Voronoi Diagrams and Delaunay Triangulations

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Computaional Geometry
Summer semester 2020

max planck institut
informatik

Overview

- Voronoi diagrams – definition and properties

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- Fortune's algorithm (1987)

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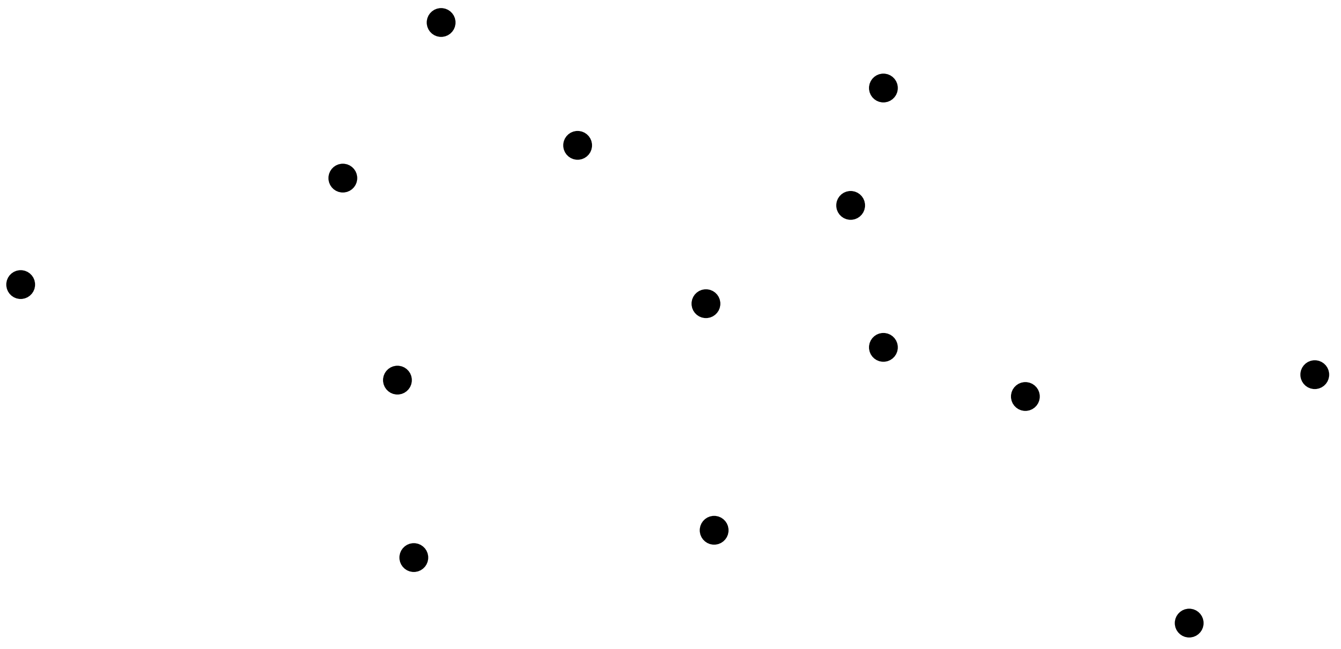
- Lifting to a paraboloid; computation via convex hull

Next lecture!

Motivation – nearest neighbor

Given: $P \subset \mathbb{R}^2$.



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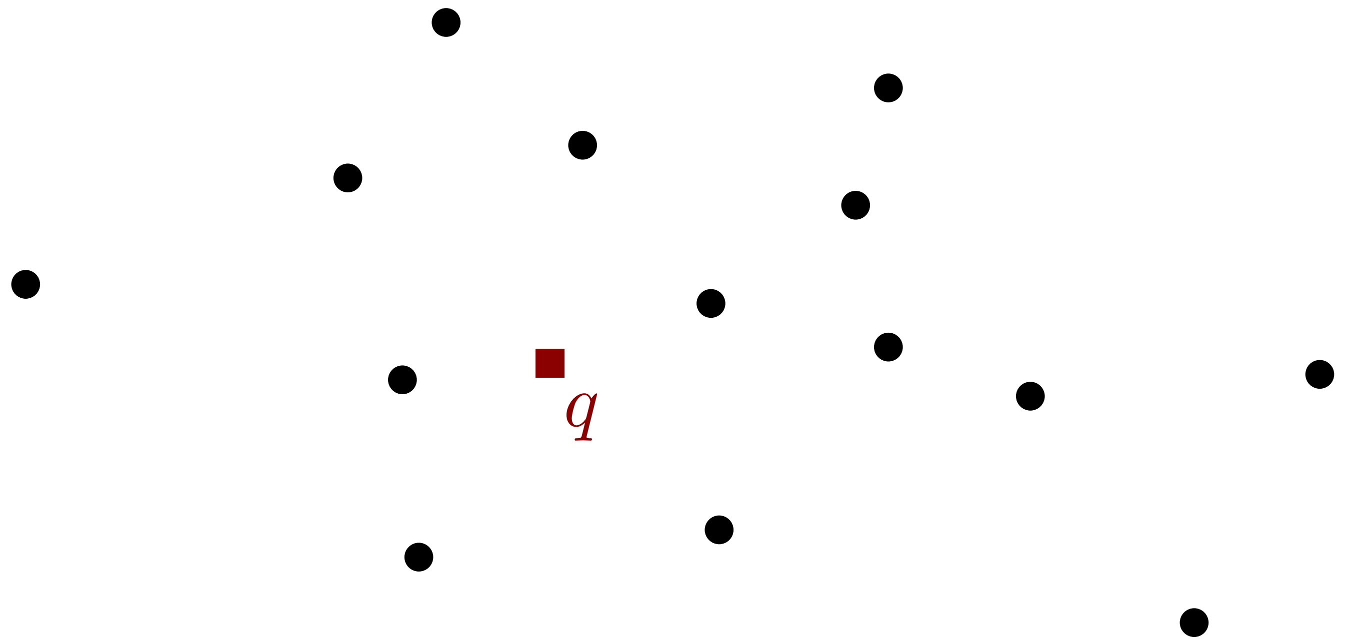

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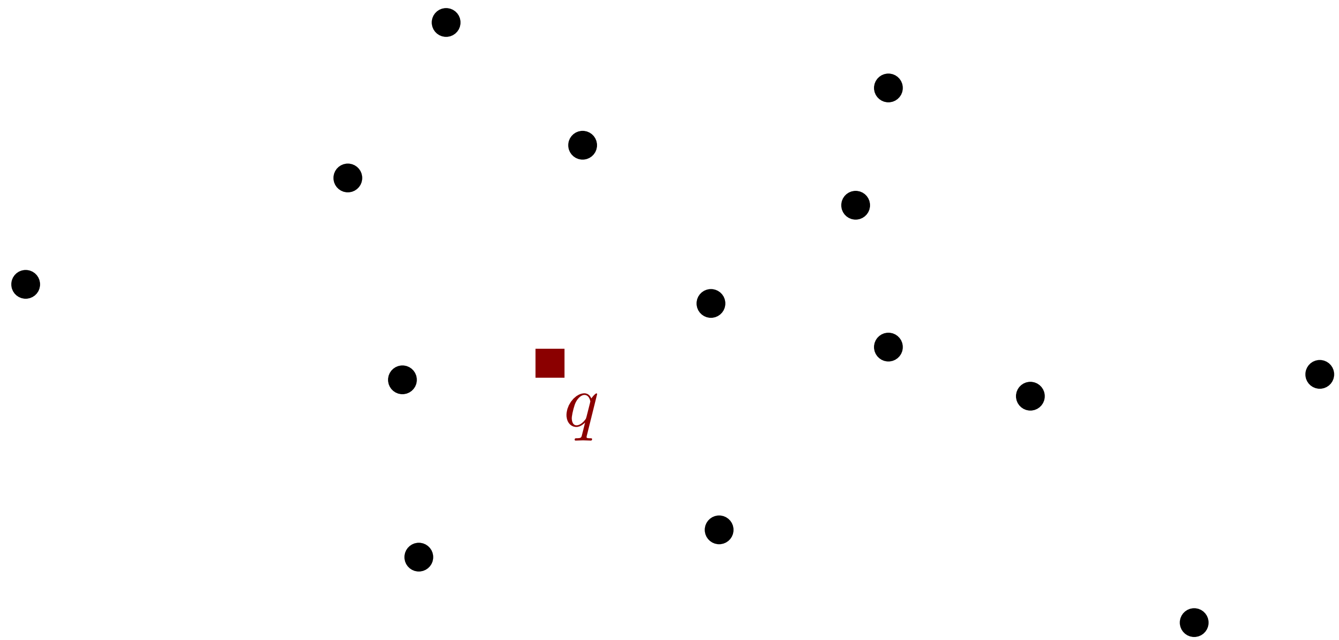

What is the nearest point in P to a given query point $q \in \mathbb{R}^2$?

- Where in P should I get my ice cream if I'm at q?
- Accident at q. Which hospital in P should send helicopter?

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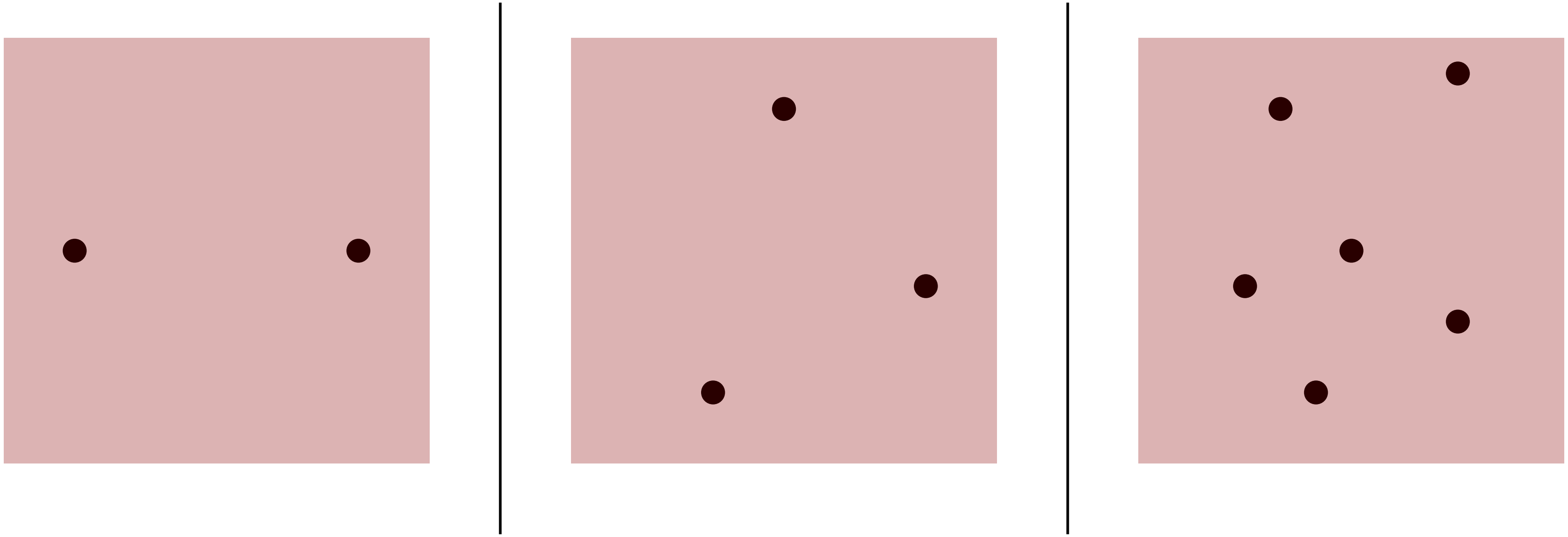

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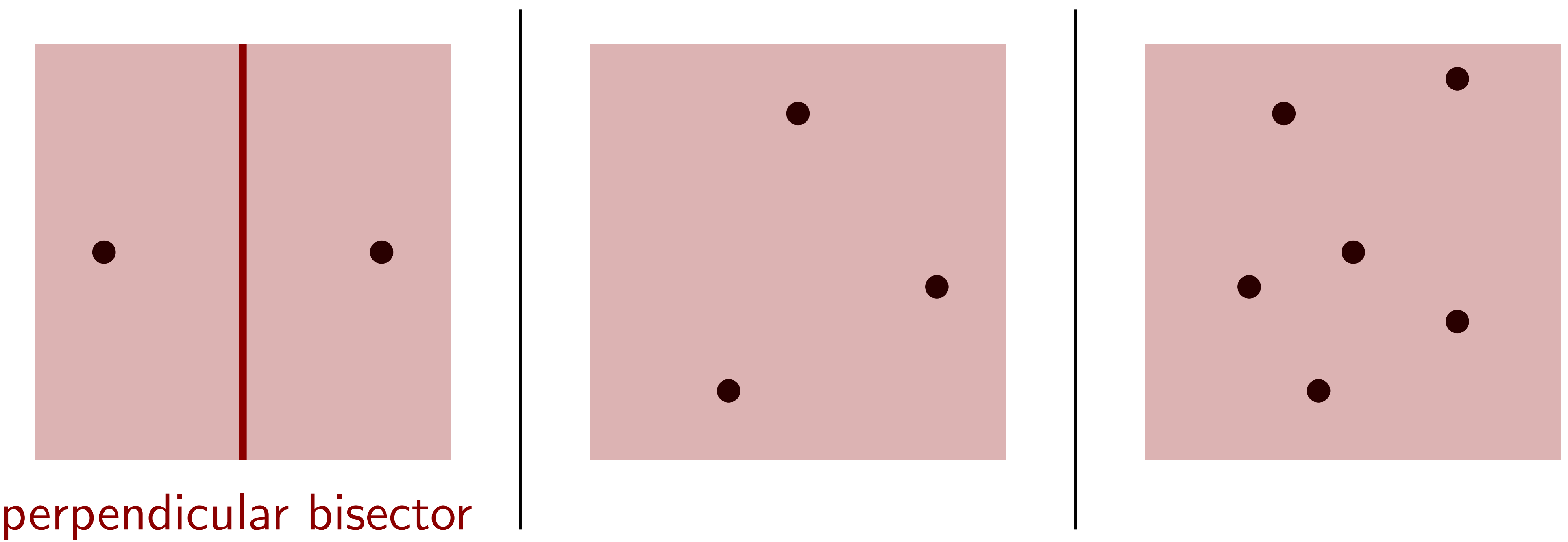

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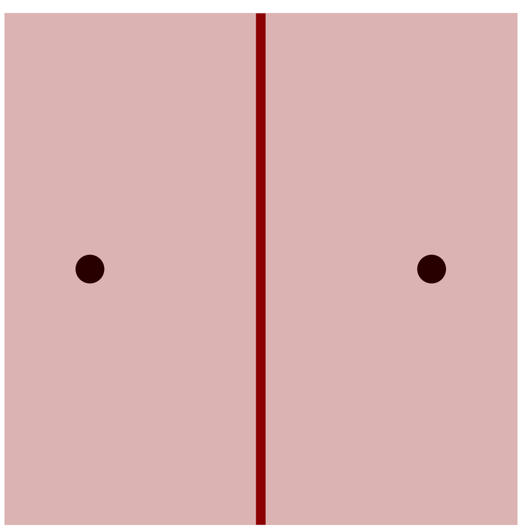
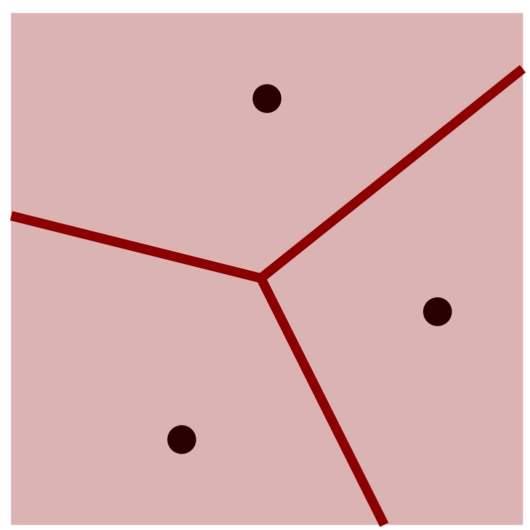
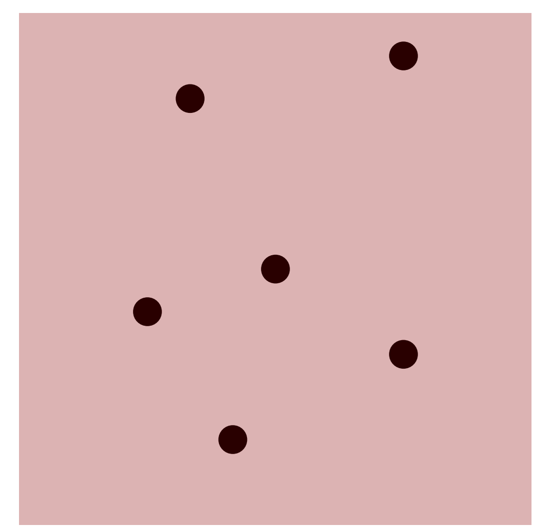


perpendicular bisector





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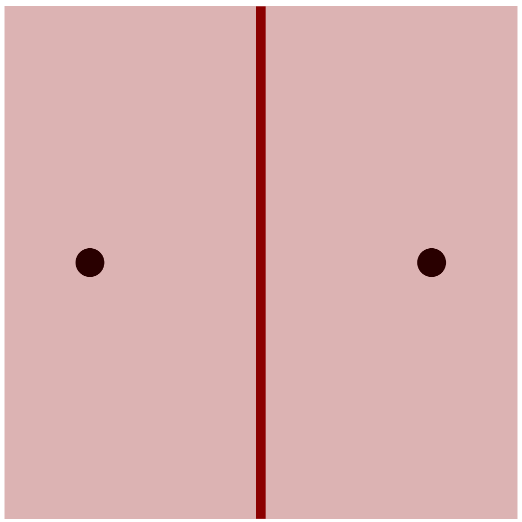

