# Local Search for Hitting Set and Set Cover

*Sándor Kisfaludi-Bak*

Computaional Geometry
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max planck institut
informatik

# Overview

- r-divisions in planar graphs

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- Locality condition for halfspaces

# Balanced separator for a subset

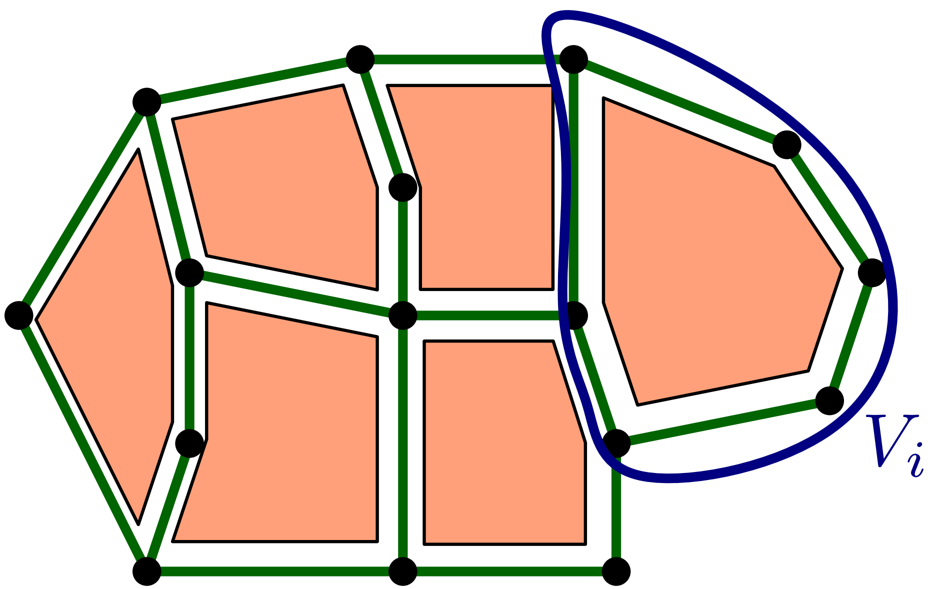
**Observation.** Given planar graph $G = (V, E)$ and a vertex set $W \subset V$, $G$ has separator of size $O(\sqrt{n})$ s.t. each side has $\leq 36/37|W|$ vertices from $W$.

*Proof.*
Start proof with smallest square that encloses $\geq |W|/37$ disks from $W$.

# $r$-divisions

**Theorem (Frederickson 1987)** For any $r \in \mathbb{Z}_+$ and planar graph $G$, there are $O(n/r)$ vertex sets $V_1, V_2, \ldots$ satisfying

- every edge is induced by some $V_i$
- $|V_i| \le r$
- small boundaries: $\partial V_i = V_i \cap (\bigcup_{j \ne i} V_j)$, $|\partial V_i| = O(\sqrt{r})$
- small total boundary set: $\sum_i |\partial V_i| = O(n/\sqrt{r})$



# Computing an $r$-division

*Proof sketch.* Use planar separator theorem.

Recursively divide until size $\leq r$

$X :=$ union of separators throughout.

$V_i$: final group+neighborhood

group size ✓group number ✓edge covering ✓

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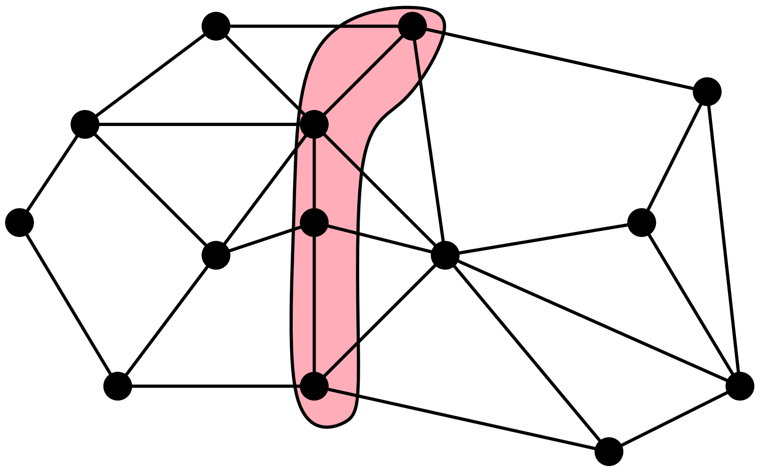

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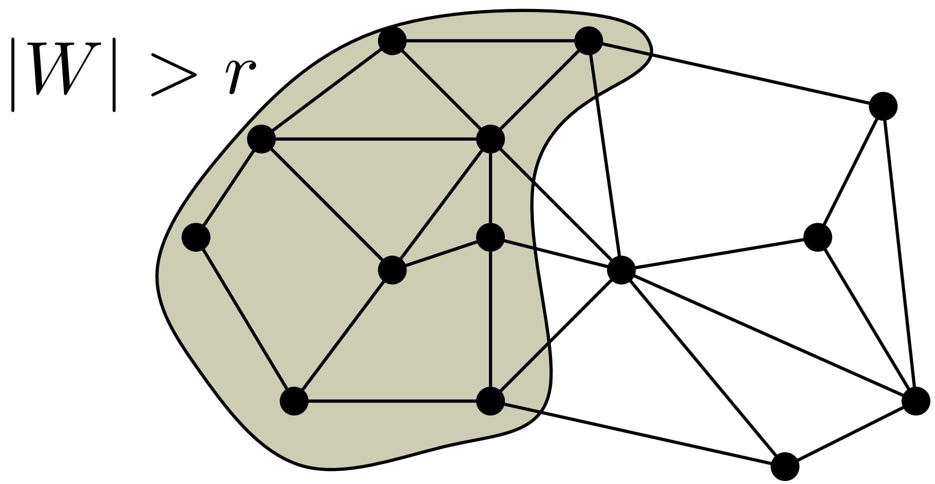

