Packing and covering: planar separator and shifting

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Computational Geometry
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Overview

- Planar separator theorem (slides by Mark de Berg)
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- Independent set in planar graphs (slides by MdB)
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• Exact algorithms for packing and covering
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• Exact algorithms for packing and covering

• Shifting strategy: approximation schemes
Planar graphs: graphs that can be drawn without crossing edges
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**Planar Separator Theorem (Lipton, Tarjan 1979)**

For any planar graph \( G = (V, E) \) there is a separator \( S \subset V \) of size \( O(\sqrt{n}) \) such that \( V \setminus S \) can be partitioned into subsets \( A \) and \( B \), each of size at most \( \frac{2}{3}n \) and with no edges between them.
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**Planar Separator Theorem (Lipton, Tarjan 1979)**

For any planar graph $G = (V, E)$ there is a separator $S \subset V$ of size $O(\sqrt{n})$ such that $V \setminus S$ can be partitioned into subsets $A$ and $B$, each of size at most $\frac{2}{3}n$ and with no edges between them.

Such a $(2/3)$-balanced separator can be computed in $O(n)$ time.

*slide by Mark de Berg*
Fact: Any planar graph is the contact graph of a set of interior-disjoint disks.
A geometric proof of the Planar Separator Theorem

**Fact:** Any planar graph is the contact graph of a set of interior-disjoint disks.

**Proof idea:** Find a square $\sigma$ intersecting $O(\sqrt{n})$ disks that is a balanced separator.
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**Theorem.** For any contact graph of $n$ interior-disjoint disks, there is an $\alpha$-balanced separator of size $O(\sqrt{n})$, where $\alpha = 36/37$.

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The smallest square containing at least $n/37$ disks.
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\[ \sqrt{n} \text{ squares } \sigma_1, \ldots, \sigma_{\sqrt{n}} \]

at distance $1/\sqrt{n}$ from each other
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Constructing the separator:
Select a square $\sigma_i$ that intersects $O(\sqrt{n})$ disks and put these disks into the separator.
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**Constructing the separator:**
Select a square \(\sigma_i\) that intersects \(O(\sqrt{n})\) disks and put these disks into the separator.

**Things to check**
- separator is \((36/37)\)-balanced
- does square \(\sigma_i\) with the desired property actually exist??
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![Diagram showing the smallest square containing at least \( n/37 \) disks, with a (36/37)-balanced separator indicated.](slide by Mark de Berg)
**Theorem.** For any contact graph of $n$ interior-disjoint disks, there is an $\alpha$-balanced separator of size $O(\sqrt{n})$, where $\alpha = 36/37$.

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- Smallest square containing at least $n/37$ disks
- The separator is $(36/37)$-balanced
  - At least $n/37$ disk inside
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**Proof.**

Does \( \sigma_i \) intersecting \( O(\sqrt{n}) \) disks exist?

The total number of disk-square intersections is

\[
\leq \sum_{i=1}^{n_{\text{small}}} (1 + \text{diam}(D_i) \cdot \sqrt{n})
\]

\[
\leq n_{\text{small}} + O(\sqrt{n}) \cdot \sum_{i=1}^{n_{\text{small}}} \sqrt{\text{area}(D_i)}
\]

\[
= O(n)
\]

The last step uses

- \( \sum_{i=1}^{n_{\text{small}}} \text{area}(D_i) = O(1) \) (sort of . . .)