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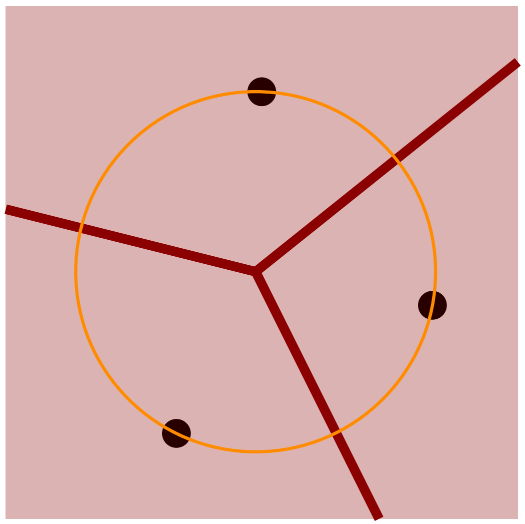
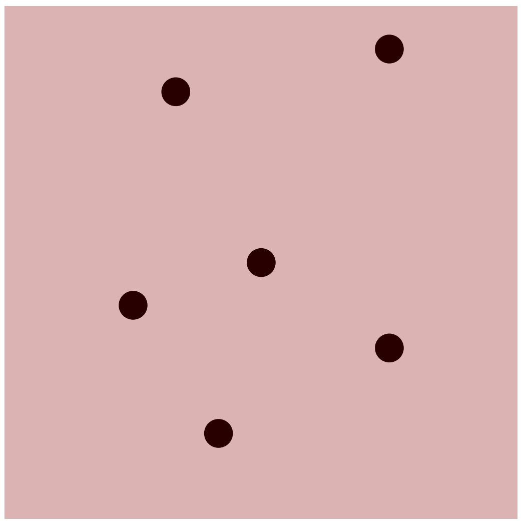


bisectors, center of circumcircle



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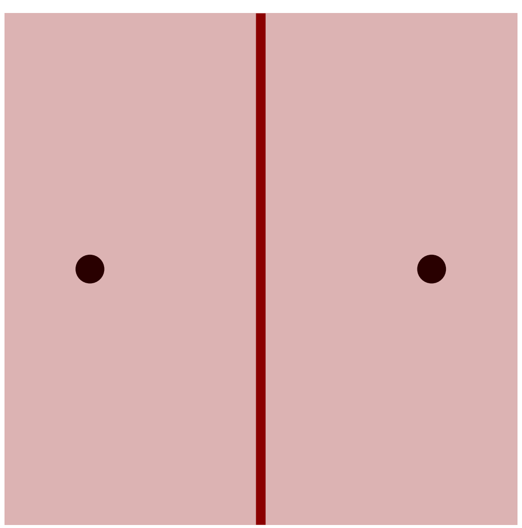

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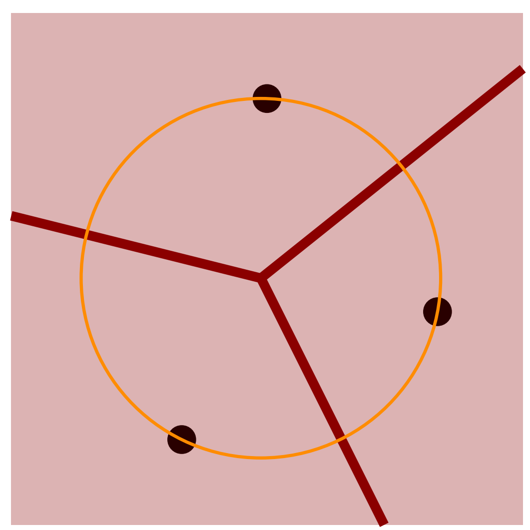

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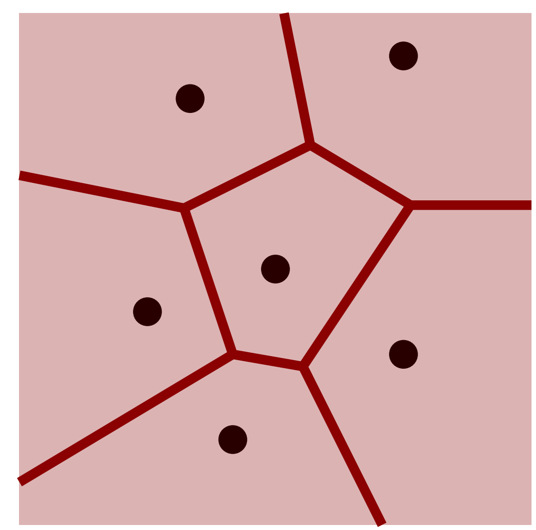

Historical notes

Voronoi diagram = Dirichlet tessellation

Goes back to Descartes

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Voronoi diagram = Dirichlet tessellation

1907 1850

Goes back to Descartes 1644

Complexity and properties in $\mathbb{R}^2$

Each Cell(p) is intersection of half-planes.
Each cell is convex (bounded or unbounded) polygon.

Vor(P): collection of segments and rays on cell boundaries

If P has 3 non-collinear pts
$\Rightarrow$ Vor(P) is connected

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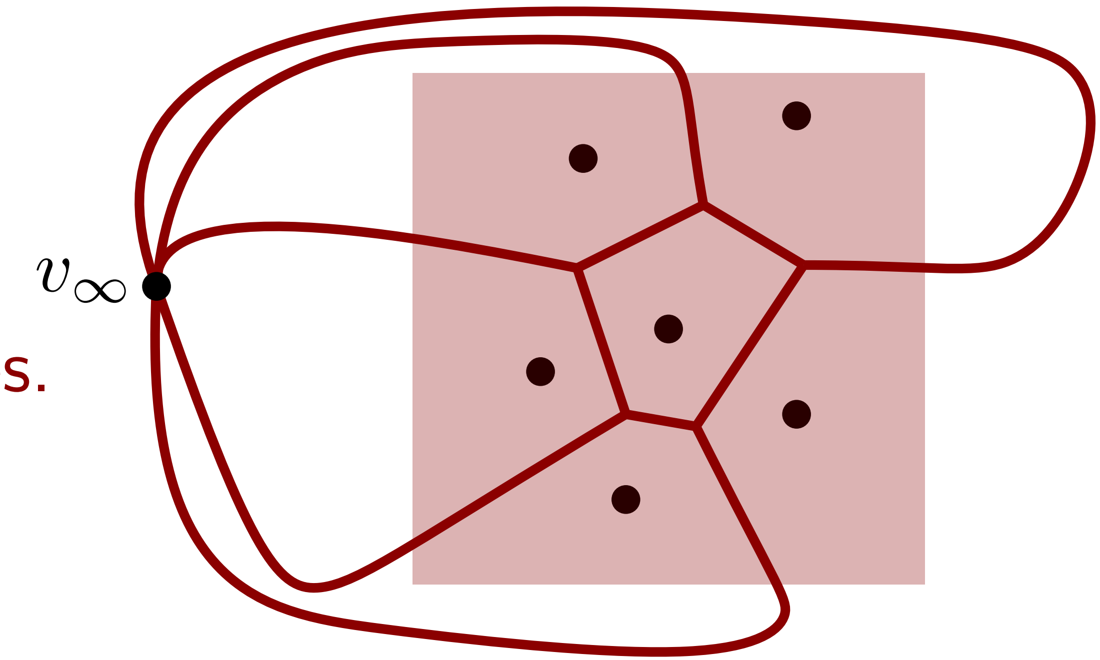
Proof. There are n cells.
Euler's formula
$\Rightarrow O(n)$ edges, $O(n)$ vertices.



Circumcircles in Voronoi diagrams

$C(q)$: largest circle around q whose interior has no pts from P

Lemma

(i) q is a vertex of Vor(P) iff $C(q)$ has at least 3 points of P

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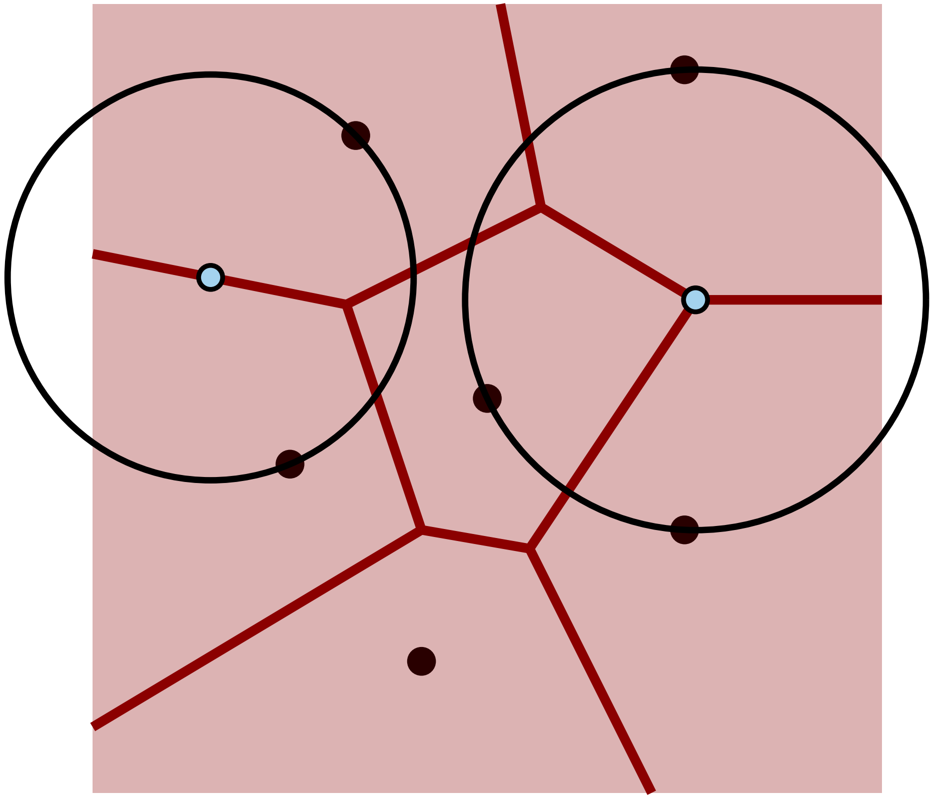
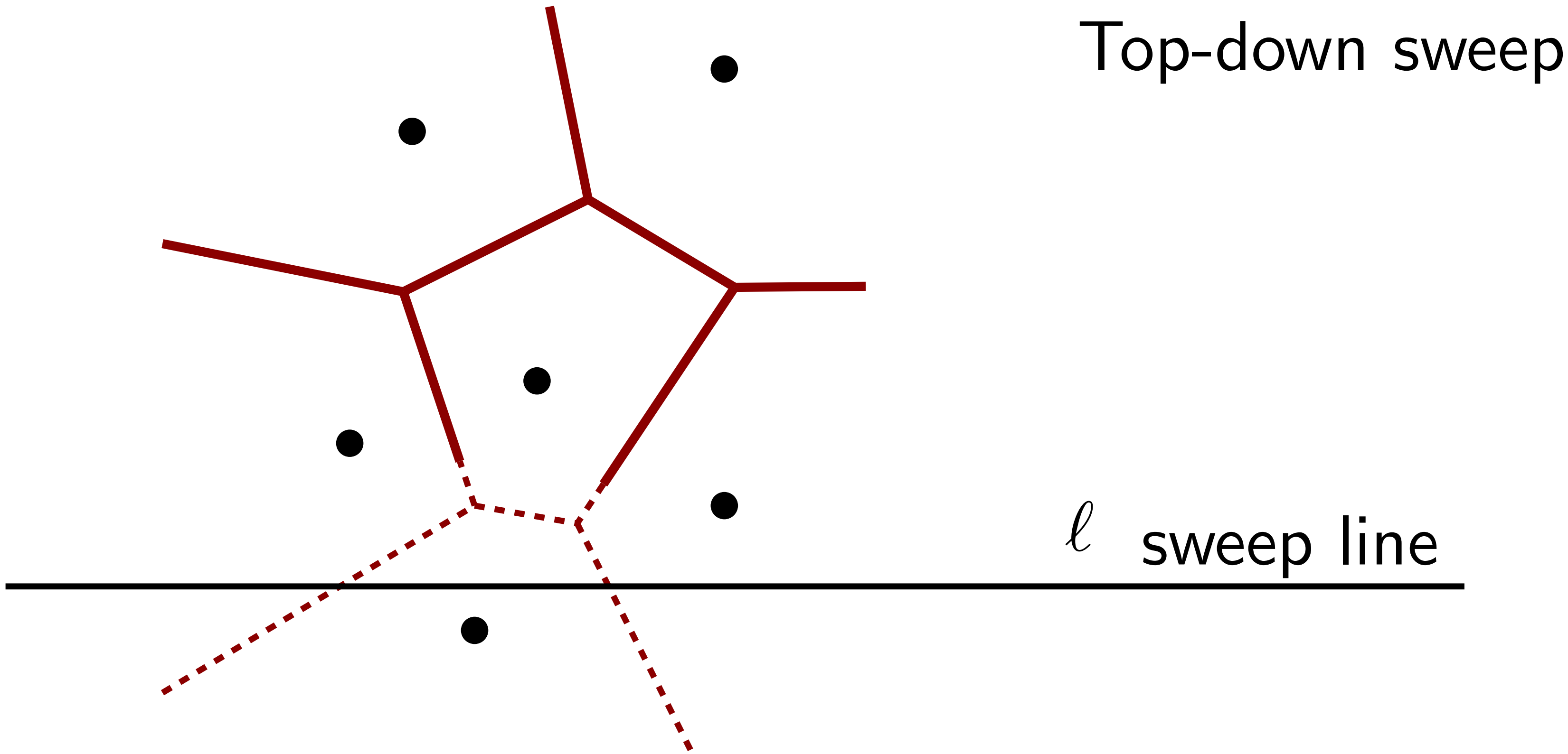


Fortune's algorithm (1987)

Sweeping with a wavefront



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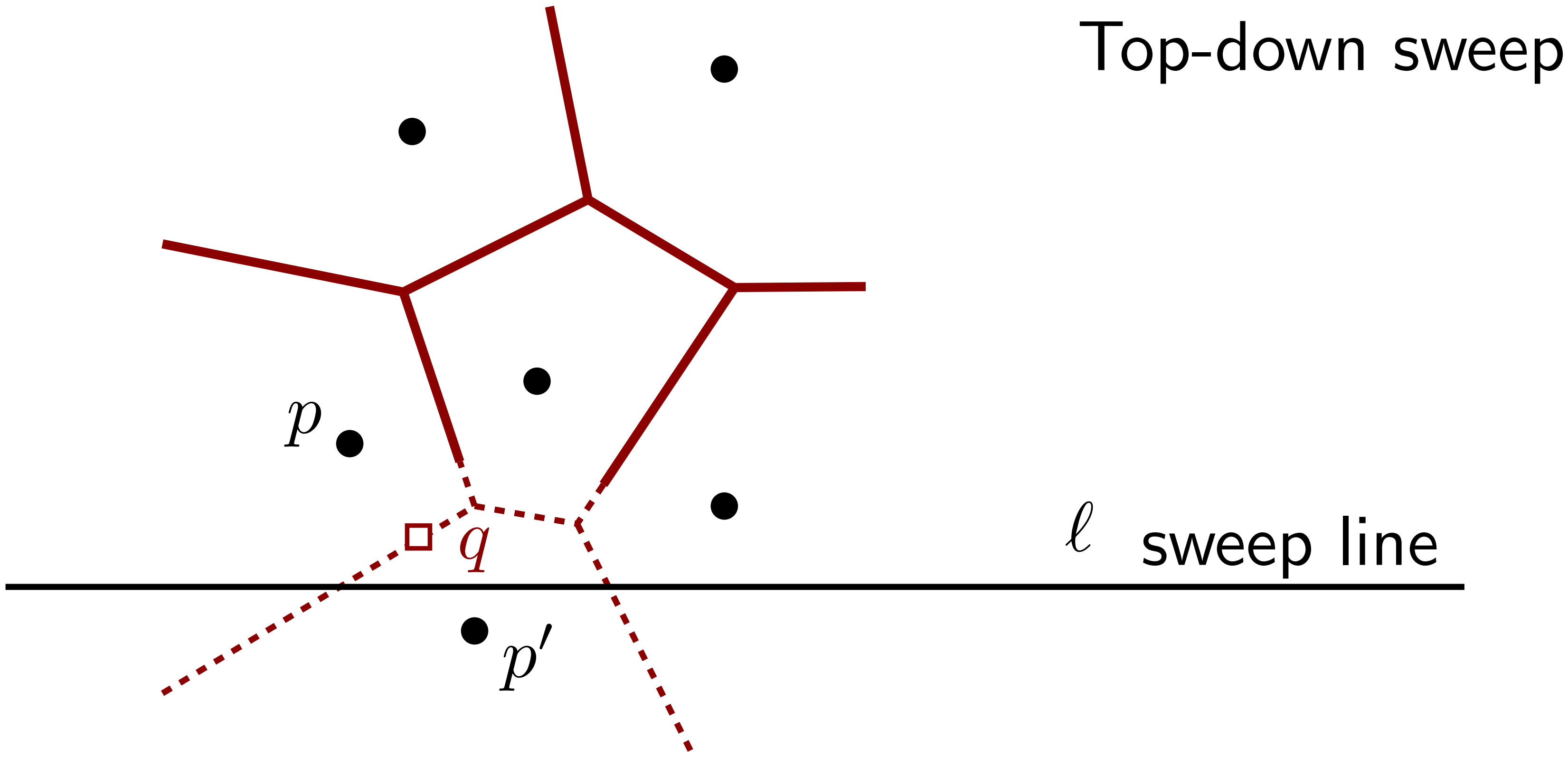