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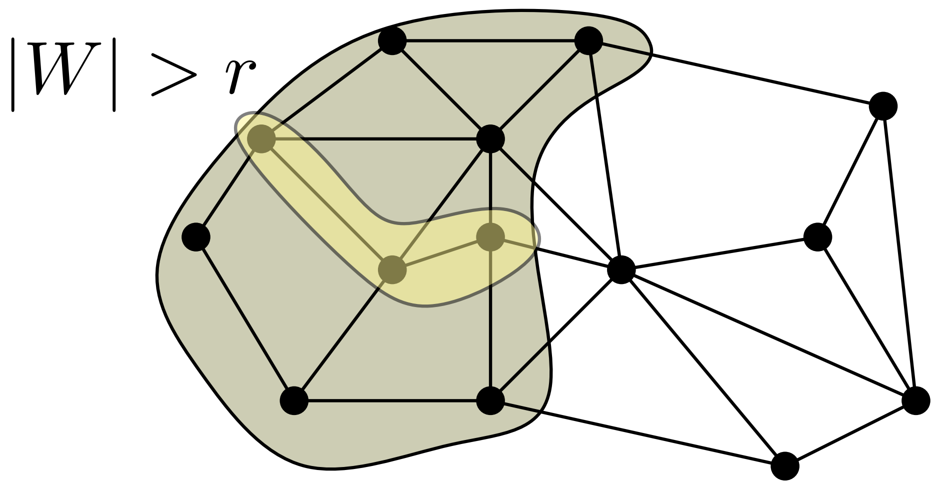

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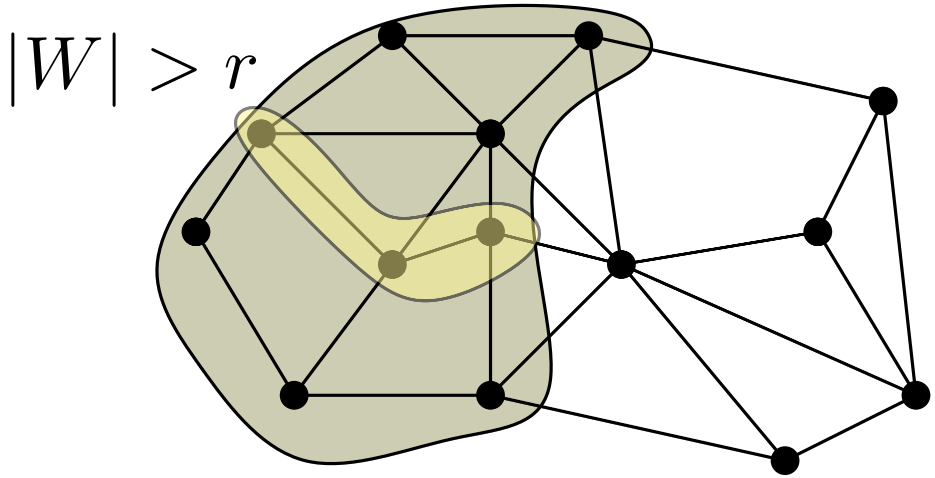


Still need $\partial W := W \cap X$ is small!

Idea: if $\partial W > c\sqrt{|W|}$, separate $W$ with balance wrt. $\partial W$.

# Hitting set via local search
# (Mustafa–Ray; Chan–Har-Peled 2008)

# Hitting set for halfspaces

## Hitting set

Given a set $P \subset \mathbb{R}^d$ of points and a set $\mathcal{D} \subset 2^{\mathbb{R}^d}$ of ranges, find minimum size $Q \subset P$ such that all ranges are "hit": for any $D \in \mathcal{D}$, $D \cap Q \neq \emptyset$.

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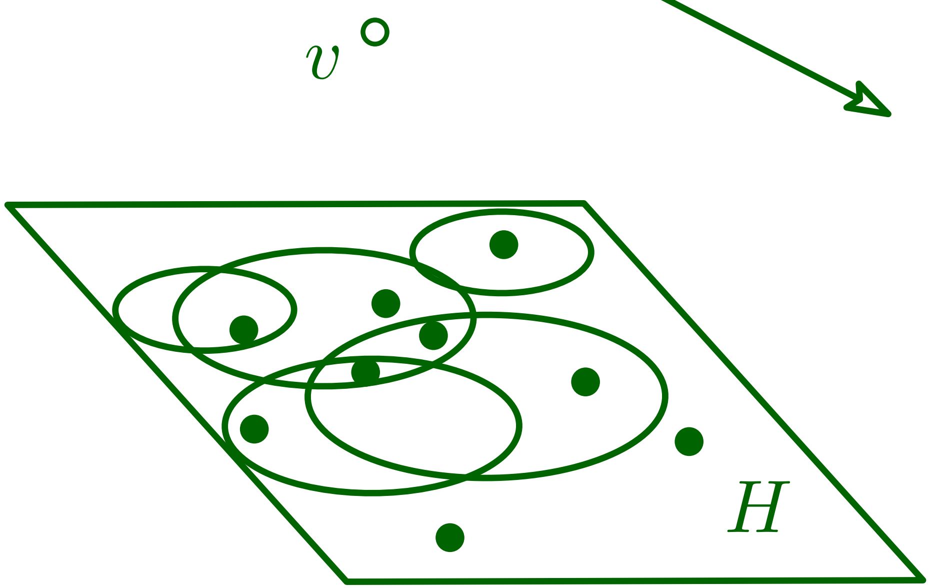
APX-hard even for fat triangles

E.g.: hitting disks, hitting triangles, hitting halfspaces in $\mathbb{R}^3$




For each disk $D \in \mathcal{D}$, take ball $B$ touching $v$ and $B \cap H = D$

Inversion with center $v$ maps each ball to halfspace.

Point-disk containment is preserved

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Running time of $k$-local search is $O(n^{2k+1}m)$ (or better)

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**Theorem (Mustafa–Ray 2010).** There is a $c > 0$ such that the $(c/\varepsilon^2)$-locally optimal hitting set for halfspaces in $\mathbb{R}^3$ is a $(1+\varepsilon)$-approximation of the minimum hitting set.

# The locality condition

# Locality condition

**Definition.** A range space $(P, \mathcal{D})$ has the locality condition if for any pair of disjoint sets $R, B \subset P$ there is a planar bipartite graph $G$ bewteen $R$ and $B$ s.t. for any $D \in \mathcal{D}$ intersecting both $R$ and $B$ we have some $uv \in E(G)$ with $u \in D \cap R$ and $v \in D \cap R$.

