Local Search for Hitting Set and Set Cover

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Computational Geometry
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Overview

- $r$-divisions in planar graphs
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- Hitting set and set cover via local search
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- The locality condition
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- r-divisions in planar graphs
- Hitting set and set cover via local search
- The locality condition
- Locality condition for halfspaces
Observation. Given planar graph $G = (V, E)$ and a vertex set $W \subset V$, $G$ has separator of size $O(\sqrt{n})$ s.t. each side has $\leq \frac{36}{37}|W|$ vertices from $W$.

Proof. Start proof with smallest square that encloses $\geq \frac{|W|}{37}$ disks from $W$. 
Theorem (Frederickson 1987) For any $r \in \mathbb{Z}_+$ and planar graph $G$, there are $O(n/r)$ vertex sets $V_1, V_2, \ldots$ satisfying

- every edge is induced by some $V_i$
- $|V_i| \leq r$
- small boundaries: $\partial V_i = V_i \cap (\bigcup_{j \neq i} V_j)$, $|\partial V_i| = O(\sqrt{r})$
- small total boundary set: $\sum_i |\partial V_i| = O(n/\sqrt{r})$
Computing an \( r \)-division

*Proof sketch.* Use planar separator theorem.

Recursively divide until size \( \leq r \)

\[ X := \text{union of separators throughout.} \]

\( V_i \): final group + neighborhood

- group size ✓
- group number ✓
- edge covering ✓
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Still need $\partial W := W \cap X$ is small!

Idea: if $\partial W > c \sqrt{|W|}$, separate $W$ with balance wrt. $\partial W$. 
Hitting set via local search
(Mustafa–Ray; Chan–Har-Peled 2008)
Hitting set for halfspaces

Hitting set

Given a set $P \subset \mathbb{R}^d$ of points and a set $\mathcal{D} \subset 2^{\mathbb{R}^d}$ of ranges, find minimum size $Q \subset P$ such that all ranges are “hit”: for any $D \in \mathcal{D}$, $D \cap Q \neq \emptyset$. 
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E.g.: hitting disks, hitting triangles, hitting halfspaces in \( \mathbb{R}^3 \)
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E.g.: hitting disks, hitting triangles, hitting halfspaces in \( \mathbb{R}^3 \)

APX-hard even for fat triangles

For each disk \( D \in D \), take ball \( B \) touching \( v \) and \( B \cap H = D \)

Inversion with center \( v \) maps each ball to halfspace.

Point-disk containment is preserved
Local search for hitting set / set cover
Dualized hitting set: find minimum set of halfspaces to hit all points = Geometric Set Cover!
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Local search:
Given a feasible hitting set $Q$, a valid local search step removes $k$ elements of $Q$ and adds $k - 1$ other elements so that the result is still a feasible hitting set.

A feasible hitting set $Q$ is $k$-locally optimal if there are no valid local search steps.
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$n = |P|$, $m = |Q|$

Running time of $k$-local search is $O(n^{2k+1}m)$ (or better)
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**Theorem (Mustafa–Ray 2010).** There is a $c > 0$ such that the $(c/\varepsilon^2)$-locally optimal hitting set for halfspaces in $\mathbb{R}^3$ is a $(1 + \varepsilon)$-approximation of the minimum hitting set.
The locality condition
Definition. A range space $(P, \mathcal{D})$ has the locality condition if for any pair of disjoint sets $R, B \subset P$ there is a planar bipartite graph $G$ between $R$ and $B$ s.t. for any $D \in \mathcal{D}$ intersecting both $R$ and $B$ we have some $uv \in E(G)$ with $u \in D \cap R$ and $v \in D \cap R$. 
**Locality condition**

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**Example:** disks in the plane  
\(G\): subgraph of Delaunay triangulation of \(P' = R \cup B\)
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Claim. For any disk \(D \subset \mathbb{R}^2\), \(DT(P')|_{P' \cap D}\) is connected.

For \(u \in D \cap R\) and \(v \in D \cap B\), there is a connecting path in \(DT(D \cap P')\), which contains red-blue edge.
Locality implies larger neighborhoods

**Theorem.** $(P, D)$ is range space satisfying locality condition, $R$ is optimal hitting set, $B$ is $k$-locally optimal, and $R \cap B = \emptyset$. Then there is planar $G = (R, B, E)$ s.t. for all $B' \subset B$ with $|B'| \leq k$, we have large neighborhood: $|N(B')| \geq |B'|$
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**Proof.** $B$ and $R$ are both hitting sets

$\Rightarrow$ every range has $\geq 1$ pt from both

If $B' \subset B$, then $(B \setminus B') \cup N(B')$ is hitting:

if only $B'$ hits $D$ from $B$, then some $b \in B'$ has red neighbor hitting $D$,
otherwise $B \setminus B'$ hits $D$. 
**Theorem.** \((P, D)\) is range space satisfying locality condition, \(R\) is optimal hitting set, \(B\) is \(k\)-locally optimal, and \(R \cap B = \emptyset\). Then there is planar \(G = (R, B, E)\) s.t. for all \(B' \subset B\) with \(|B'| \leq k\), we have large neighborhood: \(|N(B')| \geq |B'|\)

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But for \(|B'| \leq k\) there is no valid local search step.
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If \(R \cap B = I \neq \emptyset\), then let \(\mathcal{D}' = \text{ranges not hit by } I\).