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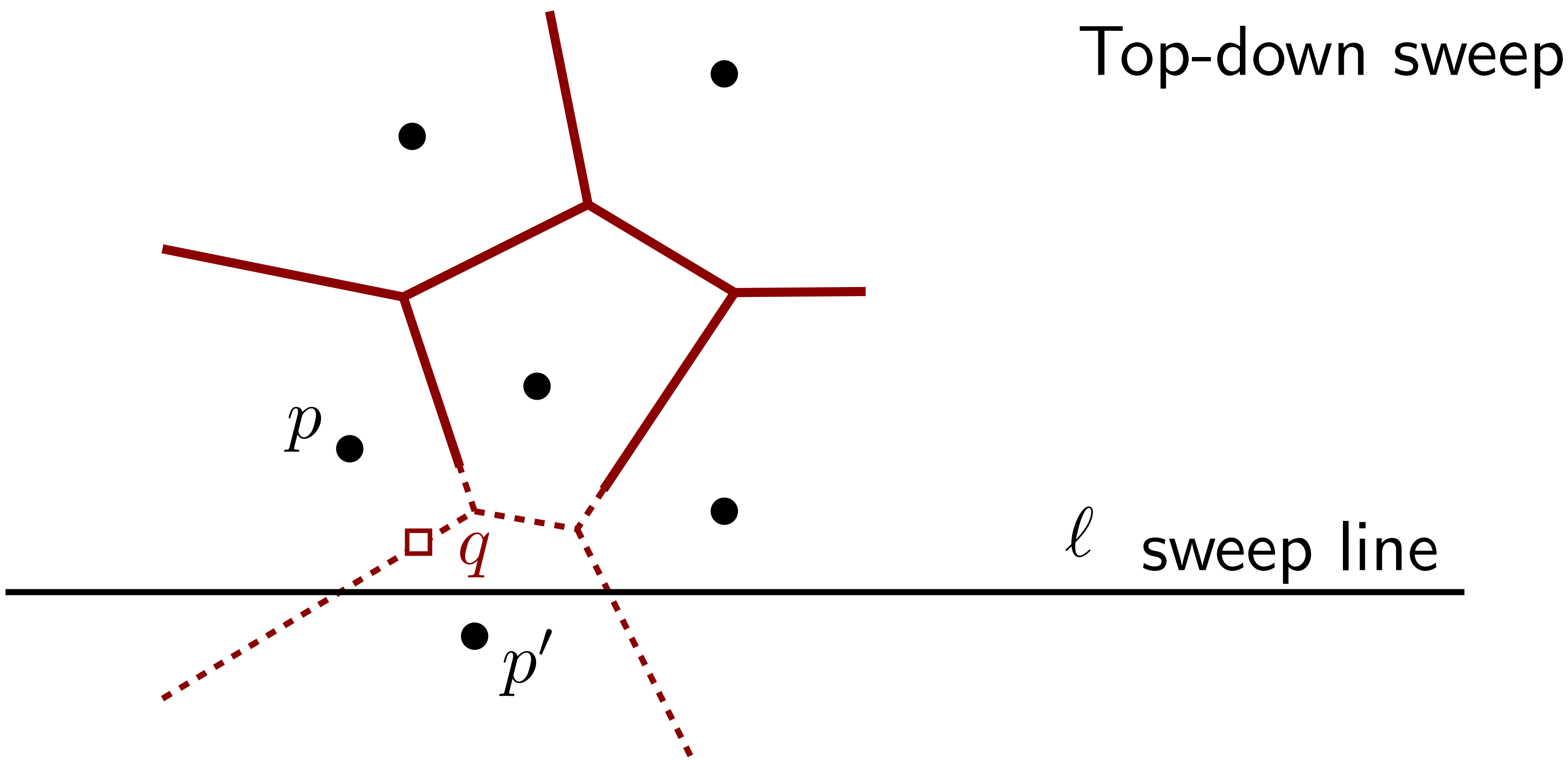

If q is on undiscovered edge btw. Cell(p) and Cell(p')

$$\mathrm{dist}(p,q) = \mathrm{dist}(p',q) \Rightarrow \mathrm{dist}(p,q) \geq \mathrm{dist}(q,\ell)$$

q is below the parabola with focus p and axis ℓ

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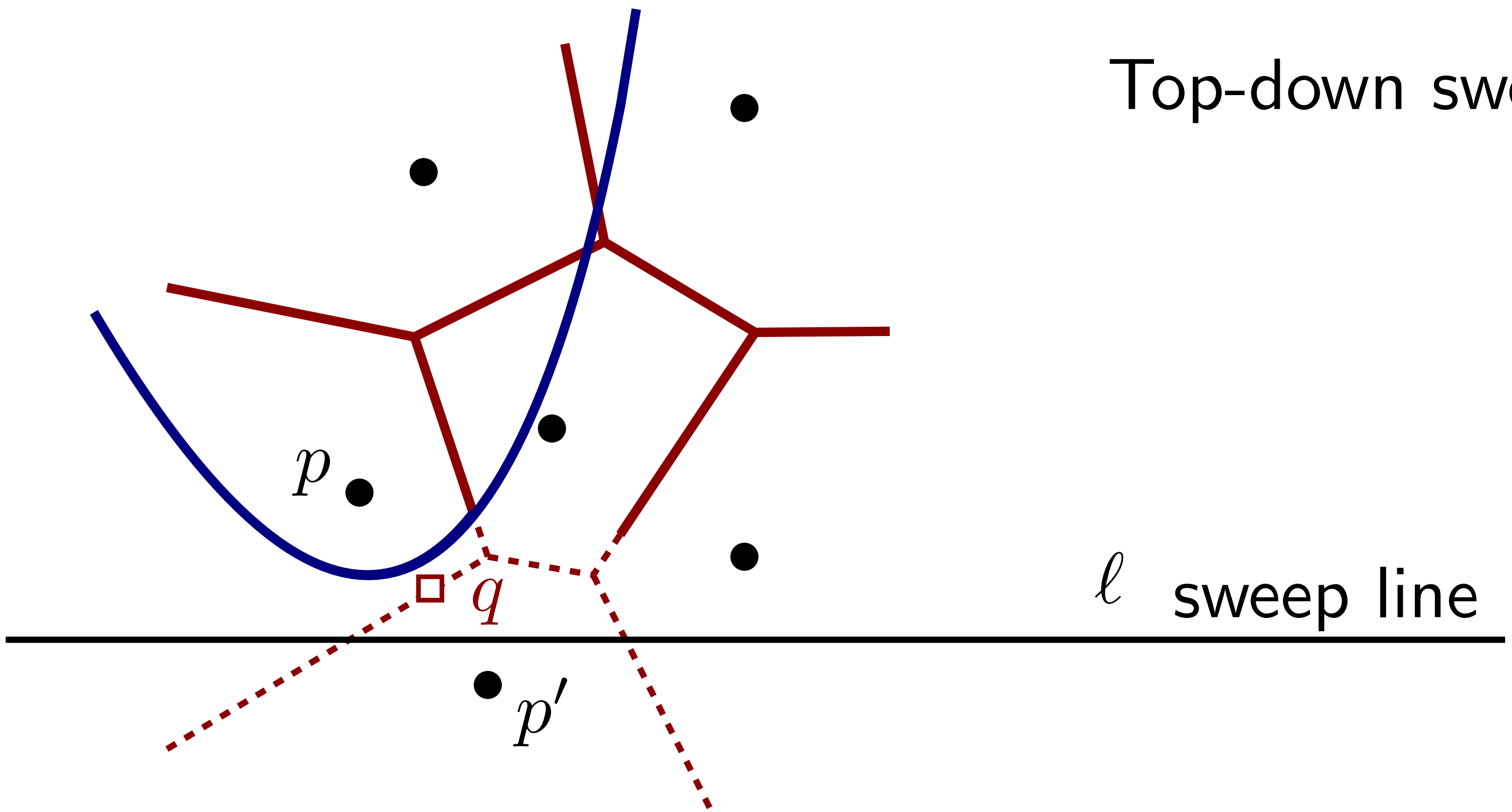

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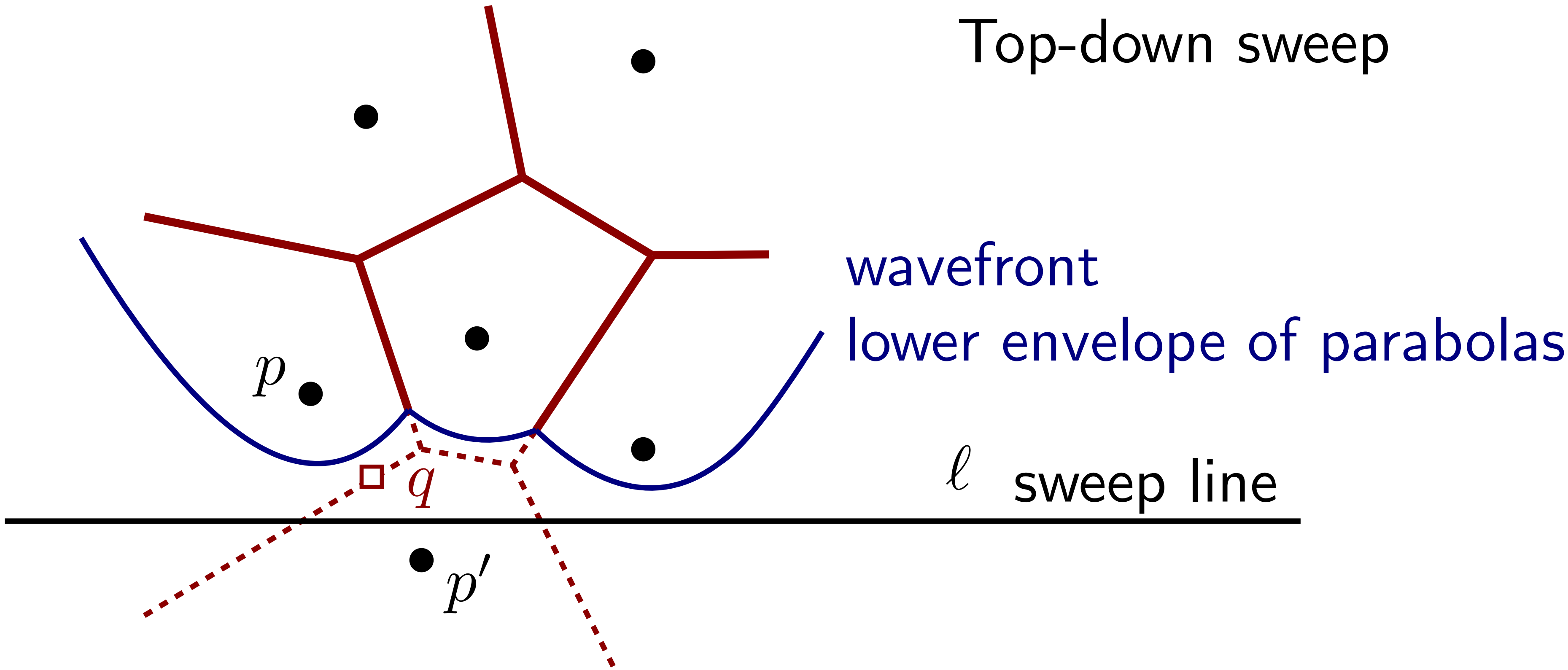

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Vor(P) above waverfront is correct

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  Part of diagram above wavefront is correctly computed

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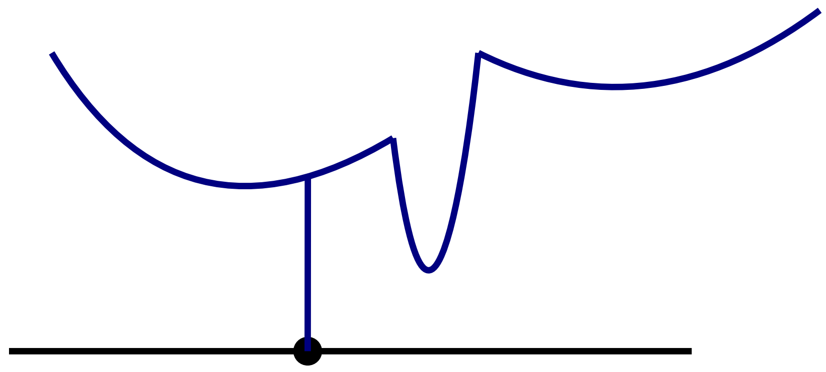

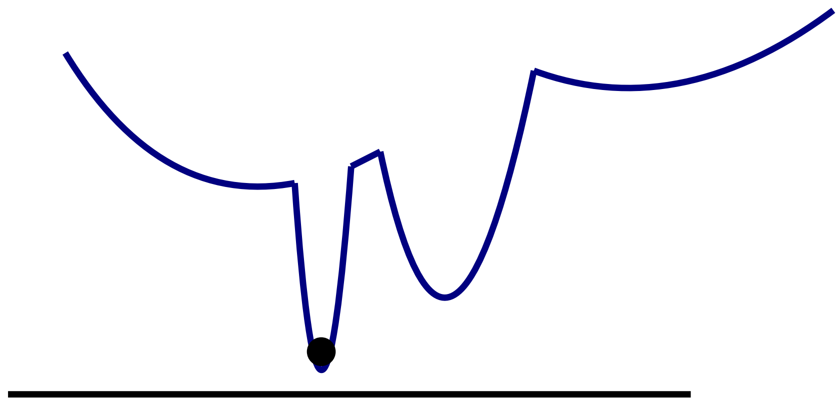

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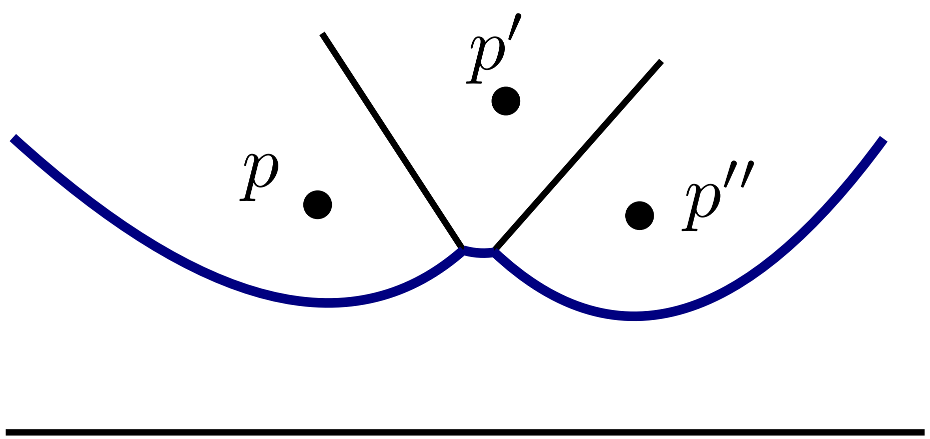

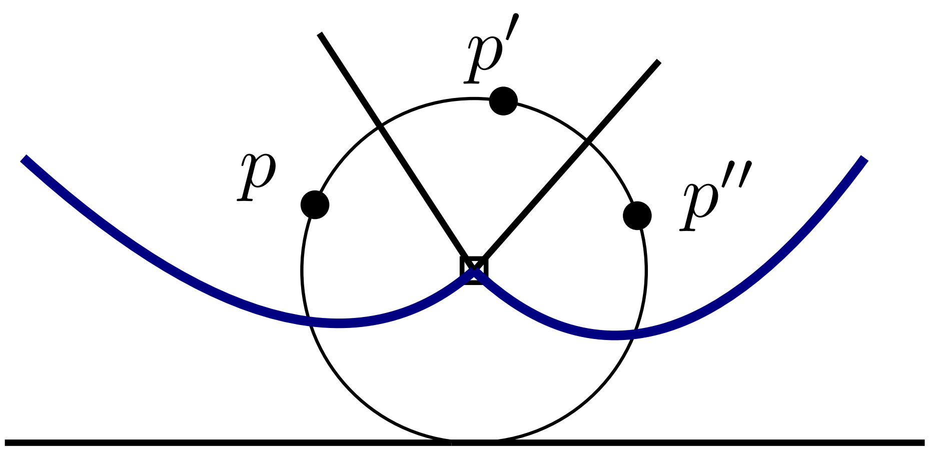

Wavefront complexity and queue maintenance

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updated with each change to wavefront.