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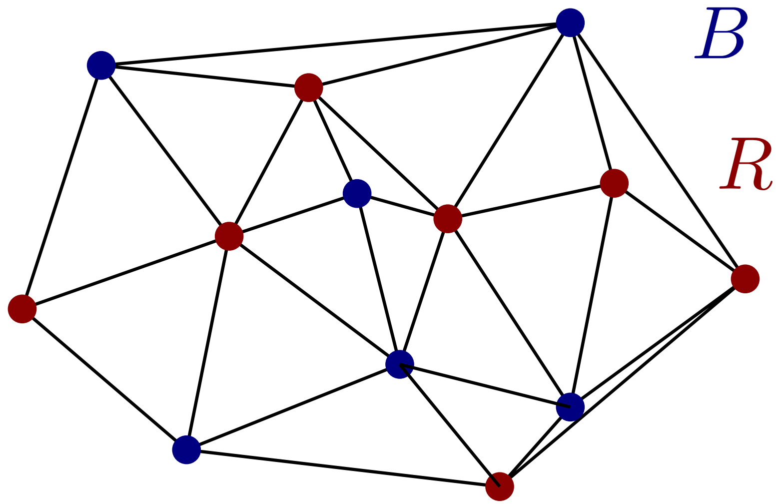
Example: disks in the plane

$G$: subgraph of Delaunay triangulation of $P' = R \cup B$



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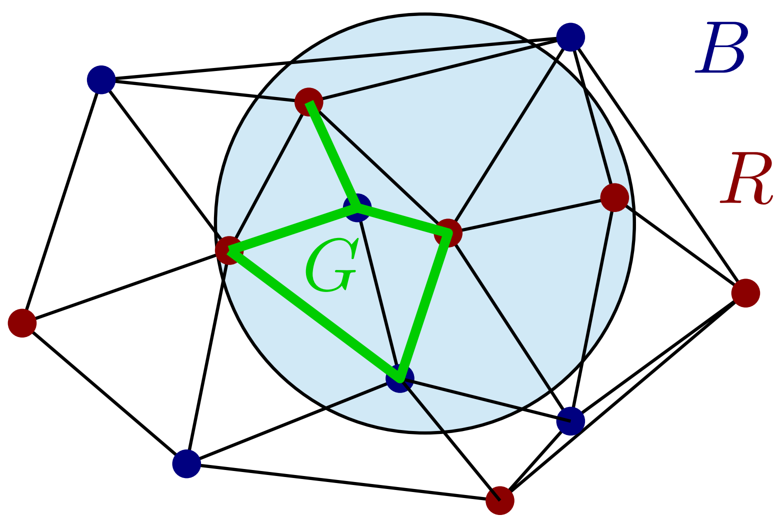


**Claim.** For any disk $D \subset \mathbb{R}^2$, $DT(P')|_{P' \cap D}$ is connected.

$\Downarrow$

For $u \in D \cap R$ and $v \in D \cap B$, there is a conencting path in DT($D \cap P'$), which contains red-blue edge

# Locality implies larger neighborhoods

**Theorem.** $(P, \mathcal{D})$ is range space satisfying locality condition, $R$ is optimal hitting set, $B$ is $k$-locally optimal, and $R \cap B = \emptyset$. Then there is planar $G = (R, B, E)$ s.t. for all $B' \subset B$ with $|B'| \leq k$, we have large neighborhood: $|N(B')| \geq |B'|$

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*Proof.* $B$ and $R$ are both hitting sets

$\Rightarrow$ every range has $\geq 1$ pt from both

If $B' \subset B$, then $(B \setminus B') \cup N(B')$ is hitting:

if only $B'$ hits $D$ from $B$, then some $b \in B'$ has red neighbor hitting $D$,

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If $R \cap B = I \neq \emptyset$, then let $\mathcal{D}' =$ ranges not hit by $I$. Use the same on $(P \setminus I, \mathcal{D}')$. If $B_0$ is $(1+\varepsilon)$-approx on $(P \setminus I, \mathcal{D}') \rightarrow B_0 \cup I$ is $(1+\varepsilon)$-approx on $(P, \mathcal{D})$

# $B$ has large neighborhoods only if relatively small

**Theorem.** Let $G = (R, B, E)$ bipartite planar, s.t. for every $B' \subset B$ of size $|B'| \leq k$, $|N(B')| \geq |B'|$. Then $|B| \leq (1 + c/\sqrt{k})|R|$ for some constant $c$.

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*Proof.* $r := |R|,\ b := |B|,$
Use $k$-division of $G$. $\to V_1, V_2, \ldots$
$V_i$ has boundary $V_i \cap (\bigcup_{j \neq i} V_j)$ and interior $V_i \setminus (\bigcup_{j \neq i} V_j)$.
$r_i^{\partial}, b_i^{\partial}, r_i^{int}, b_i^{int}$ : # red/blue in $V_i$ in boundary and interior.

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- $b_i^{int} \leq r_i^{int} + r_i^\partial$ $\quad$ ($b_i^{int} \leq k$ so it has large neighborhood)

$$b_i^{int} + b_i^\partial \leq r_i^{int} + r_i^\partial + b_i^\partial$$

$$b \leq \sum_i (b_i^{int} + b_i^\partial) \leq \sum_i r_i^{int} + \sum_i (r_i^\partial + b_i^\partial) \leq r + \gamma(r + b)/\sqrt{k}$$

# Locality condition wrap-up

$$b \le r + \gamma(r+b)/\sqrt{k}$$

If $k \ge 4\gamma^2$, then

$$b \le r\frac{1+\gamma/\sqrt{k}}{1-\gamma/\sqrt{k}} \le \cdots \le r(1+c/\sqrt{k}) \qquad \square$$

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**Theorem.** Locality condition implies PTAS for hitting set with running time $n^{O(1/\varepsilon^2)}$.