Use the same on \((P \setminus I, \mathcal{D}')\). If \(B_0\) is \((1 + \epsilon)\)-approx on \((P \setminus I, \mathcal{D}')\) \(\rightarrow B_0 \cup I\) is \((1 + \epsilon)\)-approx on \((P, \mathcal{D})\)
B has large neighborhoods only if relatively small

**Theorem.** Let \( G = (R, B, E) \) bipartite planar, s.t. for every \( B' \subset B \) of size \(|B'| \leq k\), \( |N(B')| \geq |B'| \).
Then \(|B| \leq (1 + c/\sqrt{k})|R|\) for some constant \( c \).
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Proof. \( r := |R|, \ b := |B|, \)
Use \( k \)-division of \( G \). \( \rightarrow V_1, V_2, \ldots \)
\( V_i \) has boundary \( V_i \cap (\bigcup_{j \neq i} V_j) \) and interior \( V_i \setminus (\bigcup_{j \neq i} V_j) \).
\( r_i^{\partial}, b_i^{\partial}, r_i^{\text{int}}, b_i^{\text{int}} : \# \text{ red/blue in } V_i \text{ in boundary and interior.} \)
Theorem. Let $G = (R, B, E)$ bipartite planar, s.t.
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$V_i$ has boundary $V_i \cap (\bigcup_{j \neq i} V_j)$ and interior $V_i \setminus (\bigcup_{j \neq i} V_j)$.
$r^\partial_i$, $b^\partial_i$, $r^{int}_i$, $b^{int}_i$ : $\#$ red/blue in $V_i$ in boundary and interior.

- $\sum_i (r^\partial_i + b^\partial_i) \leq \gamma (r + b)/\sqrt{k}$ ($\gamma = \text{const}$) (by $k$-division)
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Use $k$-division of $G$. $\rightarrow$ $V_1, V_2, \ldots$
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- $\sum_i (r_i^{\partial} + b_i^{\partial}) \leq \gamma (r + b)/\sqrt{k}$ \hspace{1cm} ($\gamma = \text{const}$) (by $k$-division)
- $b_i^{\text{int}} \leq r_i^{\text{int}} + r_i^{\partial}$ \hspace{1cm} ($b_i^{\text{int}} \leq k$ so it has large neighborhood)

\[
b \leq \sum_i (b_i^{\text{int}} + b_i^{\partial}) \leq \sum_i r_i^{\text{int}} + \sum_i (r_i^{\partial} + b_i^{\partial}) \leq r + \gamma (r+b)/\sqrt{k}
\]
Locality condition wrap-up

\[ b \leq r + \frac{\gamma (r + b)}{\sqrt{k}} \]

If \( k \geq 4\gamma^2 \), then

\[ b \leq r \frac{1 + \gamma/\sqrt{k}}{1 - \gamma/\sqrt{k}} \leq \ldots \leq r(1 + c/\sqrt{k}) \]
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**Theorem.** Locality condition implies PTAS for hitting set with running time \( n^{O(1/\varepsilon^2)} \).

**Theorem.** Hitting disks with points in \( \mathbb{R}^2 \) has a PTAS with running time \( n^{O(1/\varepsilon^2)} \).
Locality condition for half-spaces
Radon’s theorem

**Theorem (Radon, 1921)** Any set of \(d + 2\) points in \(\mathbb{R}^d\) can be partitioned into two subsets whose convex hulls intersect.
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**Theorem (Radon, 1921)** Any set of $d + 2$ points in $\mathbb{R}^d$ can be partitioned into two subsets whose convex hulls intersect.