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$O(n)$ events with $O(\log n)$ time per event $\Rightarrow O(n \log n)$

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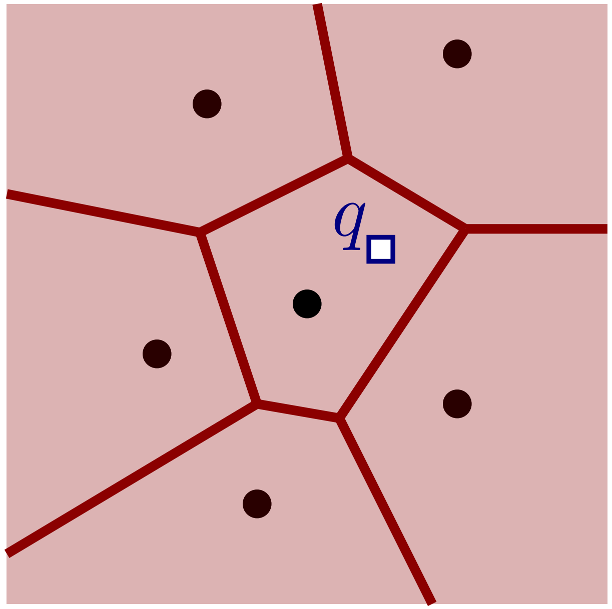

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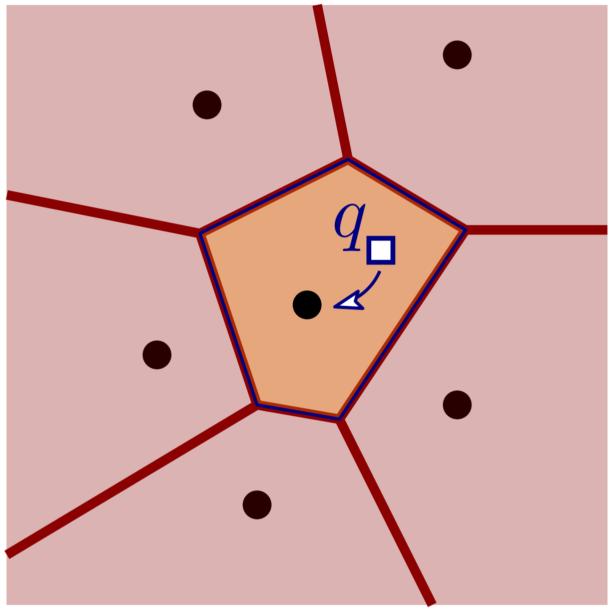

General Voronoi diagrams

Voronoi diagram in different metrics:

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Other generalizations:

- of segments
- additively/multiplicatively weighted
- power diagram
- Farthest point
- Order-k

Delaunay triangulations

Triangualtions, complexity

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"Good" triangulation?

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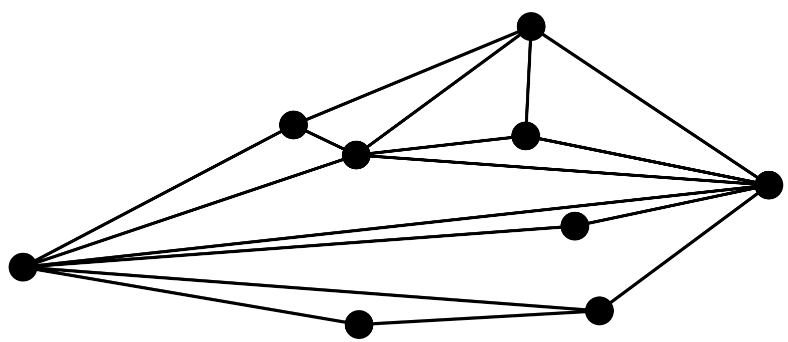

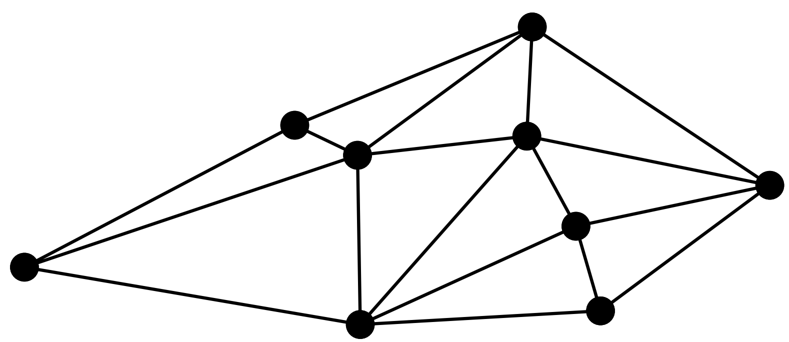

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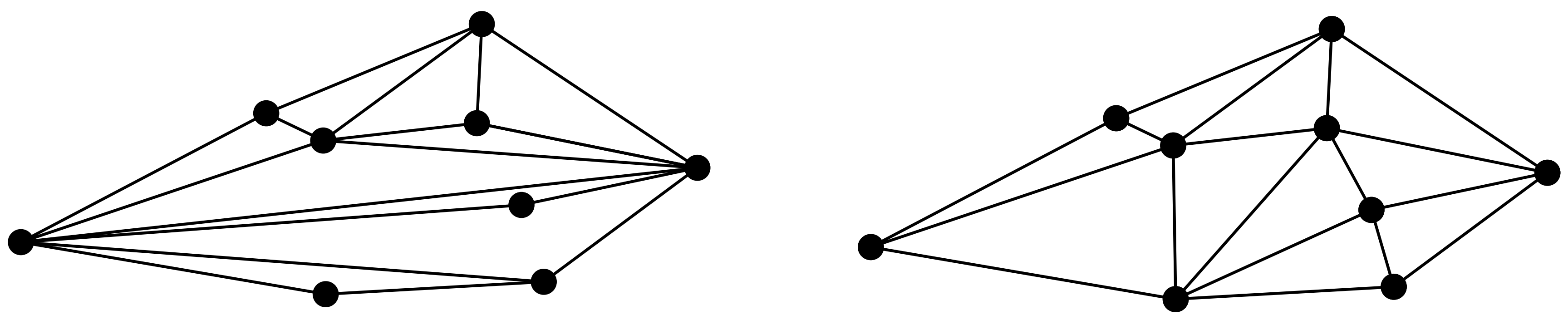

- Distance along triangualtion edges approximates Euclidean distance

Delaunay triangulation definition

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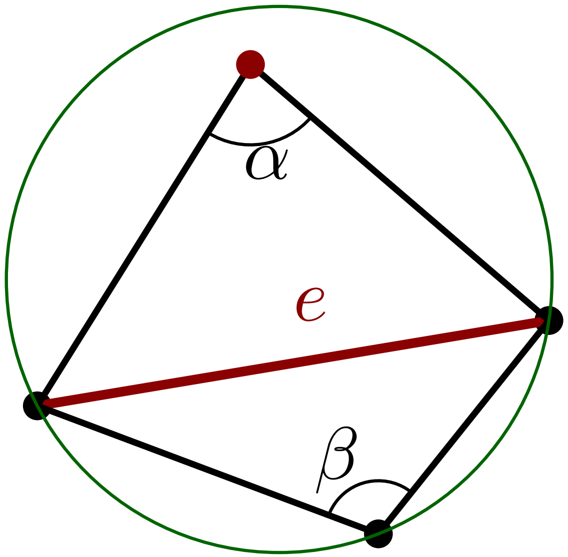

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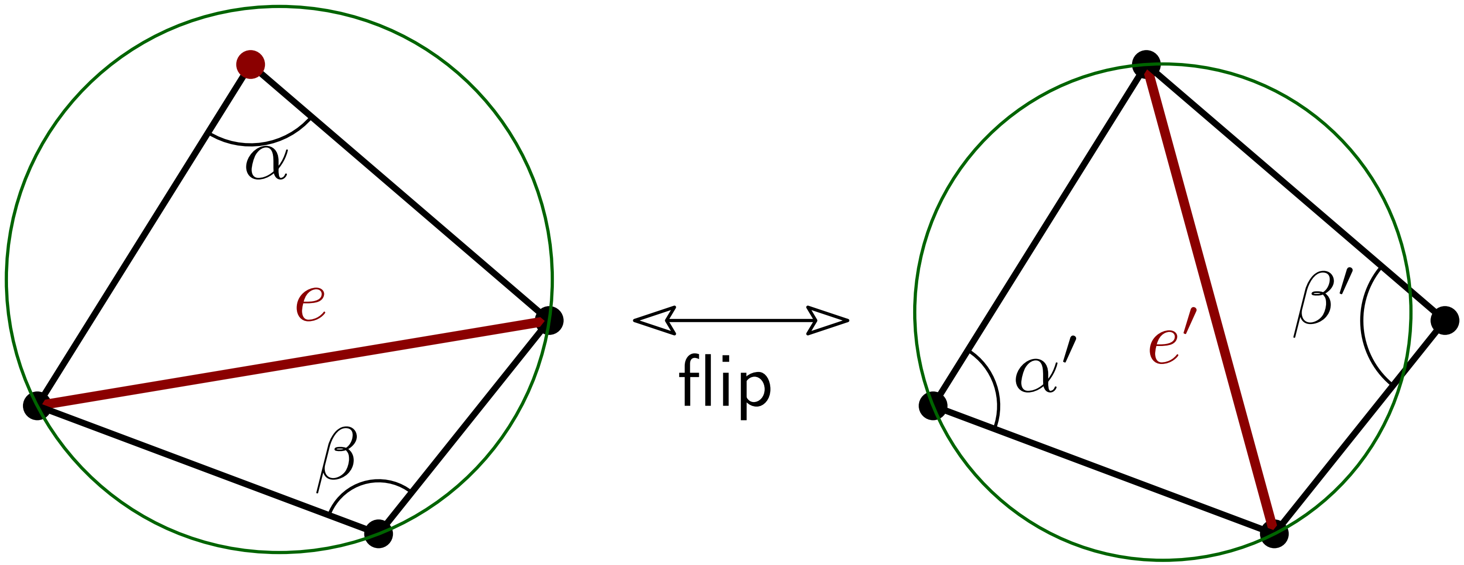

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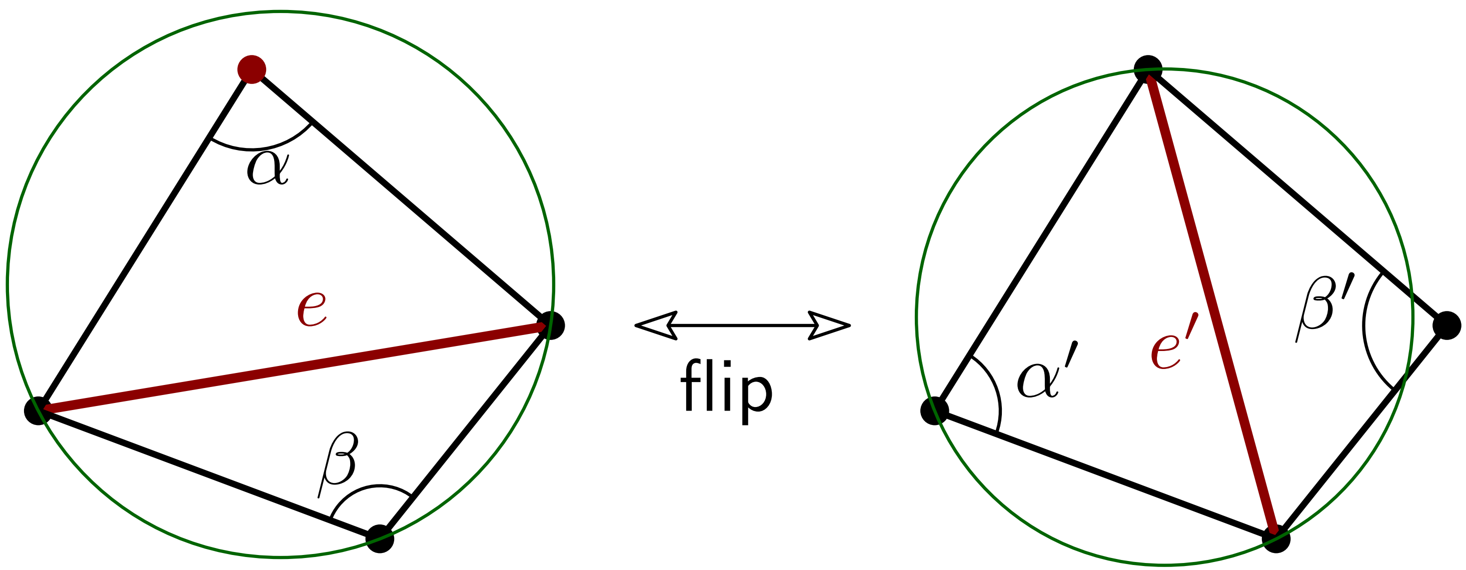

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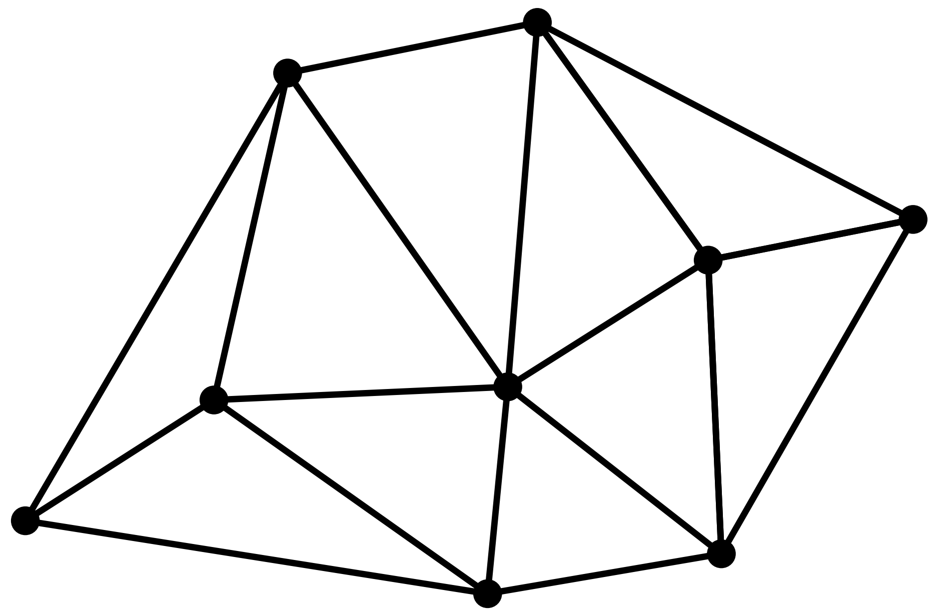
DT is a triangulation whose angles (when ordered in increasing sequence) are lexicographically maximized.

Example



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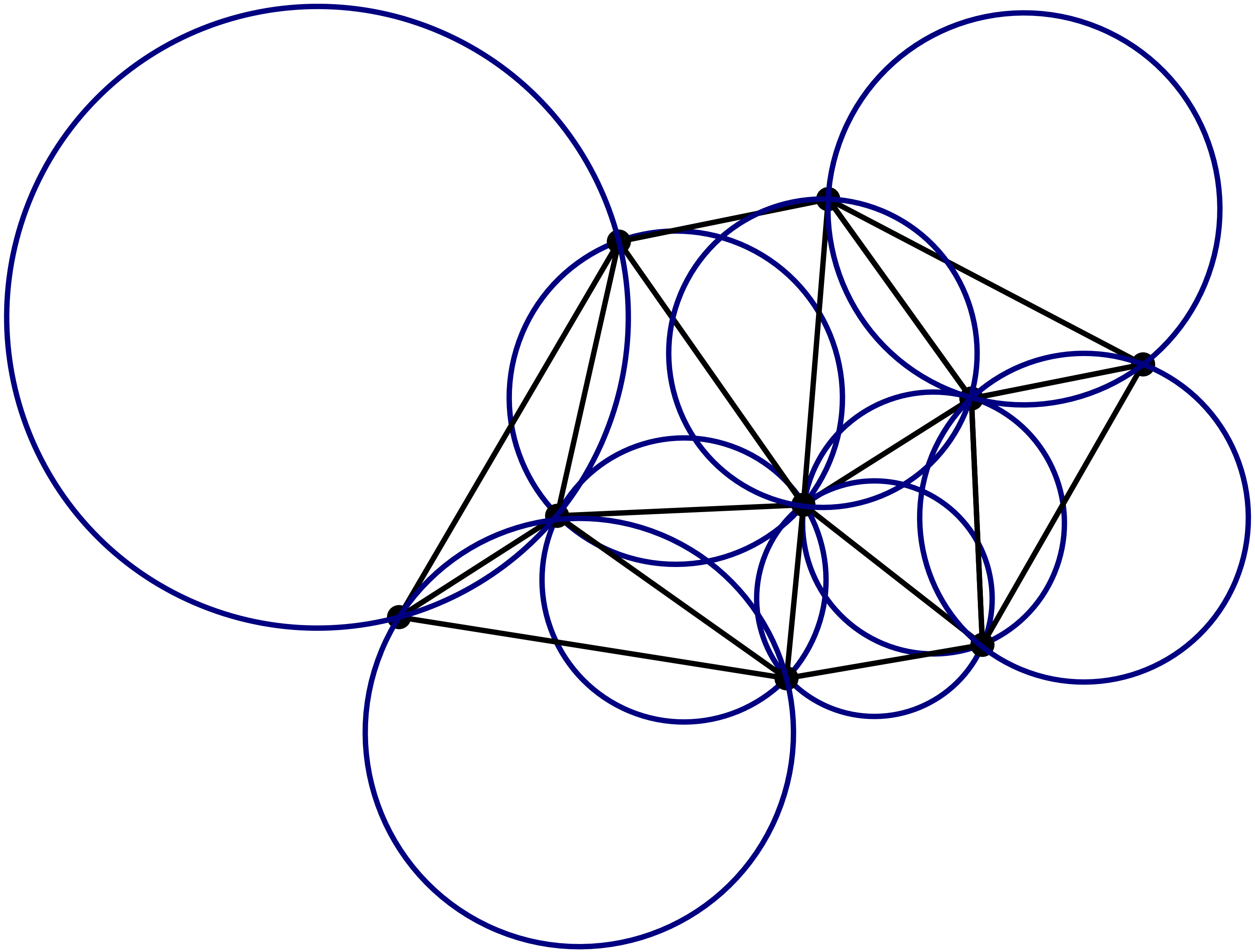
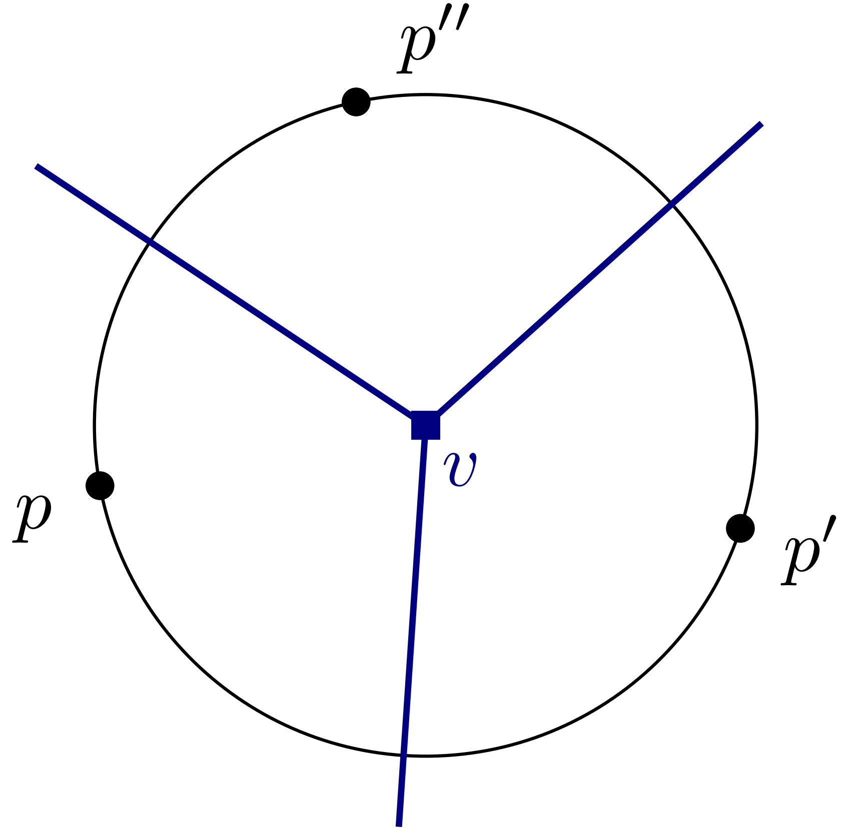