**Theorem.** Hitting disks with points in $\mathbb{R}^2$ has a PTAS with running time $n^{O(1/\varepsilon^2)}$.

# Locality condition for half-spaces

# Radon's theorem

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*Proof.* Let $P = \{p_1, \ldots, p_{d+2}\}$.
There exsists $\lambda_1, \ldots, \lambda_{d+2}$ not all $0$ s.t.

$$\sum_{i=1}^{d+2} \lambda_i p_i = 0 \text{ and } \sum_{i=1}^{d+2} \lambda_i = 0.$$

Let $I$: indices $i$ where $\lambda_i > 0$. (denote remaining indices by $J$)
Then $\sum_{i \in I} \lambda_i = -\sum_{j \in J} \lambda_j =: \mu$, thus

$$p' := \sum_{i \in I} \frac{\lambda_i}{\mu} p_i = \sum_{j \in J} \frac{-\lambda_j}{\mu} p_j \in \mathrm{conv}(P|_I) \cap \mathrm{conv}(P|_J)$$

□

# Locality for half-spaces: graph and embedding

Recall: $R$ and $B$ disjoint hitting sets for a set $\mathcal{D}$ of half-spaces. Need bipartite planar graph $G$ on $R \cup B$, s.t. for any $D \in \mathcal{D}$ containing both red and blue, there is an edge induced.

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Two stages:

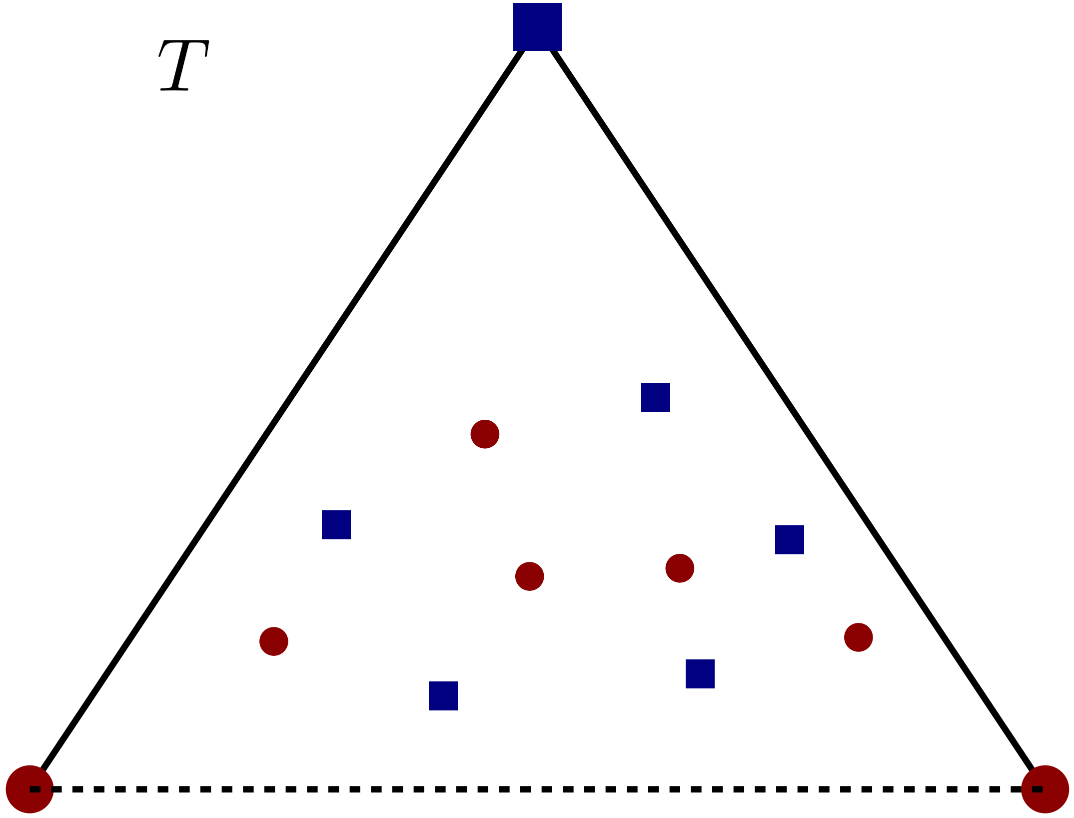
- Add all red-blue edges of $C := \partial \mathrm{conv}(R \cup B)$ to $G$, triangulate faces of $C$
- For $p \in (R \cup B) \setminus C$, let $p'$ be point where $\mathrm{ray}(o, p)$ exits $C$ Define edges of $p$ via $p'$ in a triangle of $C$.

  $\Rightarrow$ results in planar graph on $C$

# Defining $G$ in a bichromatic triangle

$T$ is a triangle of $C$.

If $T$ has 1 red and 2 blue corners (1 blue 2 red symmetric)



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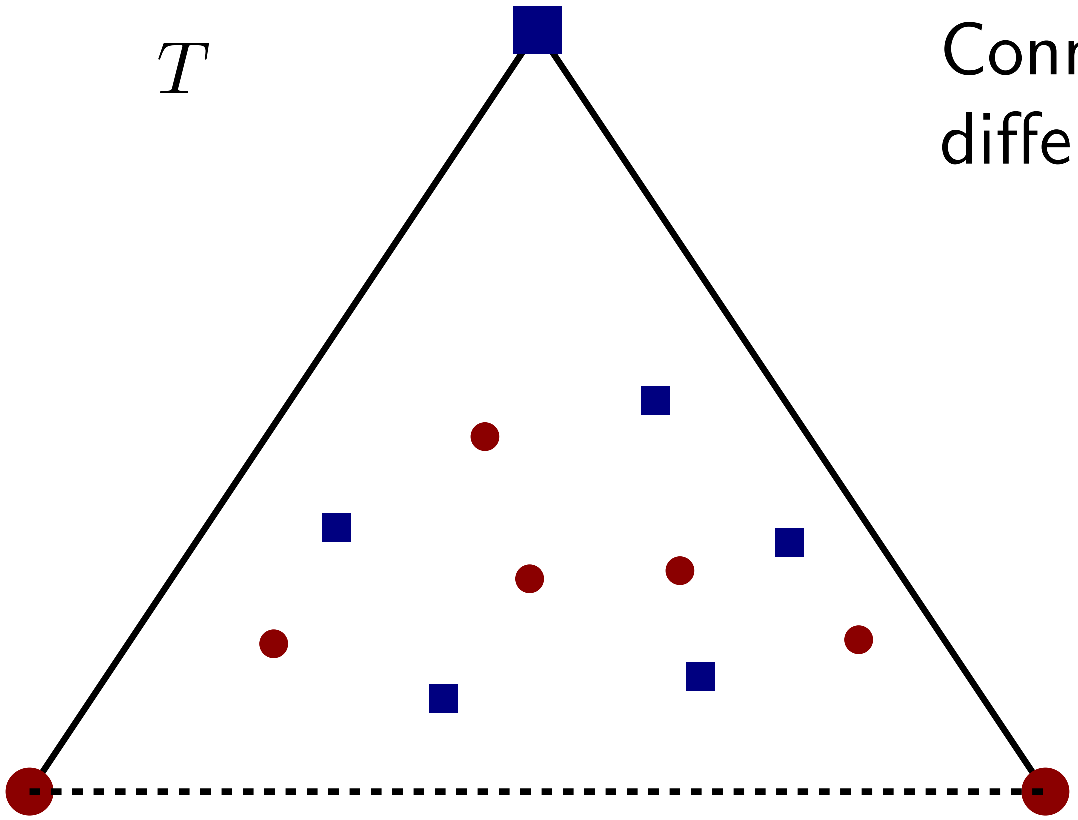