- \( \sum_{i=1}^{k} \sqrt{a_i} \leq \sum_{i=1}^{k} \sqrt{\frac{\sum_{i=1}^{k} a_i}{k}} \)
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\[
\implies \text{one of the } \sigma_i \text{'s intersects } O(\sqrt{n}) \text{ disks}
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Theorem. \textsc{Independent Set} can be solved in $2^{O(\sqrt{n})}$ time in planar graphs.
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1. Compute $(2/3)$-balanced separator $S$ of size $O(\sqrt{n})$.
2. For each independent set $I_S \subseteq S$ (including empty set) do
   (a) Recursively find max independent set $I_A$ for $A \setminus \{\text{neighbors of } I_S\}$
   (b) Recursively find max independent set $I_B$ for $B \setminus \{\text{neighbors of } I_S\}$
   (c) $I(S) := I_S \cup I_A \cup I_B$
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Subexponential algorithms on planar graphs

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**Running time**
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Running time

$$T(n) \leq O(n) + 2^{O(\sqrt{n})} \cdot T\left(\frac{2n}{3}\right) \implies T(n) = 2^{O(\sqrt{n})}$$
Overview

- Planar separator theorem (slides by Mark de Berg)
- Independent set in planar graphs (slides by MdB)
- Exact algorithms for packing and covering
- Shifting strategy: approximation schemes
Given a set $S$ of $n$ objects in $\mathbb{R}^d$, their *intersection graph* has vertex set $S$ and edge set

$$E[S] := \{ss' \mid s, s' \in S \text{ and } s \cap s' \neq \emptyset\}$$
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Planar graphs $\subset$ Disk graphs (object: disks in $\mathbb{R}^2$)
Packing: discrete vs continuous

Continuous:
Given $n$ objects, do they fit in some other object without overlap?
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Given $n$ objects, do they fit in some other object without overlap?

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Given $n$ objects, find maximum subset of non-overlapping objects
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Same as max. independent set in intersection graph
Theorem. Independent set in intersection graphs of disks can be solved in $n^{O(\sqrt{k})}$ time, where $k =$ size of max indep. set.
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Proof.
Solution $I$ has $k$ interior-disjoint disks. There is a balanced separator square $\sigma$ intersecting $O(\sqrt{k})$ disks from $I$. 
Theorem. Independent set in intersection graphs of disks can be solved in \( n^{O(\sqrt{k})} \) time, where \( k \) = size of max indep. set.

Proof.
Solution \( I \) has \( k \) interior-disjoint disks.
There is a balanced separator square \( \sigma \) intersecting \( O(\sqrt{k}) \) disks from \( I \).

Claim. Given \( S \), we can compute a family \( Y \) of \( \text{poly}(n) \) squares containing all attainable square separators of all subsets of \( S \).
for each separator $\sigma \in Y$ do
  for each intersecting $I_\sigma \subset S$ of size $O(\sqrt{k})$ do
    Remove disks in $S$ intersecting $\sigma$
    Remove neighbors of $I_\sigma$
    Recurse on disks inside $\sigma$
    Recurse on disks outside $\sigma$
  return largest indep. set found
Exact algorithm for discrete packing II

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$$T(n, k) = \text{poly}(n) \cdot n^{O(\sqrt{k})} \cdot 2T \left(n, \frac{36}{37}k\right)$$
Exact algorithm for discrete packing II

\begin{align*}
\textbf{for} & \text{ each separator } \sigma \in Y \textbf{ do} \\
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\end{align*}

\[ T(n, k) = \text{poly}(n) \cdot n^{O(\sqrt{k})} \cdot 2T \left( n, \frac{36}{37}k \right) \]

\[ T(n, k) = n^{c\sqrt{k} + c\sqrt{(36/37)k} + c\sqrt{(36/37)^2k} + \ldots} = n^{O(\sqrt{k})} \]
Is $n^{O(\sqrt{k})}$ good?

General graphs: Independent set is NP-hard, has $O(n^k k^2)$ algo
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**Exponential-Time Hypothesis (ETH).** There is $\gamma > 0$ such that there is no $2^{\gamma n}$ algorithm for 3-SAT on $n$ variables.

