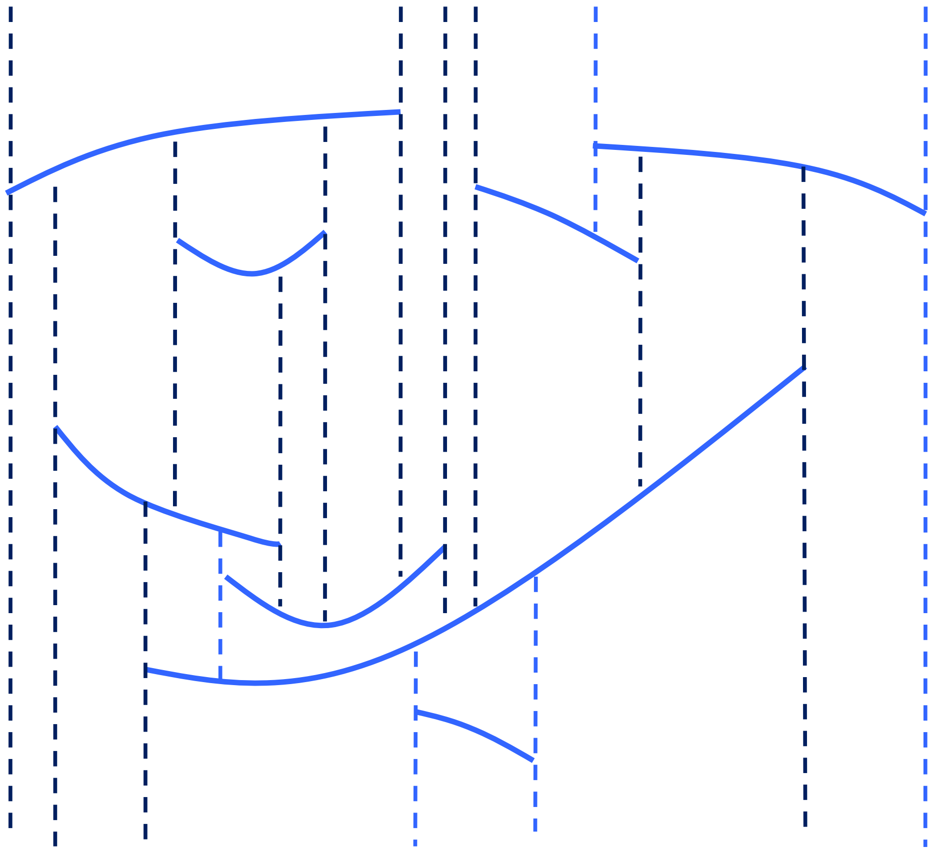

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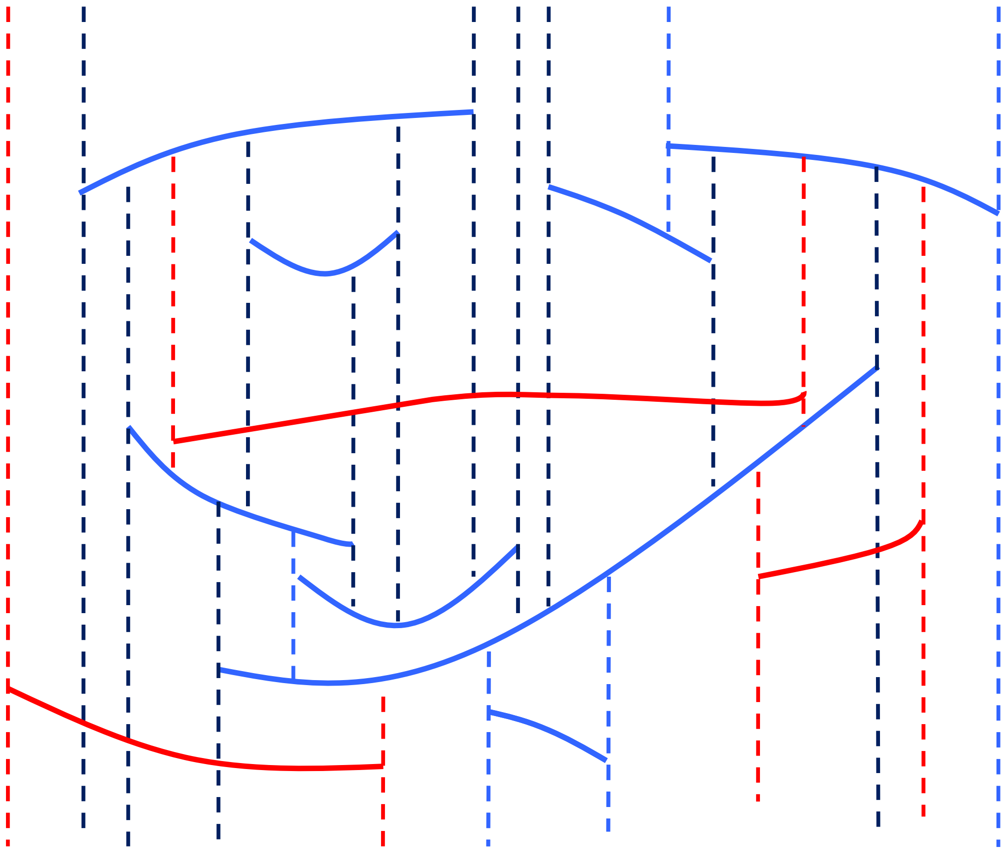

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$O(n \log n)$

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Use randomization !!!

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Randomized Planar Point Location

Idea: Use single segment removal, but remove a random segment, each with equal probability ($1/n$ for each of the n segments)

$\mathcal{T}(S)$ . . . trapezoidation for segment set S
$\mathcal{Q}(S)$ . . . query structure for segment set S

trapezoids of $\mathcal{T}(S)$ correspond 1-1 with sinks of $\mathcal{Q}(S)$.

Randomized Planar Point Location

Creating $\mathcal{T}(S)$ and $Q(S)$ from S:

1. choose a random s from S, let $S' = S \setminus \{s\}$
2. recursively construct $\mathcal{T}(S')$ and $\mathcal{Q}(S')$
3. use $\mathcal{Q}(S')$ to locate the endpoints a and b of s in $\mathcal{T}(S')$
4. split those two trapezoids verticallly by the vertical lines through a and b respecdtively
5. make the corresponding nodes in $\mathcal{Q}(S')$ to x-comparison nodes (w.r.t. a and b)
6. "Thread" segment s from a to b in $\mathcal{T}(S')$:
7. for each trapezoid cut by s make the corresponding node in $\mathcal{Q}(S')$ to a y-comparison node w.r.t s
8. Generate a sink node of $\mathcal{Q}(S')$ for each new trapezoid in the resulting trapezoidation and connect the newly created y-comparison nodes to the appropriate sink node

Randomized Planar Point Location

1. For each query point q the expected search time for q is $O(\log n)$
2. The expected size of the structures constructed is $O(n)$.
3. The expected preprocessing time is $O(n \log n)$.