Connect each $p'$ to all differently colored corners

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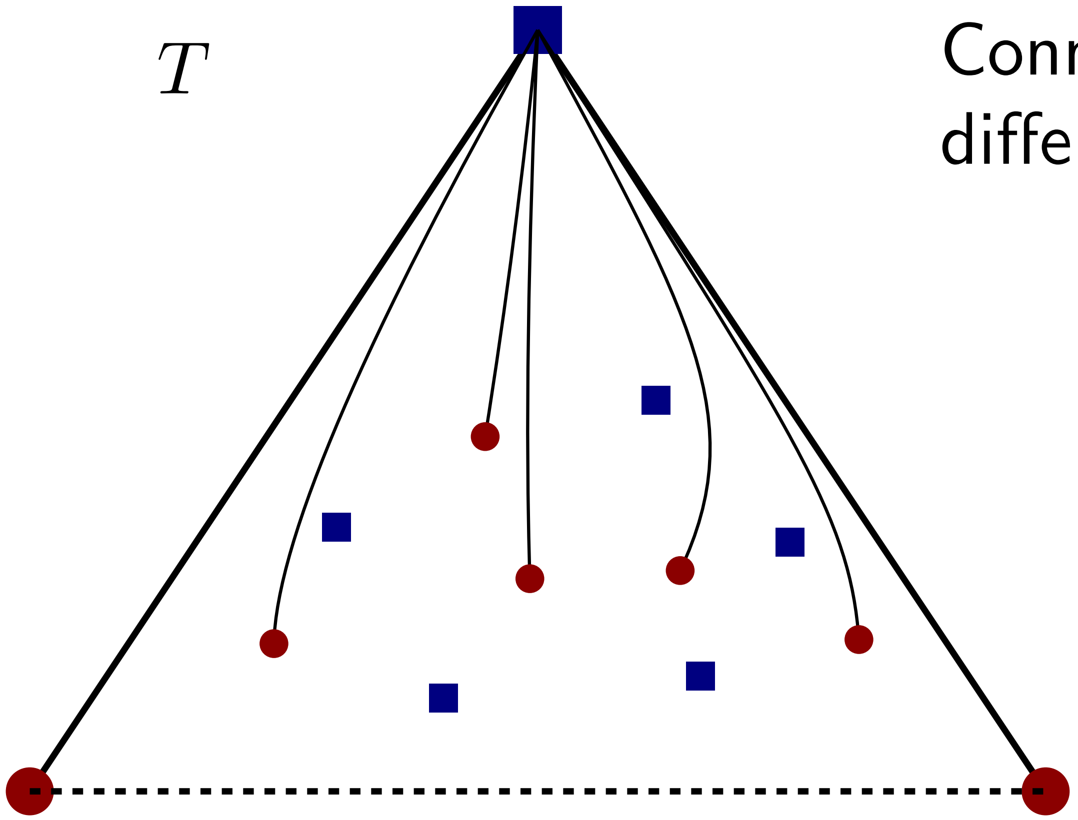


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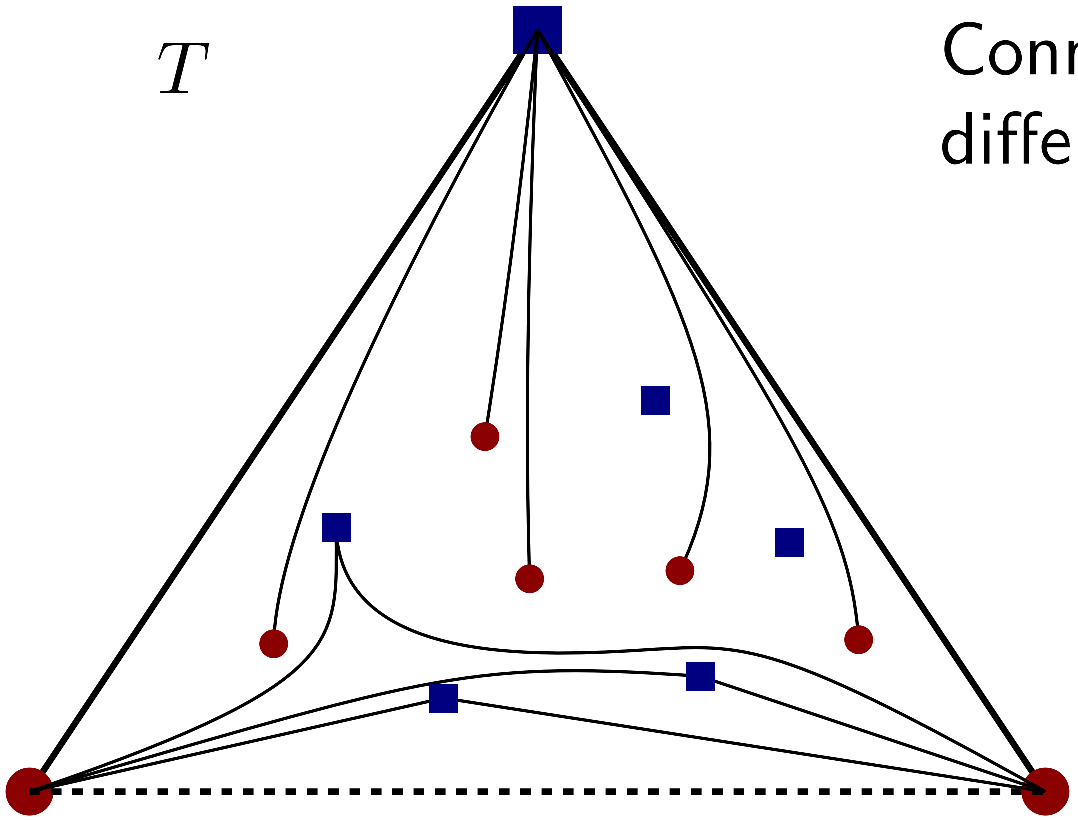