ETH $\Rightarrow$ P $\neq$ NP
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\[ \text{ETH} \Rightarrow \text{P} \neq \text{NP} \]

**Theorem.** There is no $f(k) n^{o(k)}$ algorithm for Independent Set for any computable $f$, unless ETH fails.
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**Theorem.** There is no $f(k) n^{o(\sqrt{k})}$ algorithm for Independent Set in planar graphs for any computable $f$, unless ETH fails.
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**Exponential-Time Hypothesis (ETH).** There is $\gamma > 0$ such that there is no $2^{\gamma n}$ algorithm for 3-SAT on $n$ variables.

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**Theorem.** There is no $f(k)n^{o(\sqrt{k})}$ algorithm for Independent Set **in planar graphs** for any computable $f$, unless ETH fails.

**Theorem.** There is no $f(k)n^{o(\sqrt{k})}$ algorithm for Independent Set **in disk graphs** for any computable $f$, unless ETH fails.
Geometric set cover: discrete vs continuous

*Set cover*: given $m$ subsets of $\{1, \ldots, n\}$, are there $k$ among them whose union is $\{1, \ldots, n\}$

very hard, can’t be approximated efficiently
Geometric set cover: discrete vs continuous

Set cover: given \( m \) subsets of \( \{1, \ldots, n\} \), are there \( k \) among them whose union is \( \{1, \ldots, n\} \)

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Geometric set cover:

Continuous:

Given \( P \subset \mathbb{R}^2 \), can we cover \( P \) with \( k \) unit disks?
Geometric set cover: discrete vs continuous

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similar to cont. $k$-center!
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Geometric set cover:

Continuous:
Given \( P \subset \mathbb{R}^2 \), can we cover \( P \) with \( k \) unit disks?

Discrete:
Given \( P \subset \mathbb{R}^2 \) and \( m \) unit disks \( \mathcal{D} \), can we cover \( P \) with \( k \) disks from \( \mathcal{D} \)?

similar to cont. \( k \)-center!
Theorem (Marx–Pilipczuk, 2015) Discrete geometric set cover with disks can be solved in $m^{O(\sqrt{k})}\text{poly}(n)$ time, where $k =$ size of min cover.
Theorem (Marx–Pilipczuk, 2015) Discrete geometric set cover with disks can be solved in $m^{O(\sqrt{k})}\text{poly}(n)$ time, where $k =$ size of min cover.

Proof based on guessing separator in solution’s Voronoi diagram.

Theorem (Marx–Pilipczuk, 2015). There is no $f(k)(m + n)^{o(\sqrt{k})}$ algorithm for covering points with disks for any computable $f$, unless ETH fails.
Shifting grids
Approximation schemes
Hochbaum–Maass 1985
**Definition.** A polynomial time approximation scheme (PTAS) for a minimization problem is an algorithm, which given $\varepsilon > 0$ and the input instance, outputs a feasible solution of value at most $(1 + \varepsilon)OPT$ in $\text{poly}_\varepsilon(n)$ time.

E.g. possible running time: $O(n^{1/\varepsilon})$ or $n^{O(2^{1/\varepsilon})}$
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related complexity classes: PTAS versus APX-hardness
\(P\) is APX-hard \(\Rightarrow\) \(P\) has no PTAS unless \(P=NP\)
PTASes

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$\mathcal{P}$ is APX-hard $\Rightarrow \mathcal{P}$ has no PTAS unless P=NP

**Example:** Independent set is APX-hard on general graphs.

**But!** Independent set in planar graphs has a PTAS. (Baker '83)
Theorem. The discrete packing of unit disks has a PTAS: given \( n \) unit disks, we can compute an independent set of size \((1 - \varepsilon)OPT\) in \( n^{O(1/\varepsilon)} \) time.
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Proof.

Grid of distance 2
\[ \Rightarrow \text{each (open) disk intersects } \leq 1 \text{ horizontal and } \leq 1 \text{ vertical grid line} \]
Theorem. The discrete packing of unit disks has a PTAS: given $n$ unit disks, we can compute an independent set of size $(1 - \varepsilon)OPT$ in $n^{O(1/\varepsilon)}$ time.

Proof.