The dual of Voronoi

Voronoi vertex v at circumcenter of $pp'p''$

$\Updownarrow$

circumcircle of $pp'p''$ has no point of P in its interior

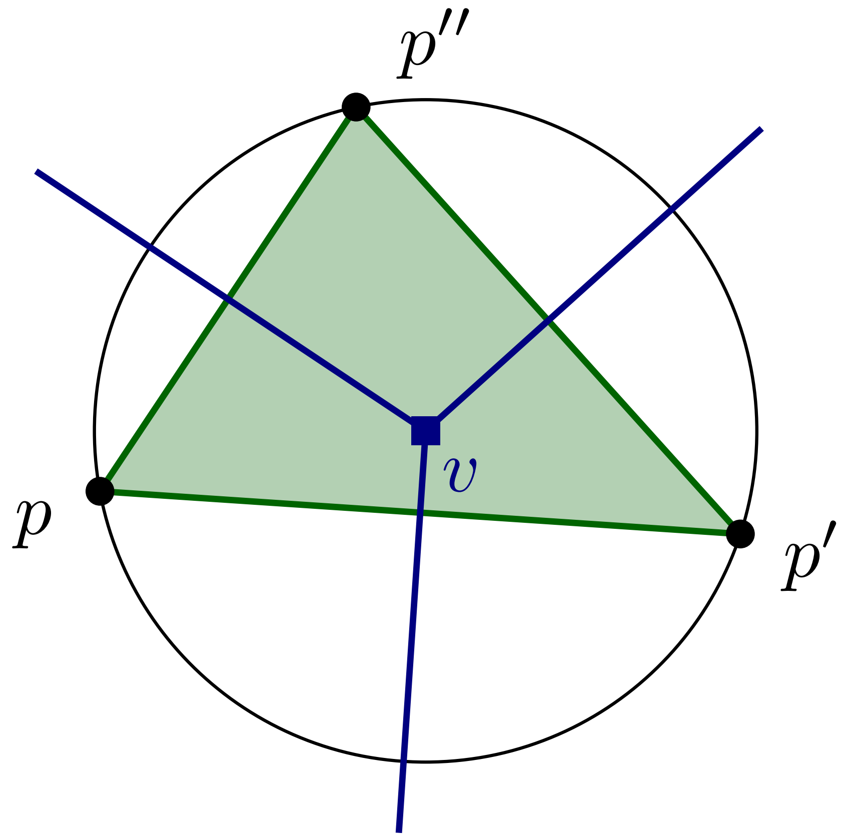


The dual of Voronoi

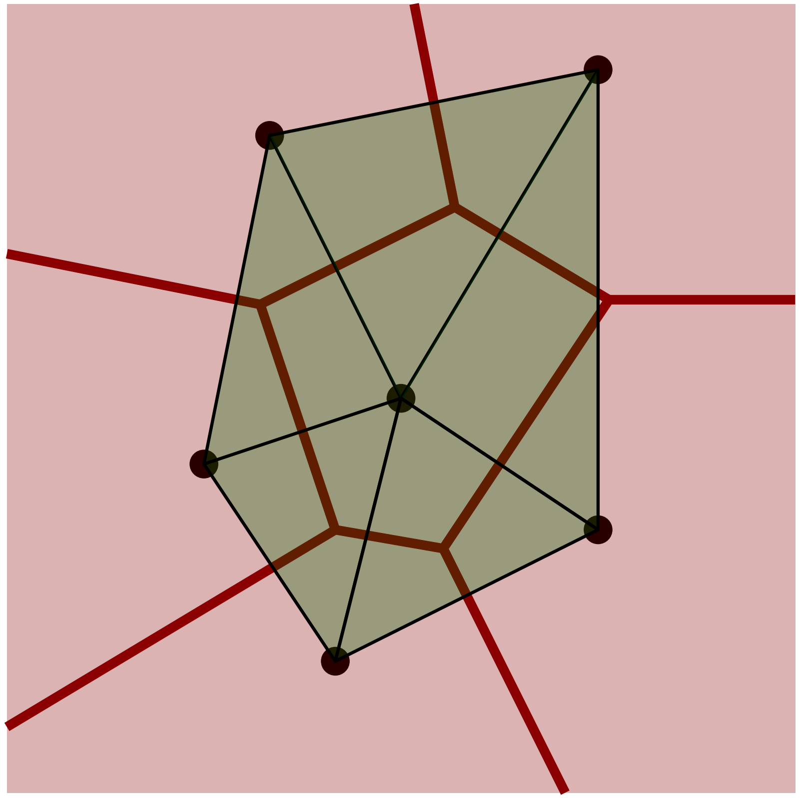
Voronoi vertex v at circumcenter of $pp'p''$

$pp'p''$ is a triangle in the Delaunay triangulation

circumcircle of $pp'p''$ has no point of P in its interior



Example: Voronoi and Delaunay



Voronoi edges

dual (Delaunay) edges

they define the Delaunay Graph

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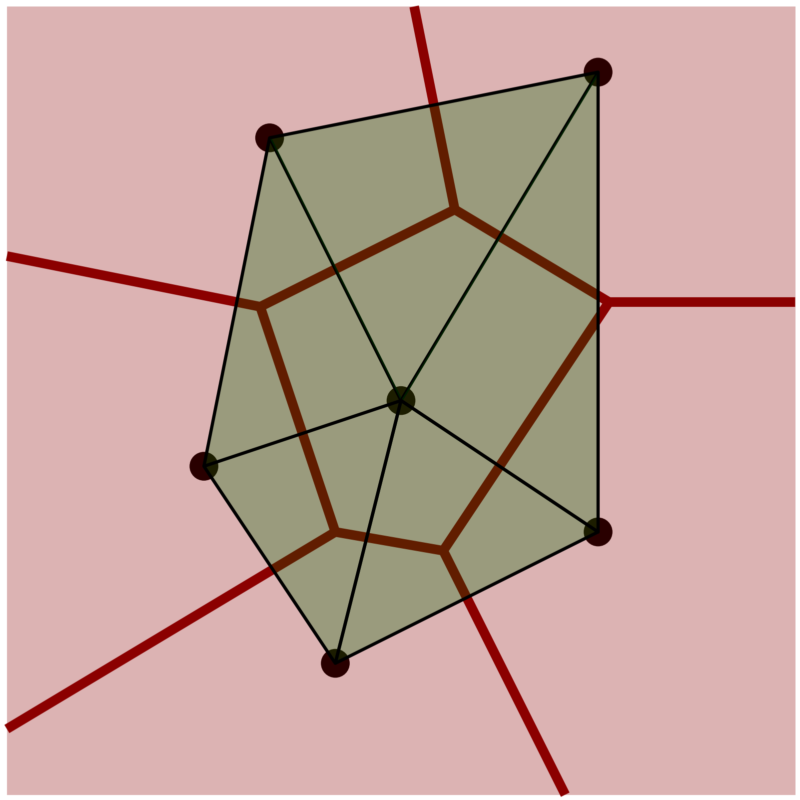


Voronoi edges

dual (Delaunay) edges

they define the Delaunay Graph

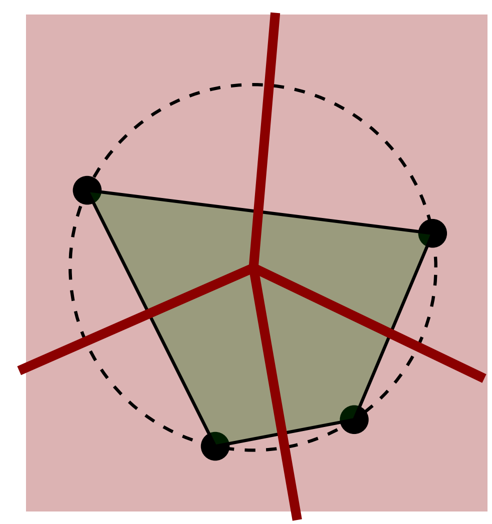
≥ 4 points on same circle $\Rightarrow$

Vor. vertex of degree ≥ 4

$\Updownarrow$

Face F of size ≥ 4 in Delaunay graph

(any triangulation of F has good triangles)



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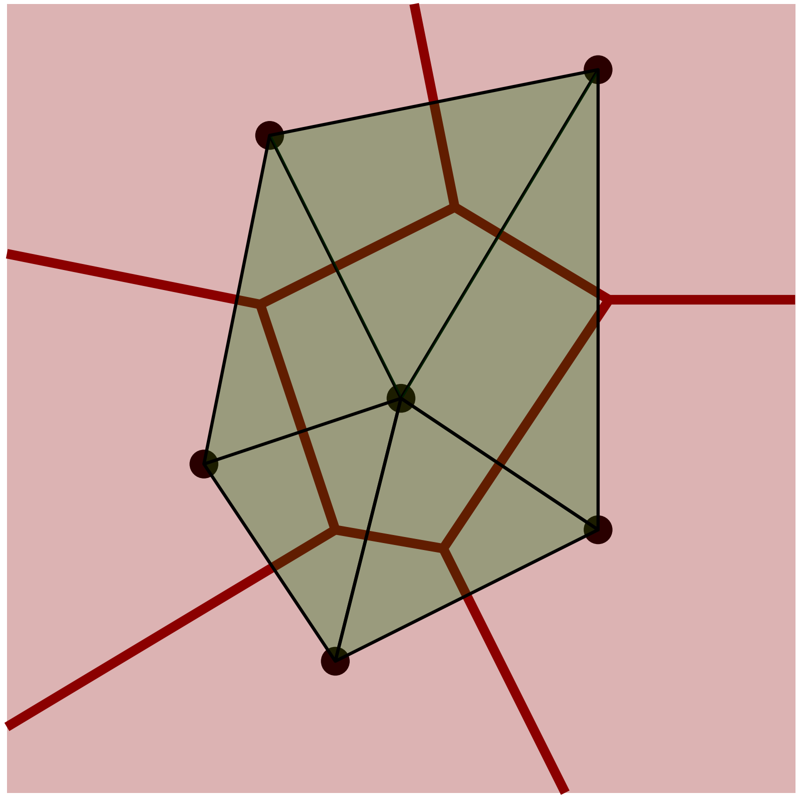


Voronoi edges

dual (Delaunay) edges

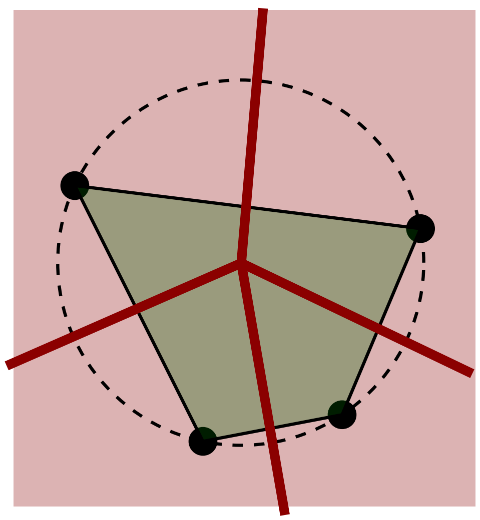
they define the Delaunay Graph

1) DG is plane graph
2) DT is unique and DT=DG
iff no 4 points on one circle

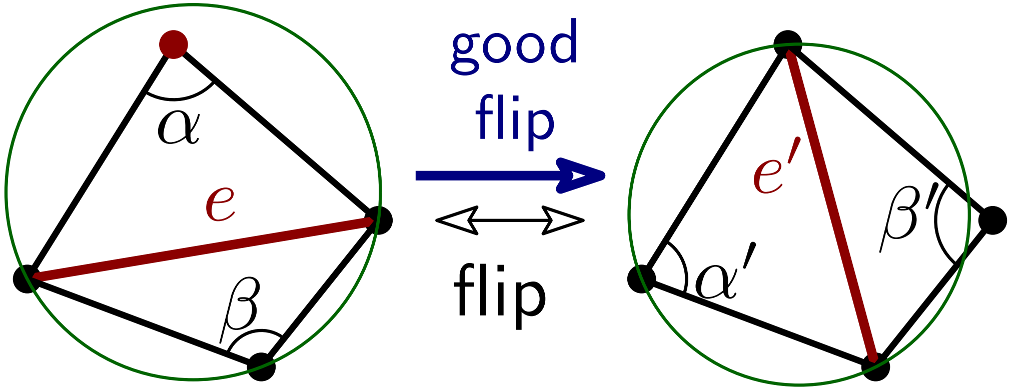
≥ 4 points on same circle $\Rightarrow$
Vor. vertex of degree ≥ 4

$\Updownarrow$

Face F of size ≥ 4 in Delaunay graph
(any triangulation of F has good triangles)



Incremental Delaunay with flips



bad triangles
$\alpha + \beta > \pi$

good triangles
$\alpha' + \beta' \leq \pi$

T is a Delaunay-tr.

$\Leftrightarrow$

No bad triangles

$\Leftrightarrow$

No bad edges to flip

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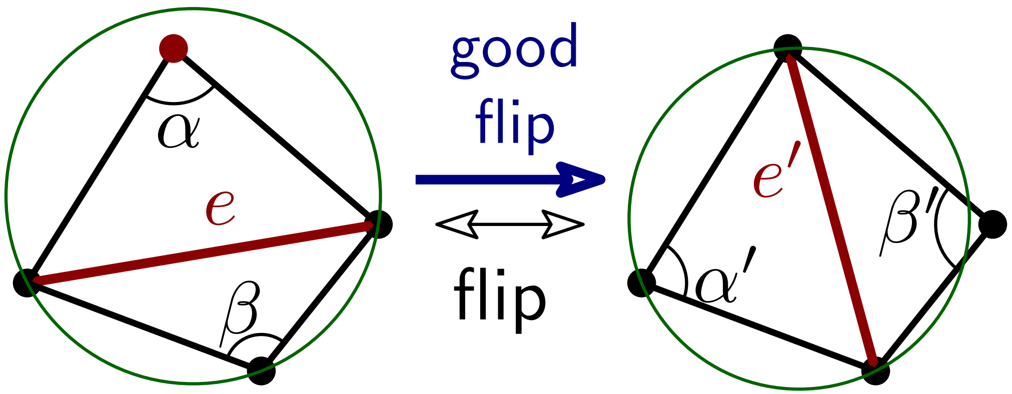

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Simple incremental algorithm:

- add points one at a time in random order
- maintain $DT(i) = DT(p_1 \ldots, p_i)$
- maintain special point location data structure on $DT(i)$

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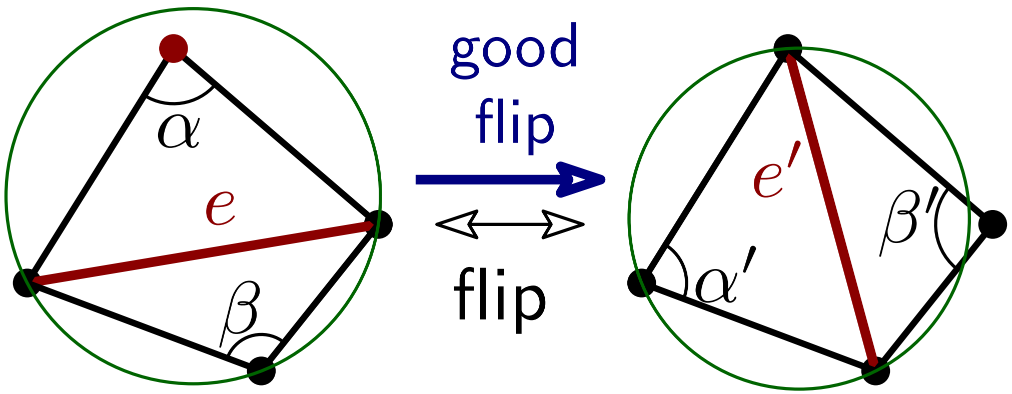

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Use flips to update triangulation

Flip algorithm

After adding p_i:

1. Find triangle $\Delta(pp'p'') \in DT(i-1)$ where $p_i \in \Delta(pp'p'')$
2. Connect p_i to p, p', p'' (to get triangulation)
3. Flip until no more bad edges,
   updating point location throughout

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Could be $\Omega(n)$ flips!