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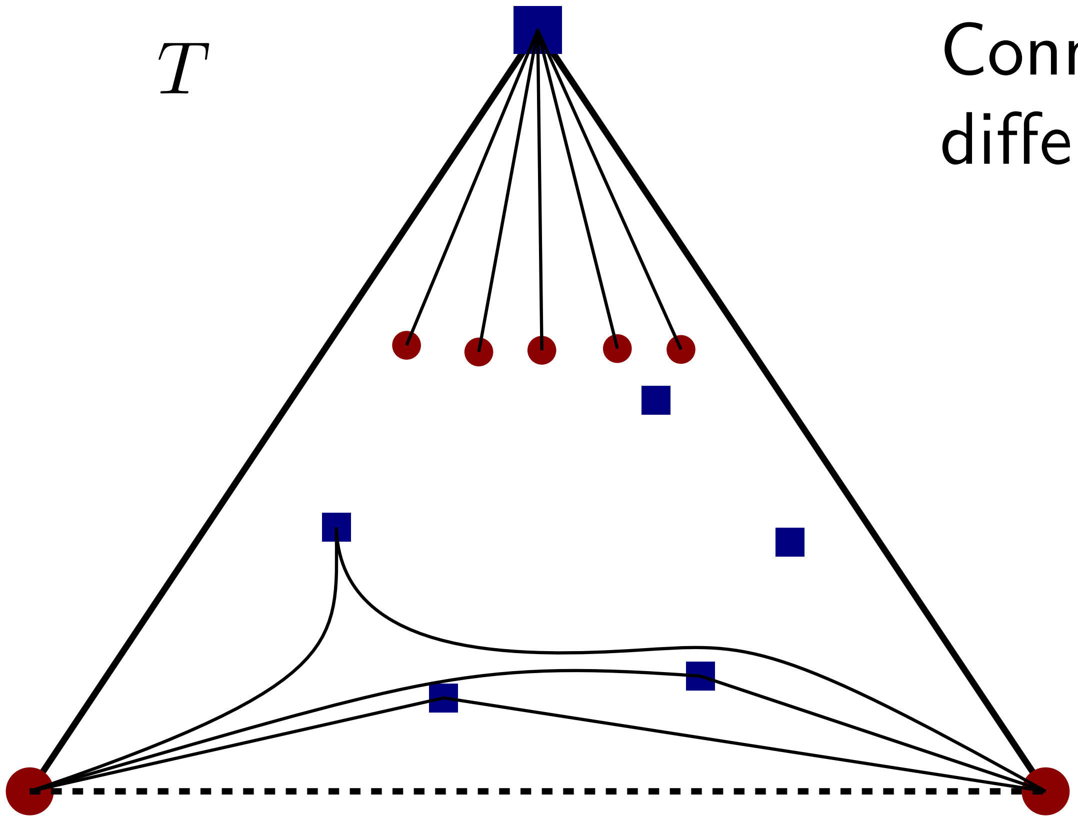


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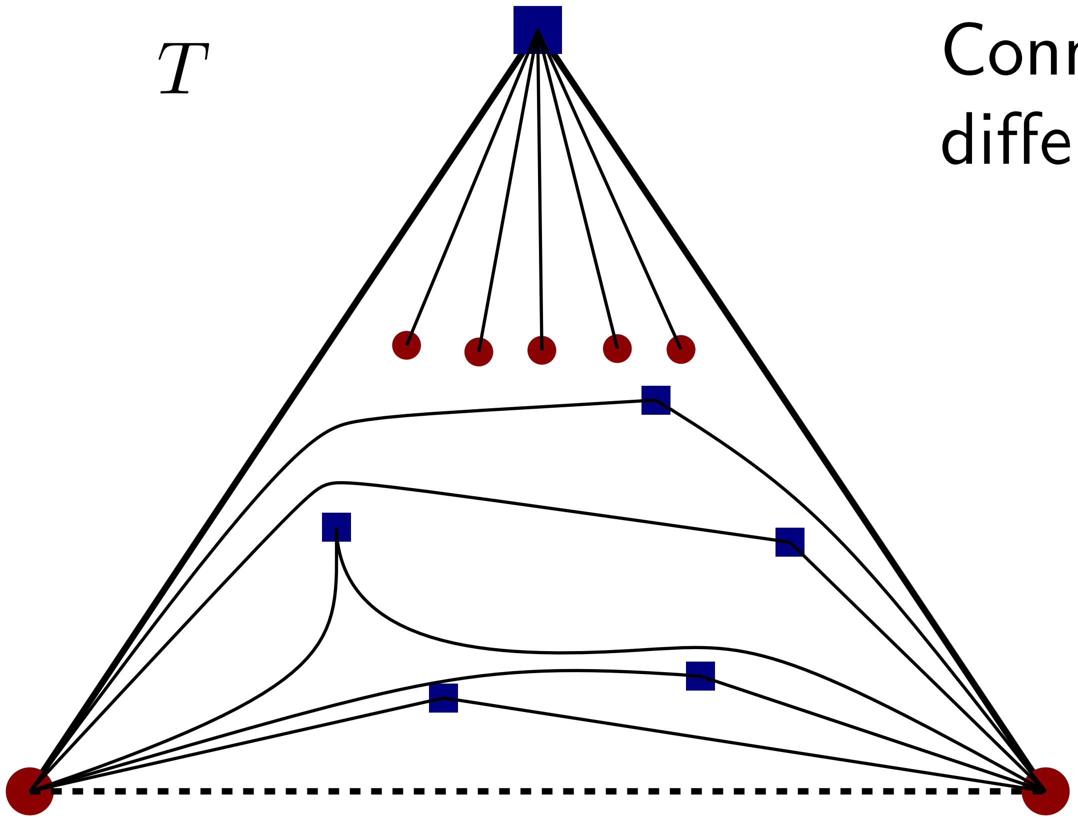