**Proof.** Let $P = \{p_1, \ldots, p_{d+2}\}$. There exists $\lambda_1, \ldots, \lambda_{d+2}$ not all 0 s.t.

$$ \sum_{i=1}^{d+2} \lambda_i p_i = 0 \quad \text{and} \quad \sum_{i=1}^{d+2} \lambda_i = 0. $$

Let $I$: indices $i$ where $\lambda_i > 0$. (denote remaining indices by $J$) Then $\sum_{i \in I} \lambda_i = -\sum_{j \in J} \lambda_j =: \mu$, thus

$$ p' := \sum_{i \in I} \frac{\lambda_i}{\mu} p_i = \sum_{j \in J} \frac{-\lambda_j}{\mu} p_j \in \text{conv}(P|_I) \cap \text{conv}(P|_J) $$
Locality for half-spaces: graph and embedding

Recall: \( R \) and \( B \) disjoint hitting sets for a set \( \mathcal{D} \) of half-spaces. Need bipartite planar graph \( G \) on \( R \cup B \), s.t. for any \( D \in \mathcal{D} \) containing both red and blue, there is an edge induced.
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Guess $o \in P$ from hitting set, remove $D \in \mathcal{D}$ that contains $o$. 
$\Rightarrow$ wlog. $o$ outside $\bigcup_{D \in \mathcal{D}} D$
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Two stages:

- Add all red-blue edges of $C := \partial \text{conv}(R \cup B)$ to $G$, triangulate faces of $C$
- For $p \in (R \cup B) \setminus C$, let $p'$ be point where ray($o, p$) exits $C$ Define edges of $p$ via $p'$ in a triangle of $C$. ⇒ results in planar graph on $C$
Defining $G$ in a bichromatic triangle

$T$ is a triangle of $C$.

If $T$ has 1 red and 2 blue corners (1 blue 2 red symmetric)
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Connect each $p'$ to all differently colored corners
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$B_T :=$ blue pts mapped to $T$

$\leq 1$ pt in $B_T$ connected to $c_1, c_2, c_3$; rest connected to two corners
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How to select neighboring corners for each $b \in B_T$?

If there is a corner $c$ s.t. there is no halfspace $h \subset \mathbb{R}^3$ containing only $b, c$ among $B_T \cup \{c_1, c_2, c_3, o\}$, then connect $b'$ to other two corners
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Claim. For all but one $b \in B_T$ there is such a corner.
Corner connections via Radon’s thm

**Claim.** For all but one $b \in B_T$ there is a corner $c$ s.t. there is no halfspace containing just $b, c$ among $B_T \cup \{o, c_1, c_2, c_3\}$.

*Proof.* by contradiction:
assume $b_1, b_2 \in B_T$ have no good corner.
There are halfspaces containing exactly
$b_i c_j$ ($i = 1, 2; j = 1, 2, 3$) among $F := \{b_1, b_2, c_1, c_2, c_3\}$
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$F$ in convex position. Radon thm gives 2:3 partition of $F$.

There is plane separating $b_1, b_2$ from corners

$\Rightarrow$ wlog. $\text{conv}(b_1, c_1) \cap \text{conv}(b_2, c_2, c_3) \neq \emptyset$. 
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$\Rightarrow$ wlog. $\text{conv}(b_1, c_1) \cap \text{conv}(b_2, c_2, c_3) \neq \emptyset$.

$\Rightarrow$ there is no halfspace containing exactly $b_1, c_1$
Lemma. Halfspaces have the locality property, and it is witnessed by $G$.

$G$ is planar bipartite. ✓

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• If \( D \in \mathcal{D} \) contains red and blue from \( C \), then there is bichromatic triangle ✓

• Let \( D' \subset D \) be halfspace parallel to \( \partial D \), smallest that contains both red and blue.
  \( D' \) has 1 blue \( b \) on its boundary.
  \( o \not\in D \Rightarrow o \not\in D' \)
  \( D' \) has \( \geq 1 \) corner \( c \) of the triangle of \( b' \).
Lemma. Halfspaces have the locality property, and it is witnessed by $G$.

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  $D'$ has 1 blue $b$ on its boundary.
  $o \notin D \Rightarrow o \notin D'$
  $\Rightarrow D'$ has $\geq 1$ corner $c$ of the triangle of $b'$.
  If $D'$ contains $c, c'$, then at least one connects to $b$.
  If $D'$ contains just $c$, then $bc \in E(G)$ by def of $G$. 

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$G$ construction correctness
Theorem. (Mustafa–Ray 2010) Hitting halfspaces with points in $\mathbb{R}^3$ has a PTAS with running time $n^{O(1/\varepsilon^2)}$. 

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- APX-hard in $\mathbb{R}^{\geq 4}$
- Locality condition can be proved for several object types in $\mathbb{R}^2$, for hitting/covering/packing
  Most general: hitting/covering/packing non-piercing objects
- Analysis is tight: $k = o(1/\varepsilon^2)$ local search doesn’t work
- General lower bounds of $n^{\Omega(1/\varepsilon)}$
Halfspace wrap-up

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Exact has matching lower bound.

**Theorem.** Hitting set of size $k$ for halfspaces with points in $\mathbb{R}^3$ can be computed in time $n^{O(\sqrt{k})}$.

In $\mathbb{R}^{\geq 4}$, there is $n^{\Omega(k)}$ lower bound.