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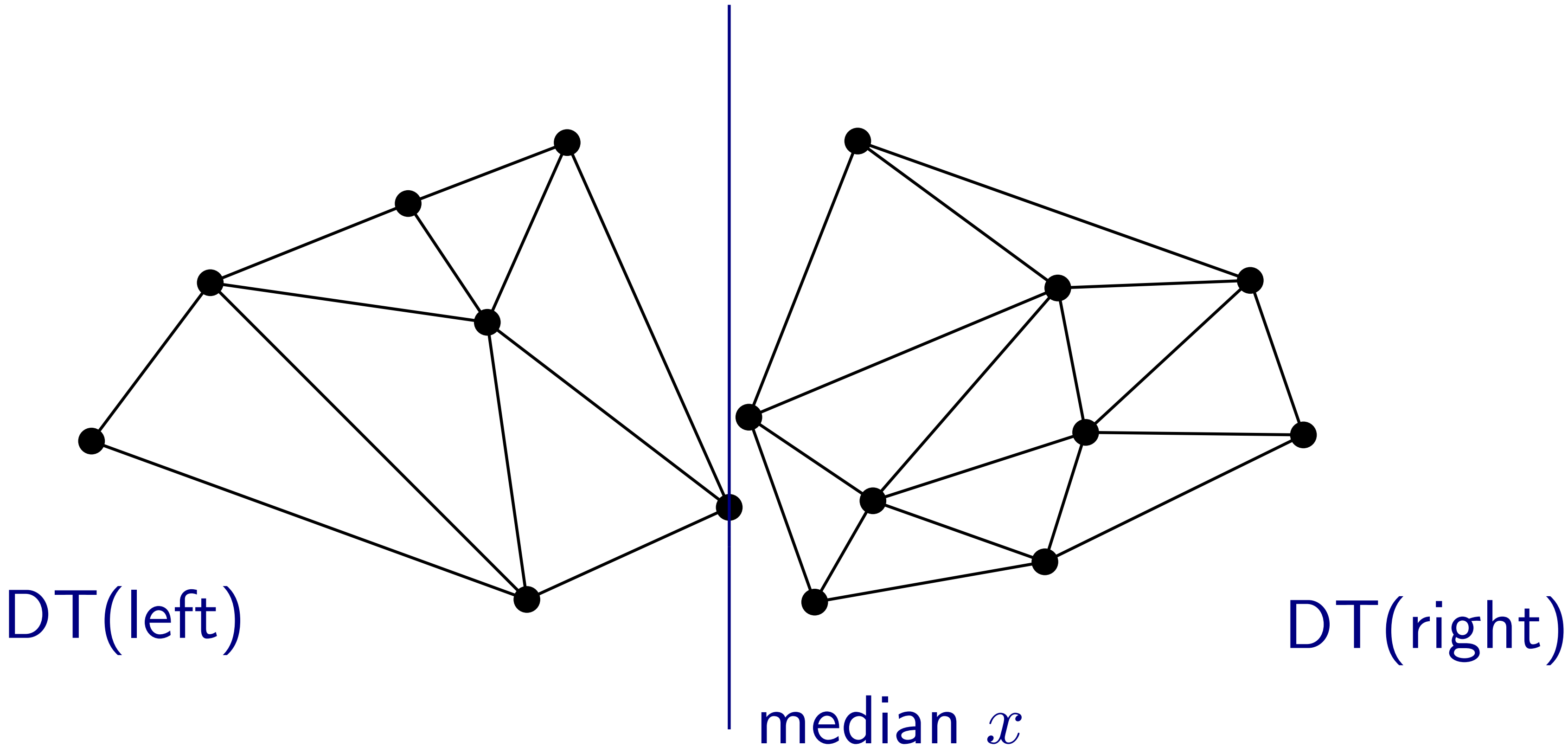
Theorem The randomized incremental construction has expected running time $O(n \log n)$ and needs $O(n)$ space in expectation.

Delaunay triangulation via divdie and conquer (Guibas and Stolfi, 1985)

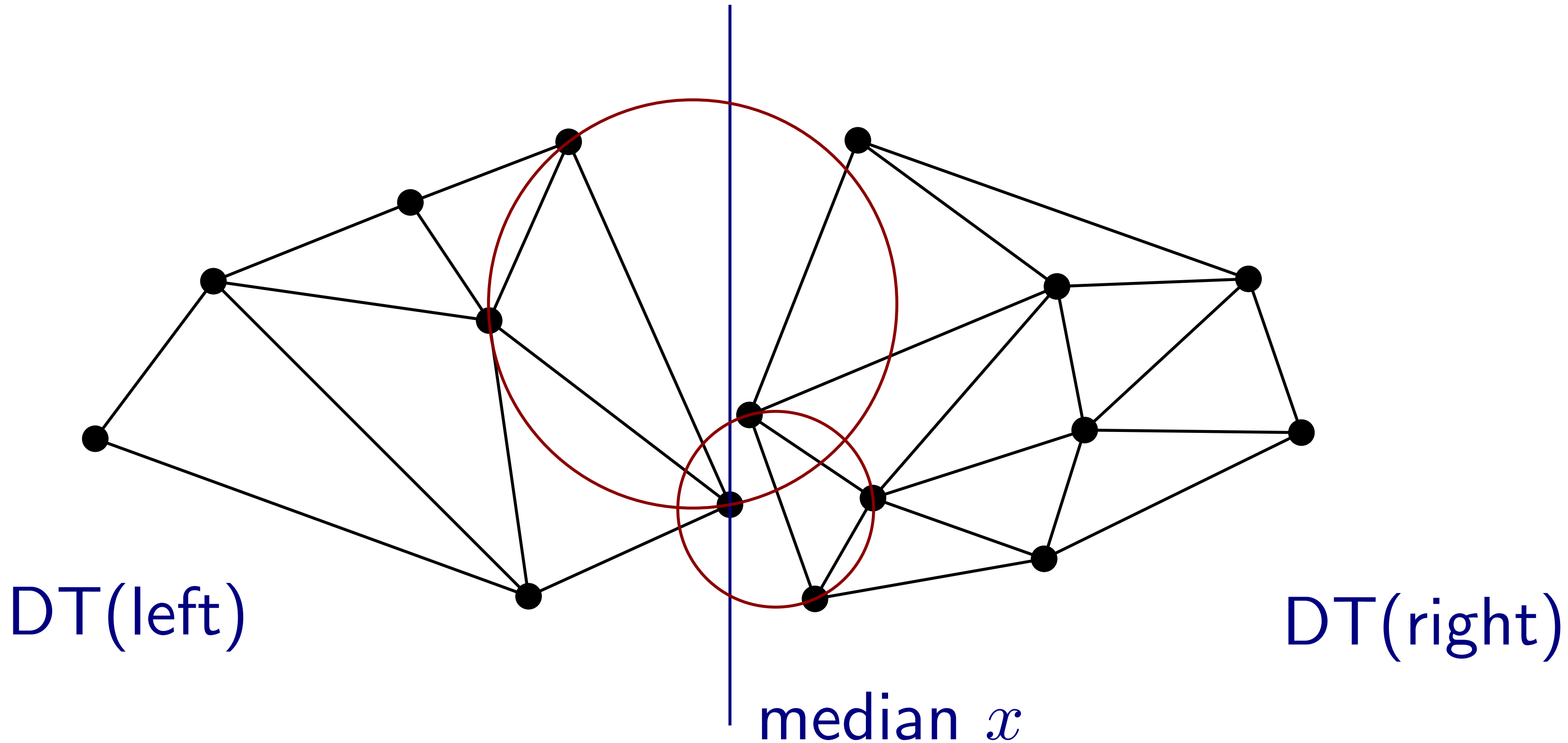
Divide and conquer DT



Task: merge in $O(n)$ time

$$T(n) = 2T(n/2) + O(n)$$

Divide and conquer DT

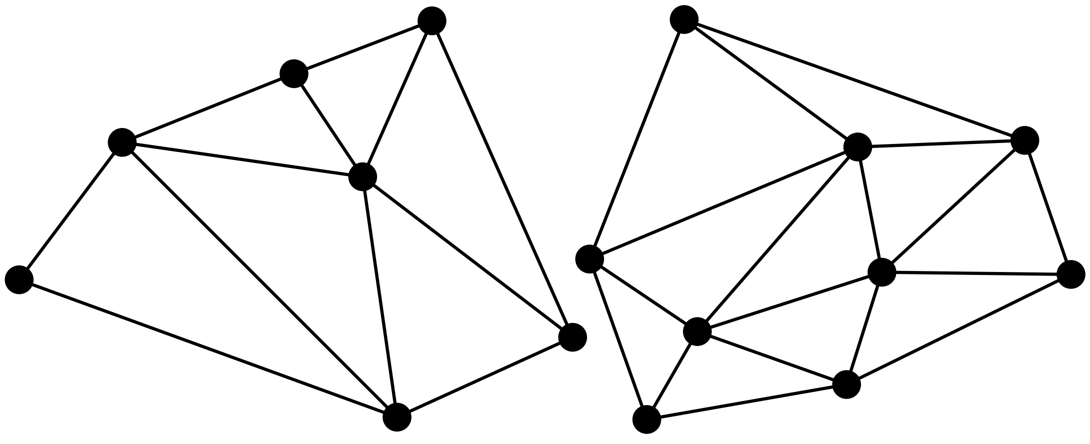


Task: merge in $O(n)$ time

$$T(n) = 2T(n/2) + O(n)$$

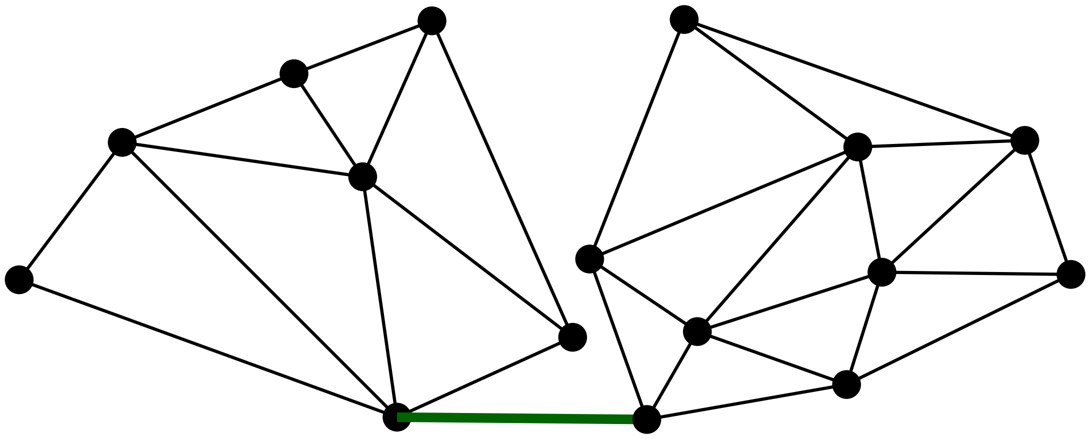
Some triangles became bad...

Bubble-up merge



Start with common lower tangent. $O(n)$

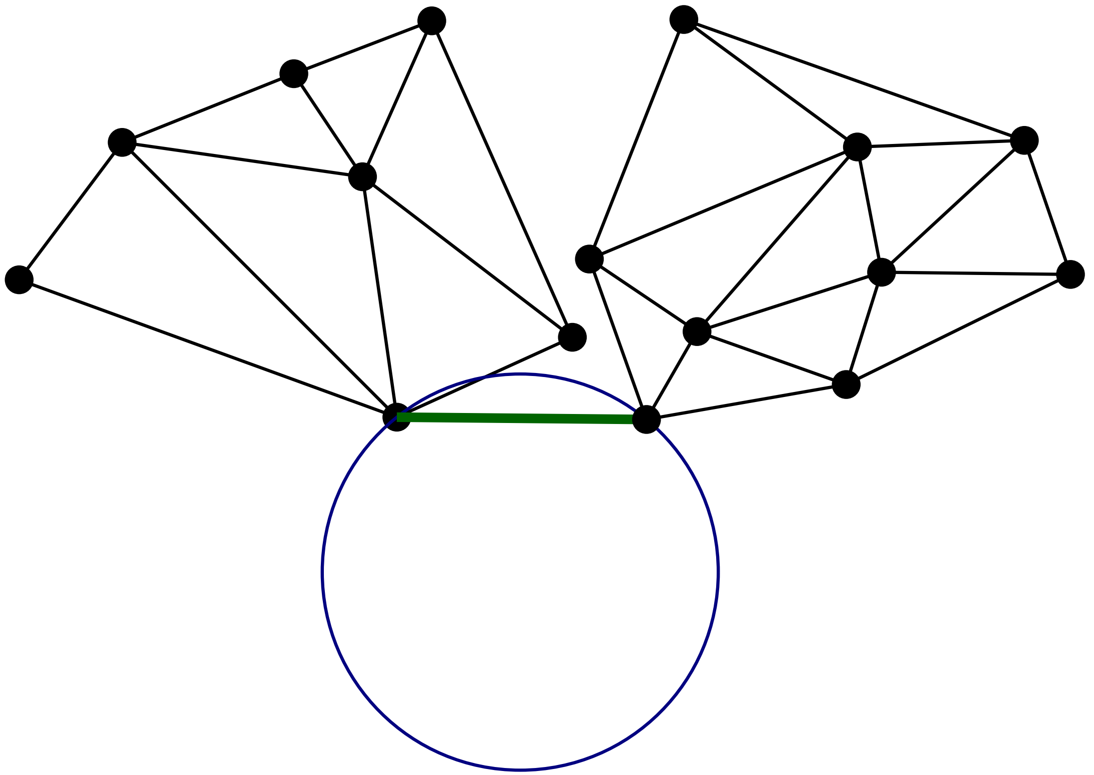
Bubble-up merge



Start with common lower tangent. $O(n)$

Push a bubble through the base edge until new vertex is hit

Bubble-up merge



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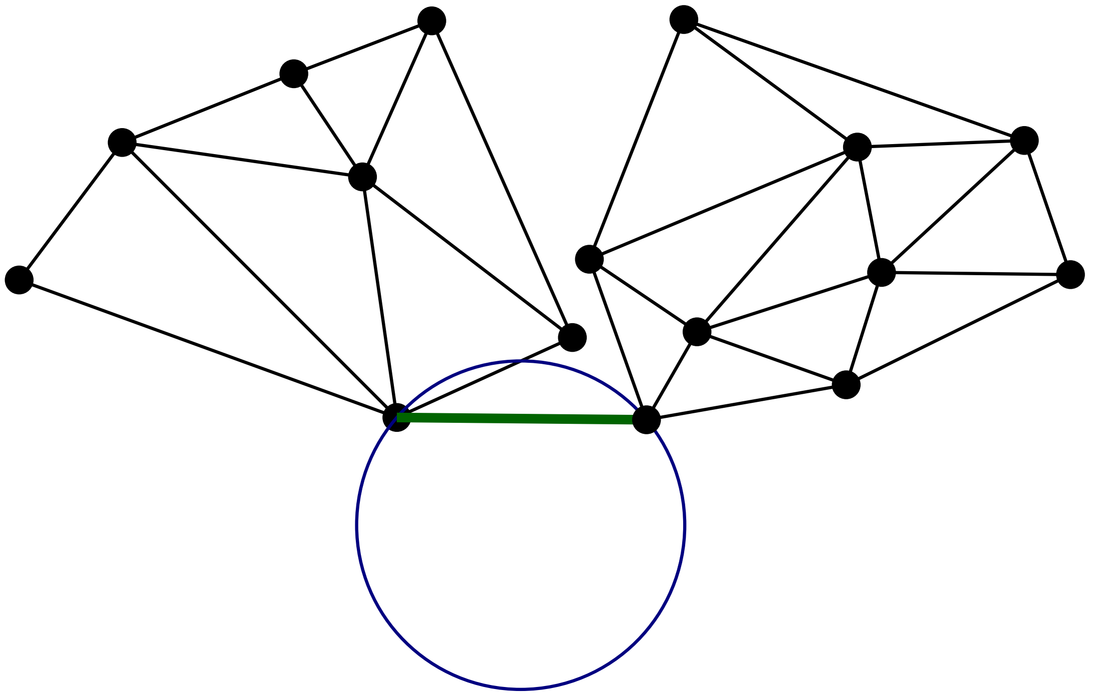