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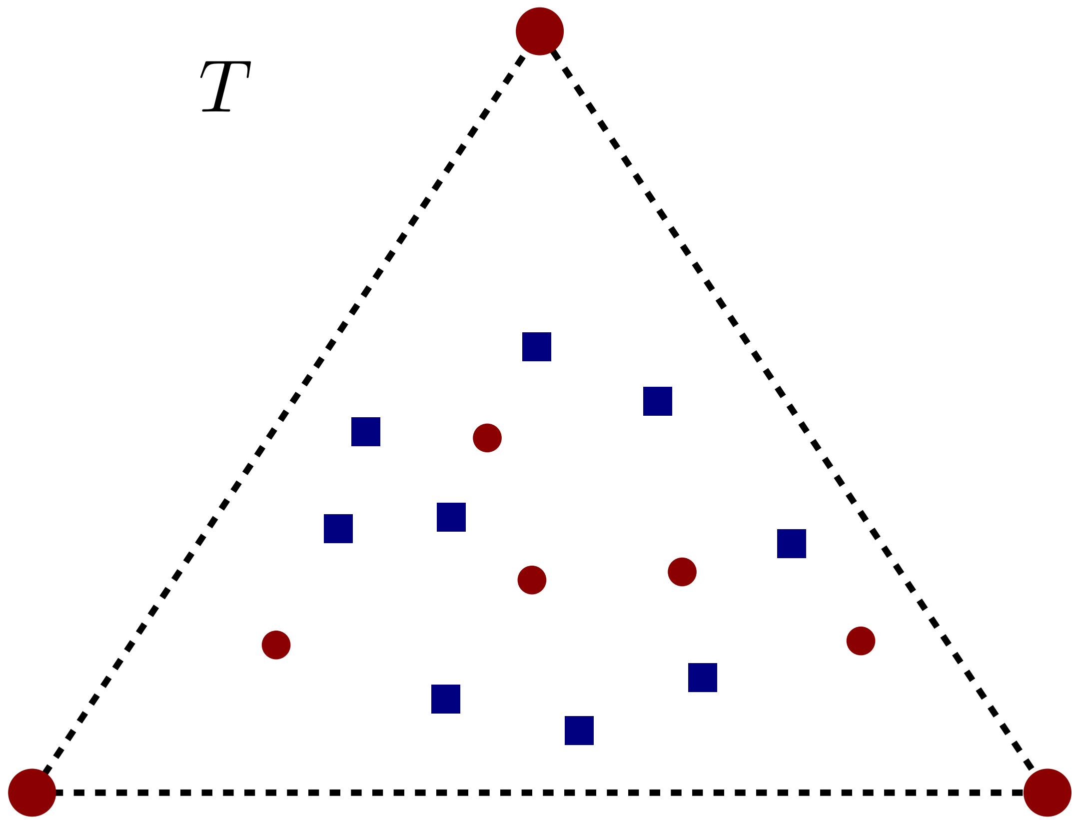

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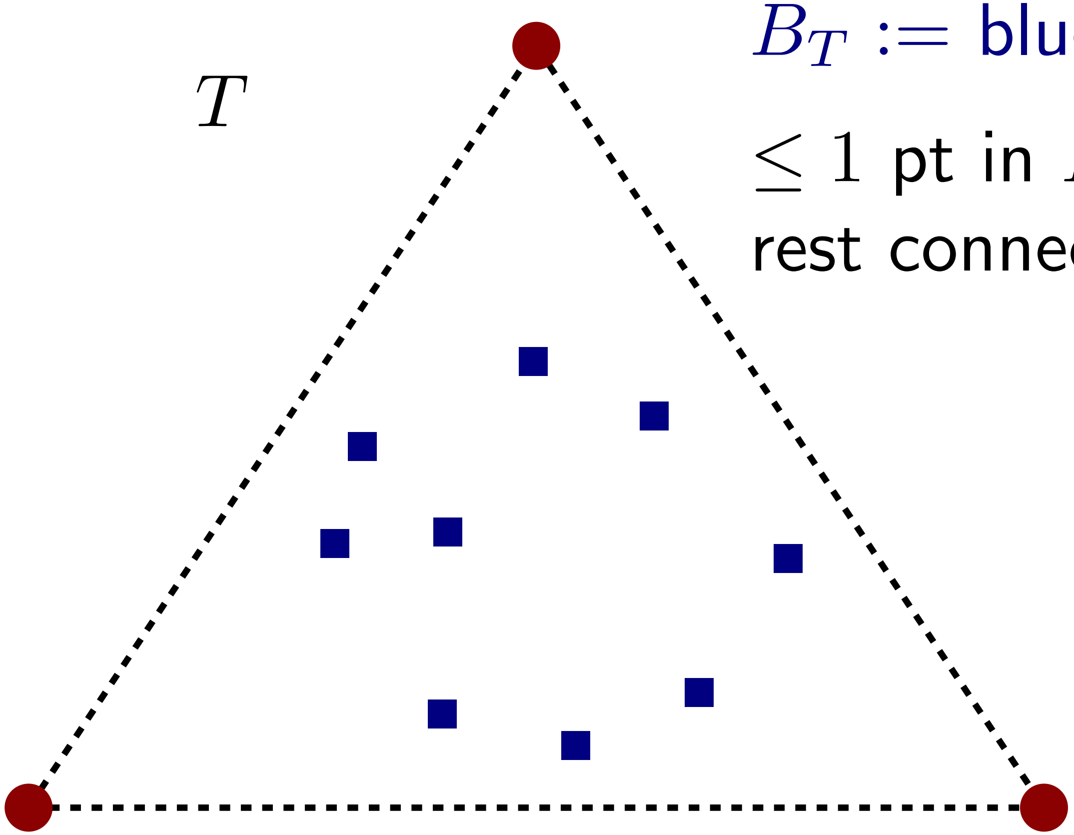