Grid of distance 2
⇒ each (open) disk intersects $\leq 1$ horizontal and $\leq 1$ vertical grid line

Let $t = \lceil 2/\varepsilon \rceil$.
For a shift $(a, b)$ ($a, b \in \{0, \ldots, t - 1\}$), select horizontal lines $a, a + t, a + 2t, \ldots$
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**Theorem.** The discrete packing of unit disks has a PTAS: given \( n \) unit disks, we can compute an independent set of size \((1 - \varepsilon)OPT\) in \( n^{O(1/\varepsilon)} \) time.

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select horizontal lines \( a, a+t, a+2t, \ldots \)

select vertical lines \( b, b+t, b+2t, \ldots \)

Remove disks intersecting selected lines
Shifting strategy: solving cells

\begin{align*}
  a & \quad b \quad b + t \quad b + 2t \\
  a + t & \\
  a + 2t &
\end{align*}
Shifting strategy: solving cells

Large cells have area $O(1/\varepsilon^2)$
$\Rightarrow$ max indep. set has size $k = O(1/\varepsilon^2)$
$\Rightarrow$ max indep. set found in $n^{O(\sqrt{k})} = n^{O(1/\varepsilon)}$ time.
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Claim. The union of large cell solutions has size at least
$(1 - \varepsilon)OPT$ for some shift $(a, b)$. 
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Claim. The union of large cell solutions has size at least $(1 - \varepsilon)OPT$ for some shift $(a, b)$.

Proof. Of the $t = \lceil 2/\varepsilon \rceil$ shifts for horizontals, there is some $a \in \{0, \ldots, t - 1\}$ intersecting $\leq \frac{\varepsilon}{2} OPT$ solution disks. Similarly there is $b$ s.t. verticals intersect $\leq \frac{\varepsilon}{2} OPT$.
⇒ $(a, b)$ works.
Discrete packing outlook

- Extends to unit balls in higher dimensions: $n^{O(1/\varepsilon^{d-1})}$

- $n^{O(1/\varepsilon)}$ is essentially tight in $\mathbb{R}^2$ (Marx 2007)

- Local search: slower PTAS for “pseudodisks” (last lecture?)
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But! major open problem:

Is there a PTAS for Independent set of axis-parallel rectangles?

or for axis parallel segments?
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Best known: $n^{O((\log \log n/\varepsilon)^4)}$ (Chuzhoy–Ene 2016)
Theorem. There is a PTAS for the continuous covering of points with unit disks with running time $n^{O(1/\varepsilon)}$. 
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Proof. A unit disk is canonical if it has 2 input points on its boundary, or its topmost point is an input point.
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Proof. A unit disk is canonical if it has 2 input points on its boundary, or its topmost point is an input point.

There is a cover of size $k \iff$ there is a canonical cover of size $k$.

2 disks per point pair $p, p' \in P$, one disk for each $p \in P$

$2\left(\binom{n}{2}\right) + n \leq n^2$ canonical disks
Shifting for set cover with unit disks

Grid of side length 2, set $t = \lceil 6/\varepsilon \rceil$

Cell disks: canonical disks inside and those intersecting the boundary
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Whole cell can be covered by $O(1/\varepsilon^2)$ (non-canoncical) disks.

$\Rightarrow$ Min cover in a cell solved in $(n^2)^{O(\sqrt{1/\varepsilon^2})} = n^{O(1/\varepsilon)}$

In C, solution $|S(C)| \leq |OPT(C)|$. Return $U := \bigcup_C S(C)$
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For some shift \( a \) blue intersects \( \leq |OPT|/t \) disks.

\[ \Rightarrow \exists (a, b) \text{ intersecting } 2|OPT|/t \leq \varepsilon|OPT|/3 \text{ disks.} \]

Each disk of OPT counted in \( \leq 4 \) cells.

\[ |U| \leq \sum_C |OPT(C)| \leq |OPT| + 3\varepsilon|OPT|/3 = (1 + \varepsilon)|OPT| \]