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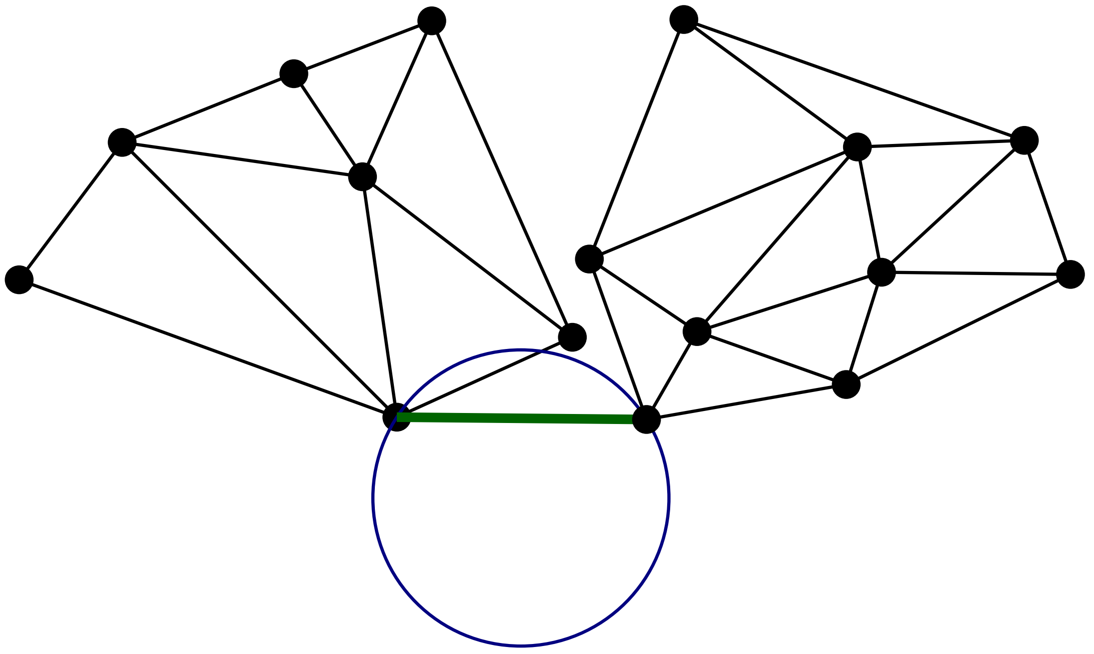

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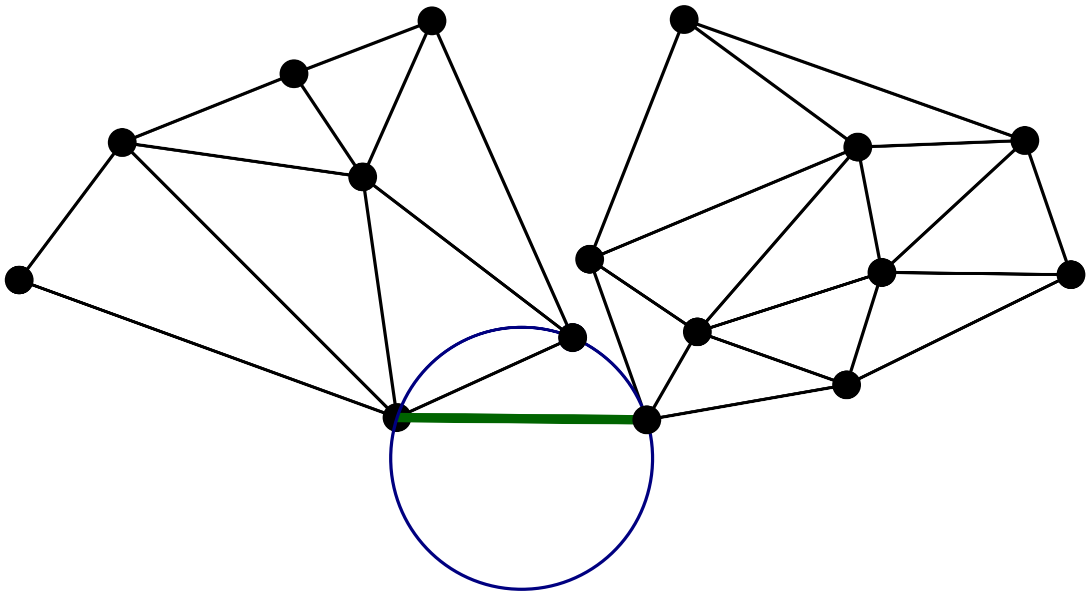

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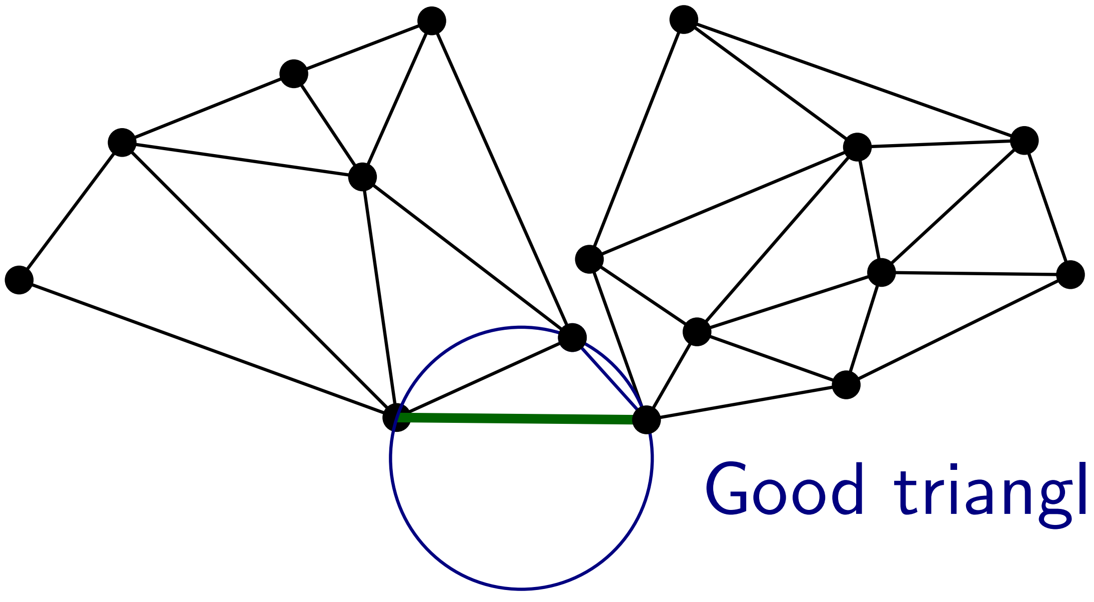

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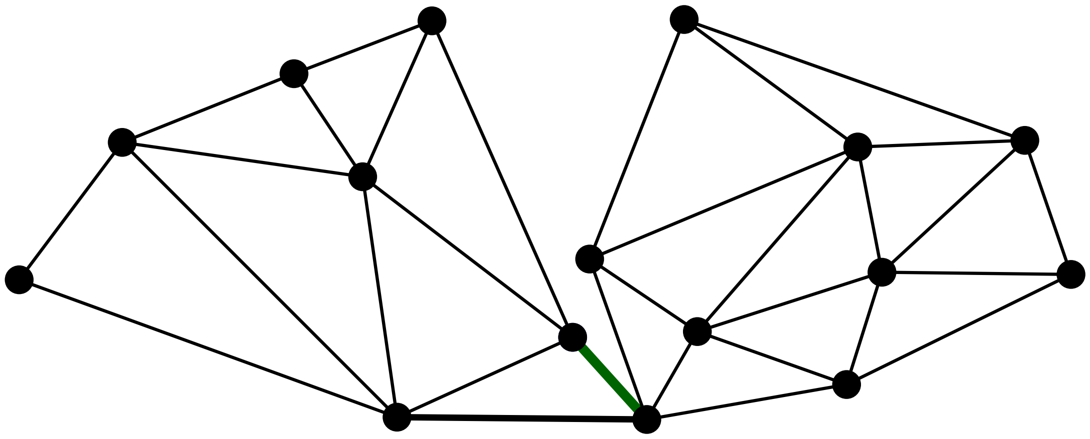

Good triangle! new edge found

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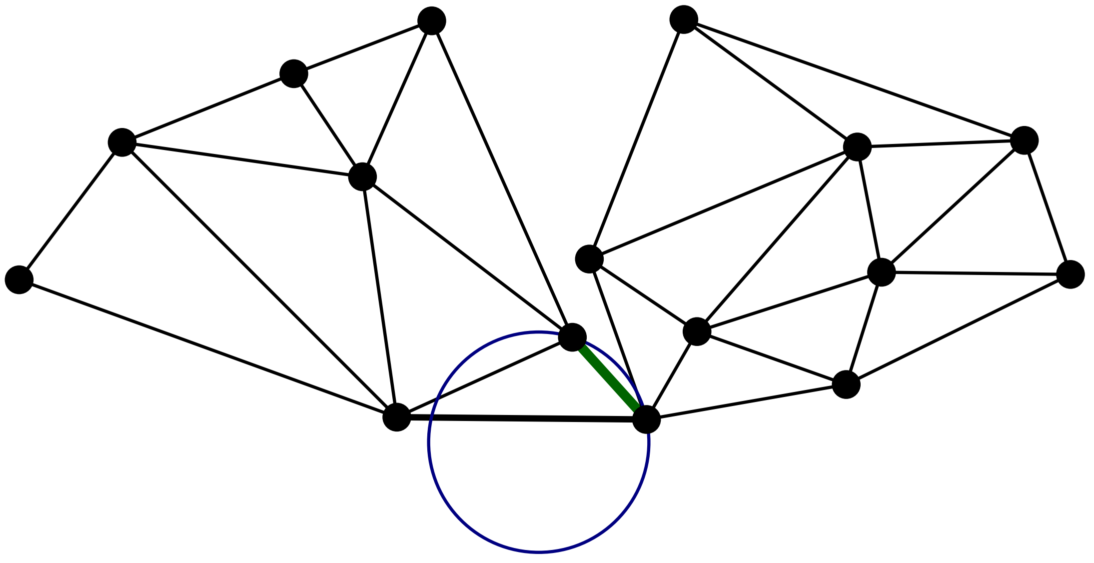

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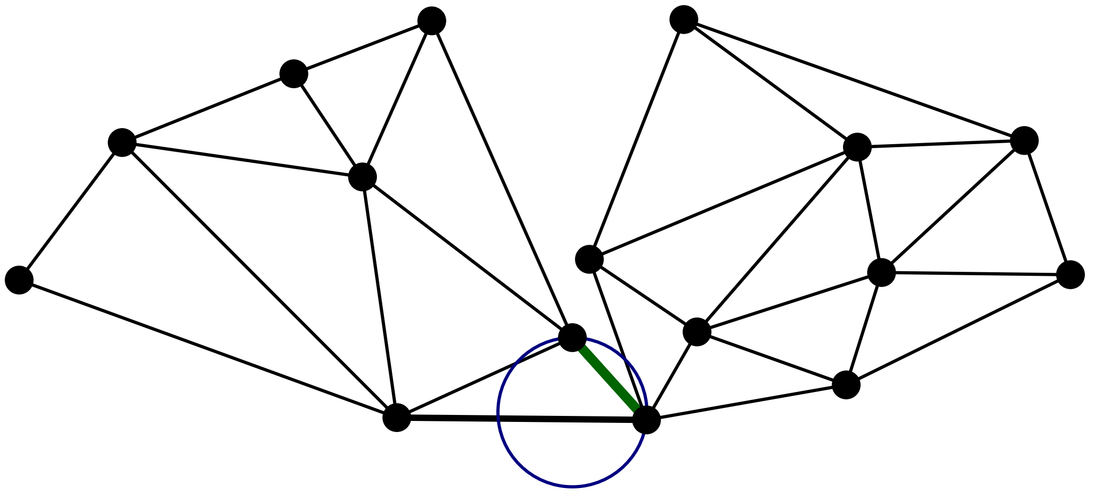

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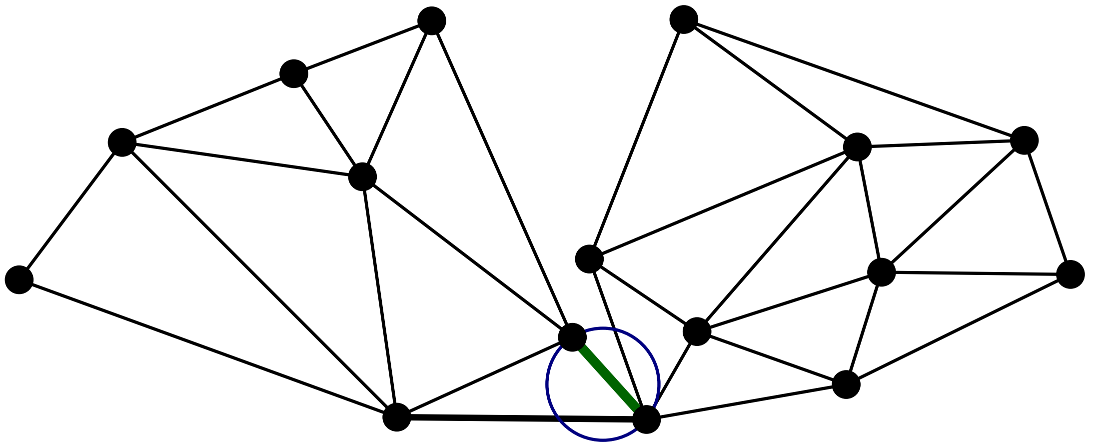

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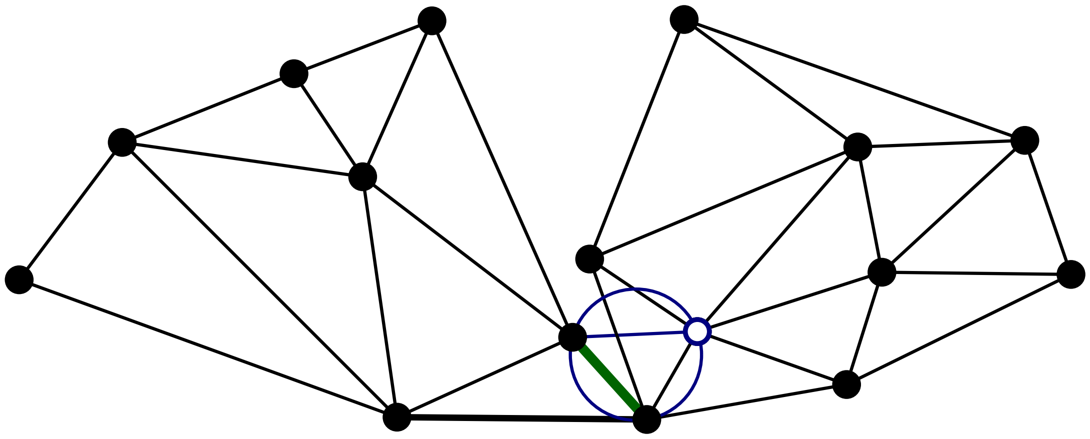

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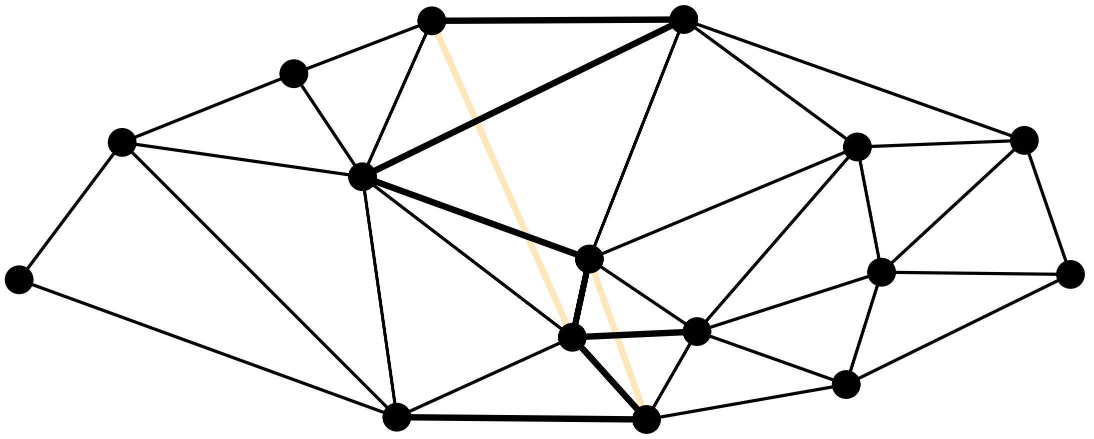

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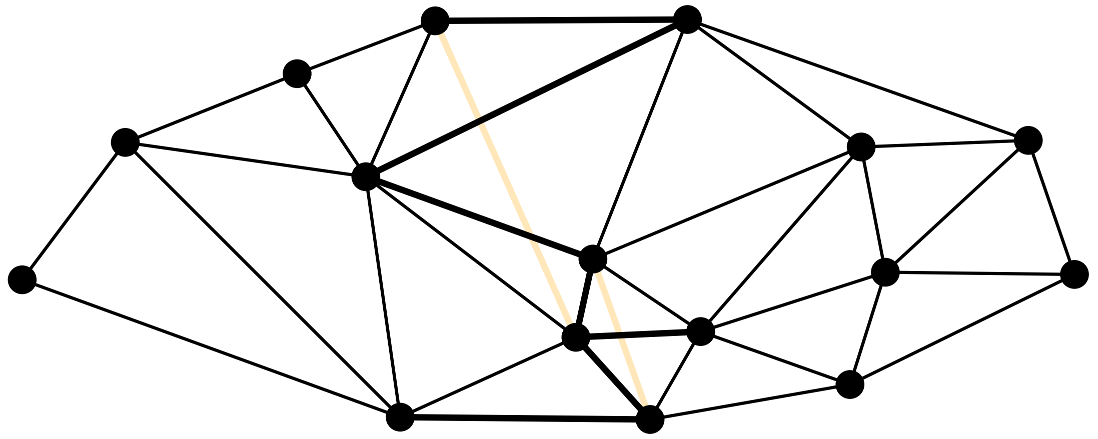

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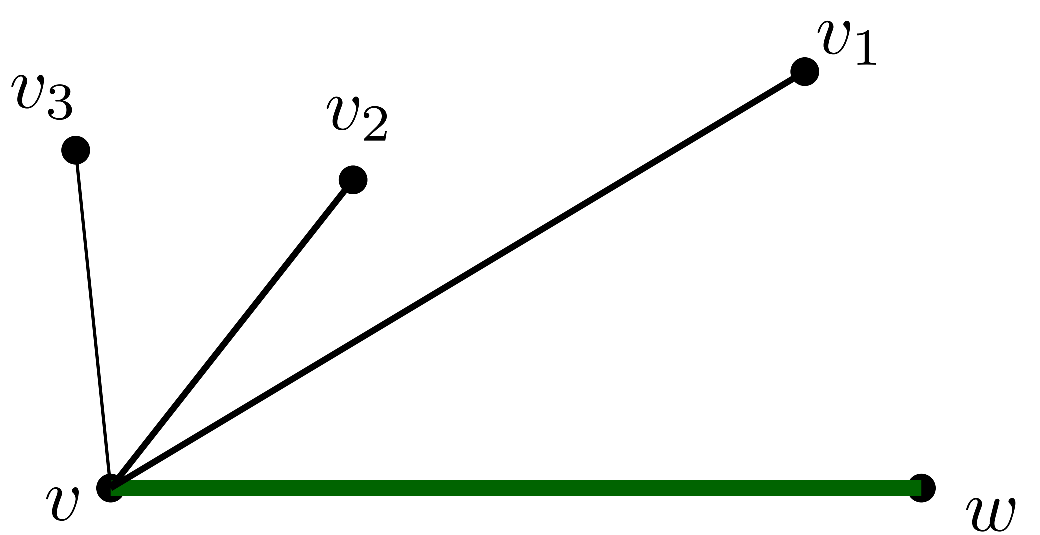
Push a bubble through the base edge until new vertex is hit

Properties:

1. All bubbles are empty
2. edge deletions are justified (they intersect some valid edge)
3. gives triangulation with only valid triangles $\Rightarrow$ gives a DT

Finding the next bubble



New vertex is DT-neighbor of v or w. (Find candidates v', w' choose best)

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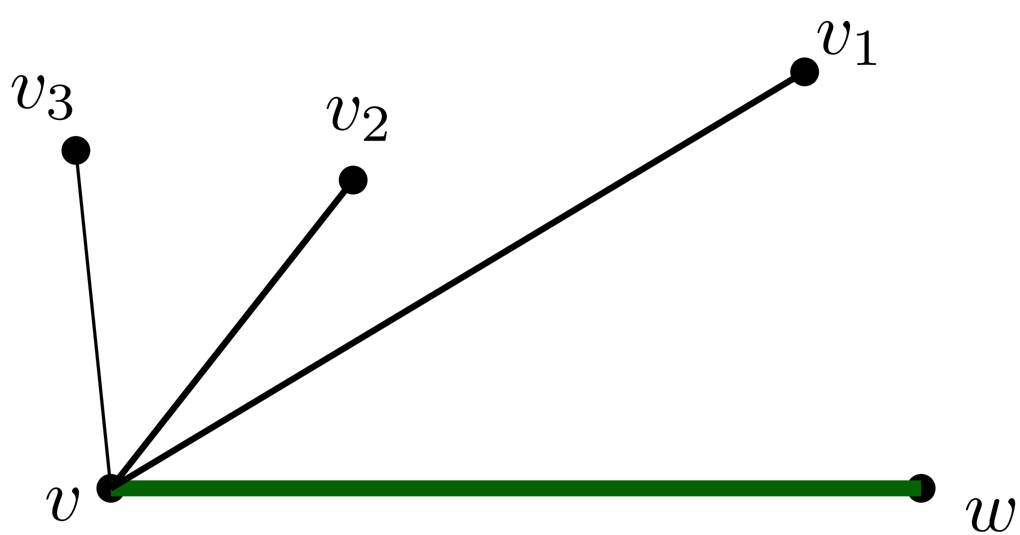

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$v_1, v_2, \ldots$: neighbors of v in CCW order after w

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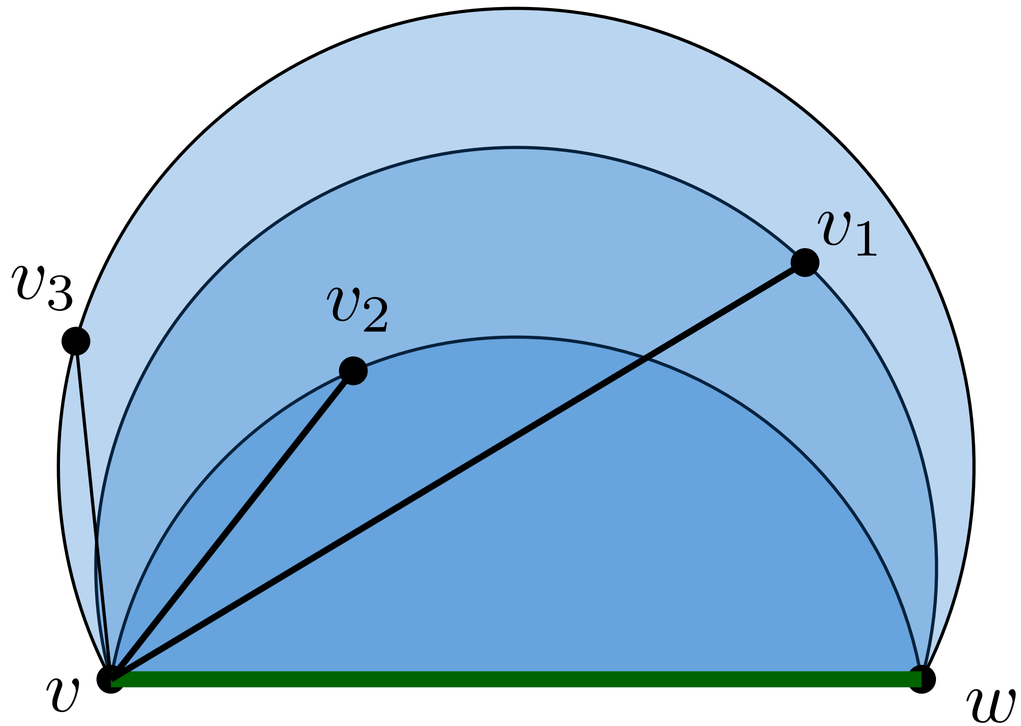

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Claim There is an i such that

$$\cdots \supset slice(vv_{i-1}w) \supset slice(vv_iw) \subset slice(vv_{i+1}w) \subset \ldots$$

Proof of unimodality

Claim There is an i such that

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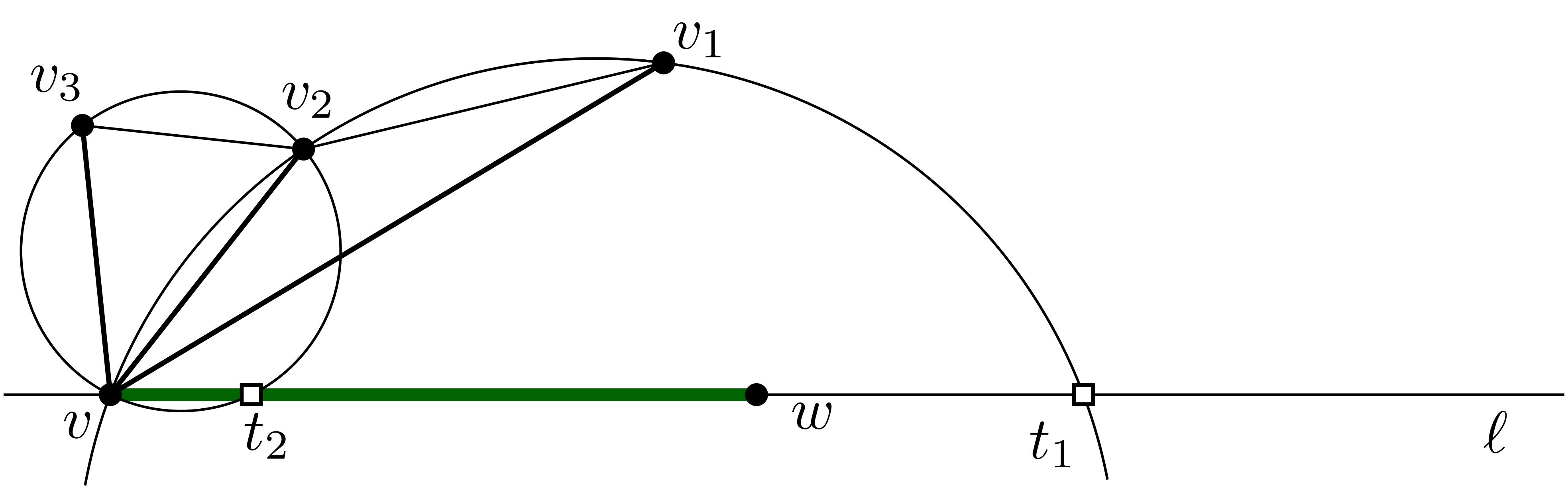
t_i := other intersection of ℓ and $circle(vv_iv_{i+1})$



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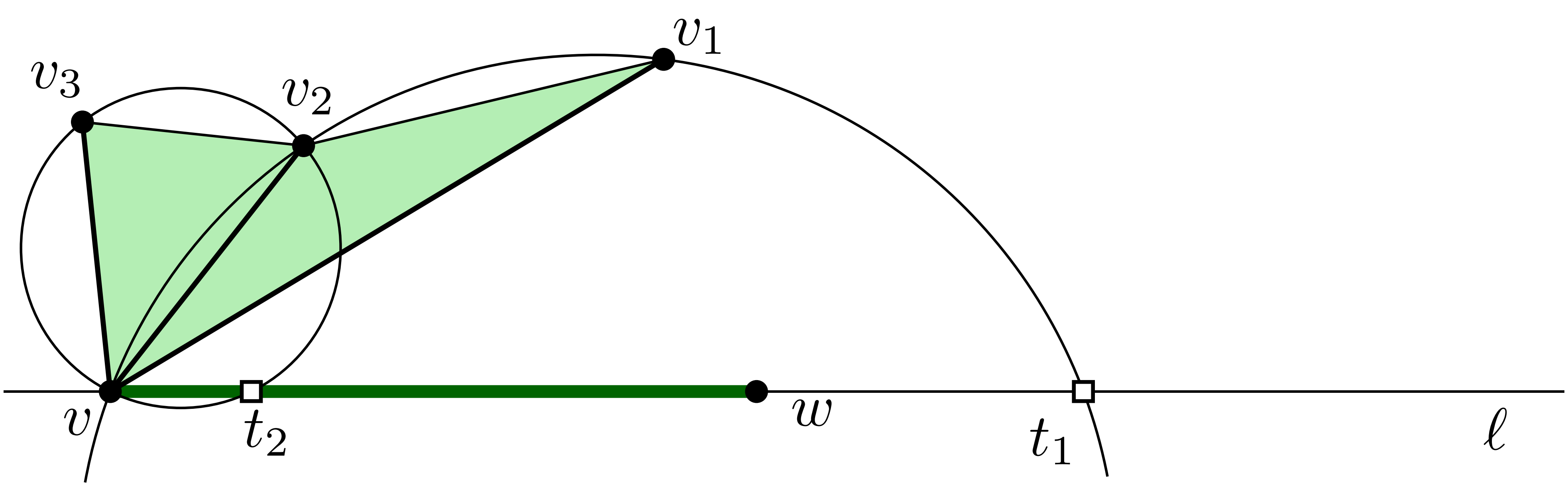
$\Delta(vv_{i-1}v_i), \Delta(vv_iv_{i+1})$ are empty triangles in DT(left)
$\Rightarrow t_1, t_2, \ldots$ moves left on ℓ



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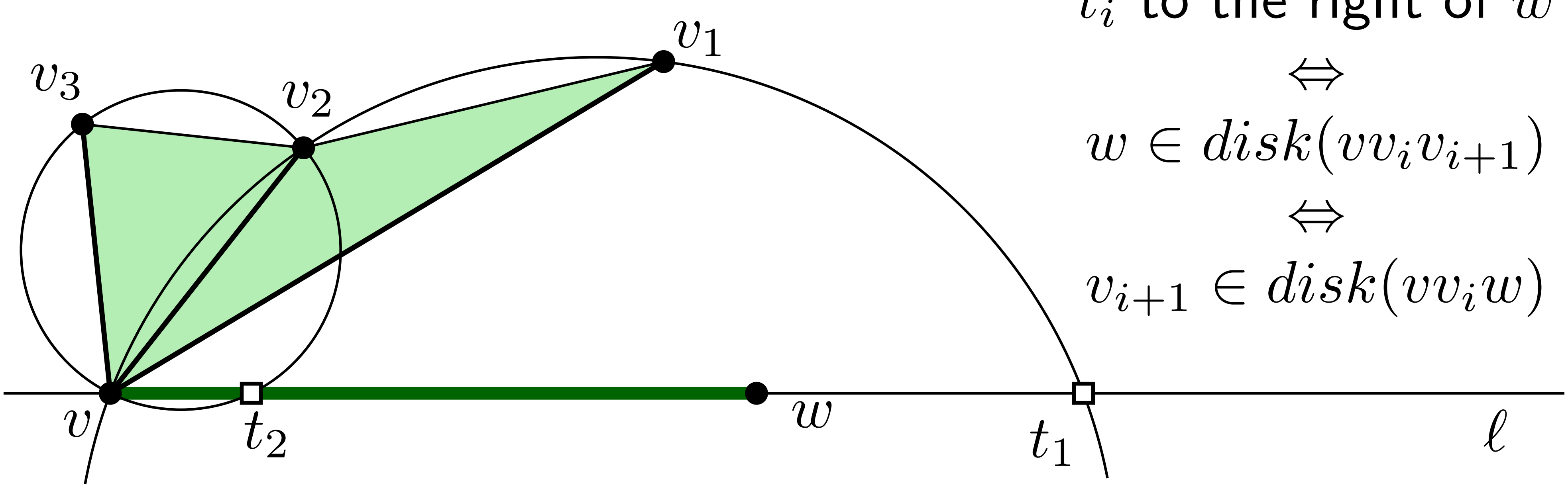
t_i to the right of w
$\Leftrightarrow$
$w \in disk(vv_iv_{i+1})$
$\Leftrightarrow$
$v_{i+1} \in disk(vv_iw)$



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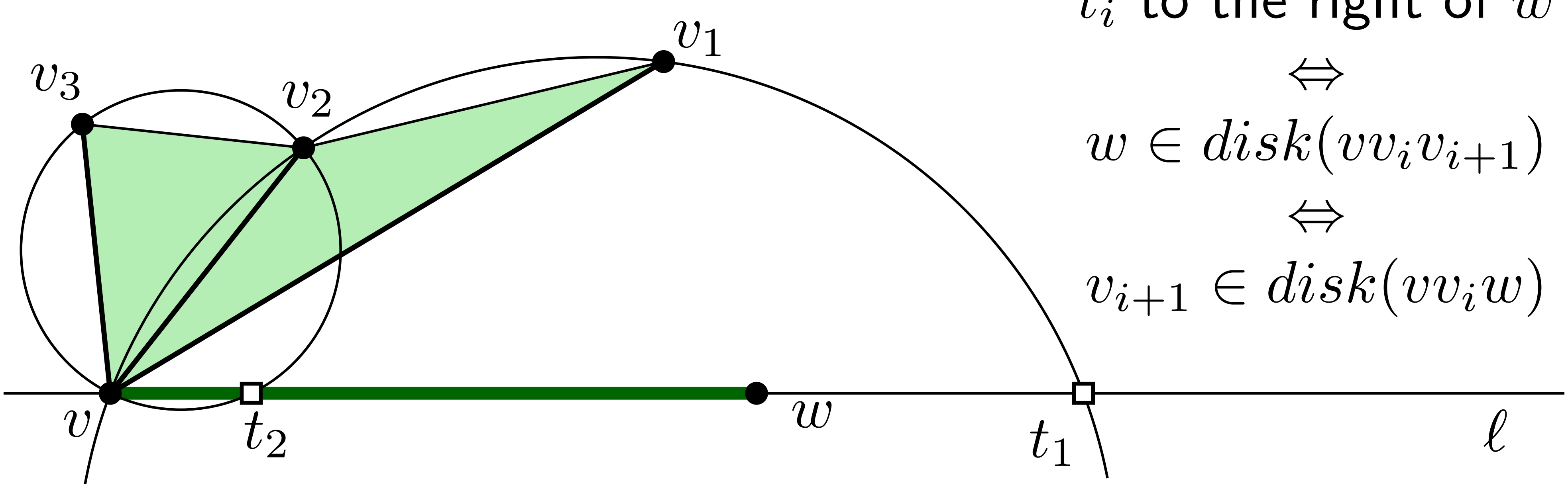

Next hit in DTleft: v_i where t_i is first to the left of w

Bubble-up merge running time

Any edge vv_i passed is not DT edge (not empty disk)
$\Rightarrow$ delete such edges

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- find common tangents
- starting at bottom tangent$= vw$:
  - find $v_i =$ next hit on left by stepping through $N(v)$ in CCW order, deleting passed edges
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  - check which of v_i,w_j works
  - set vw as new edge

  until vw is other tangent

Bubble-up merge running time

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$O(1)$ steps per deleted edge, O(n) deleted edges

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Bubble merge runs in $O(n)$