$B_T :=$ blue pts mapped to $T$

$\leq 1$ pt in $B_T$ connected to $c_1, c_2, c_3$;
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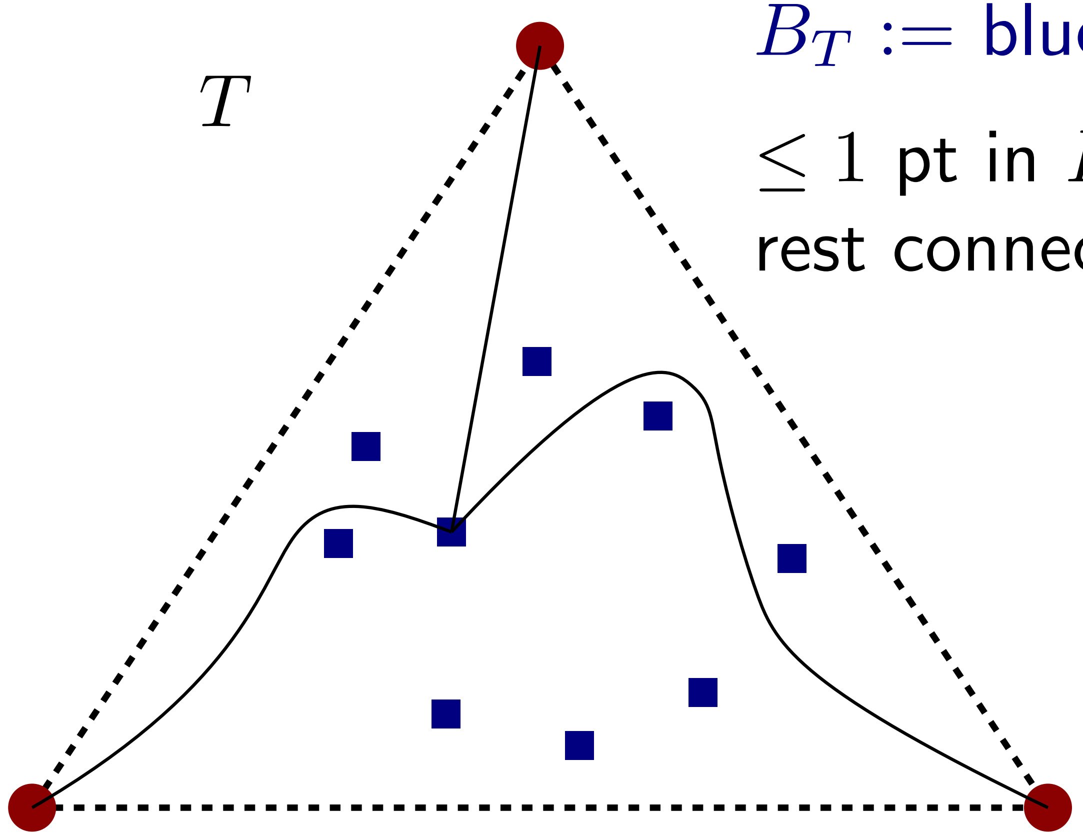


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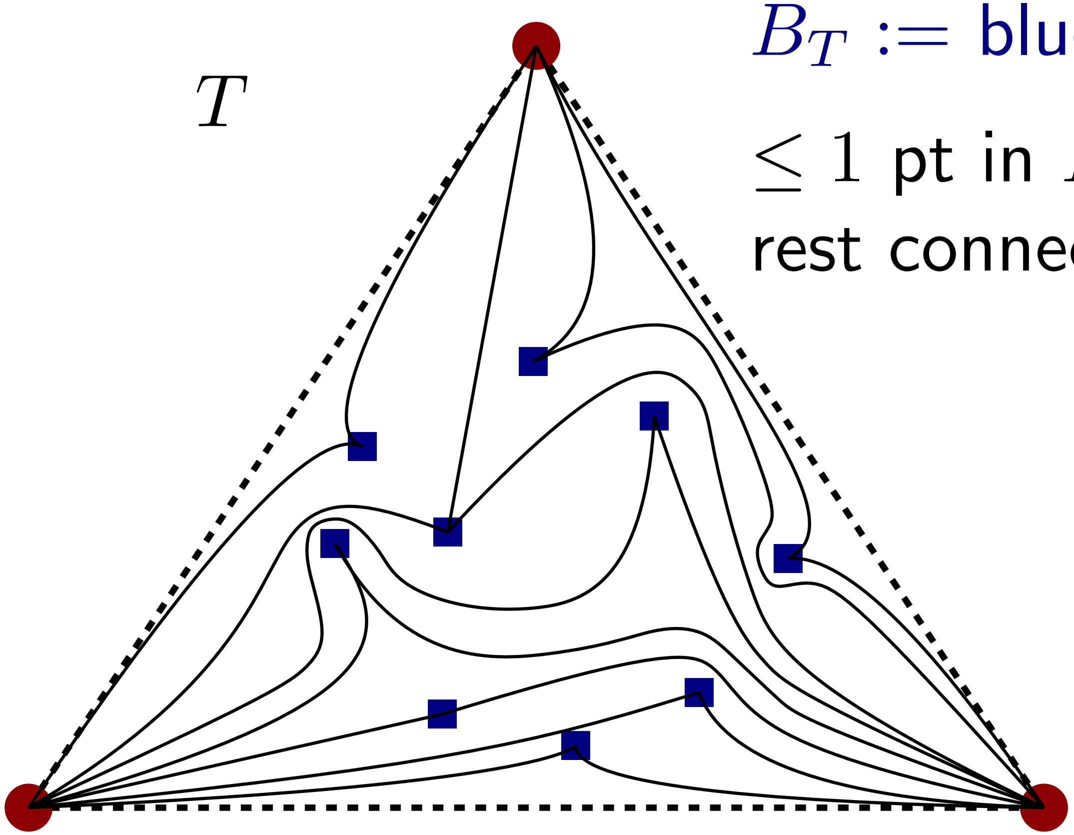


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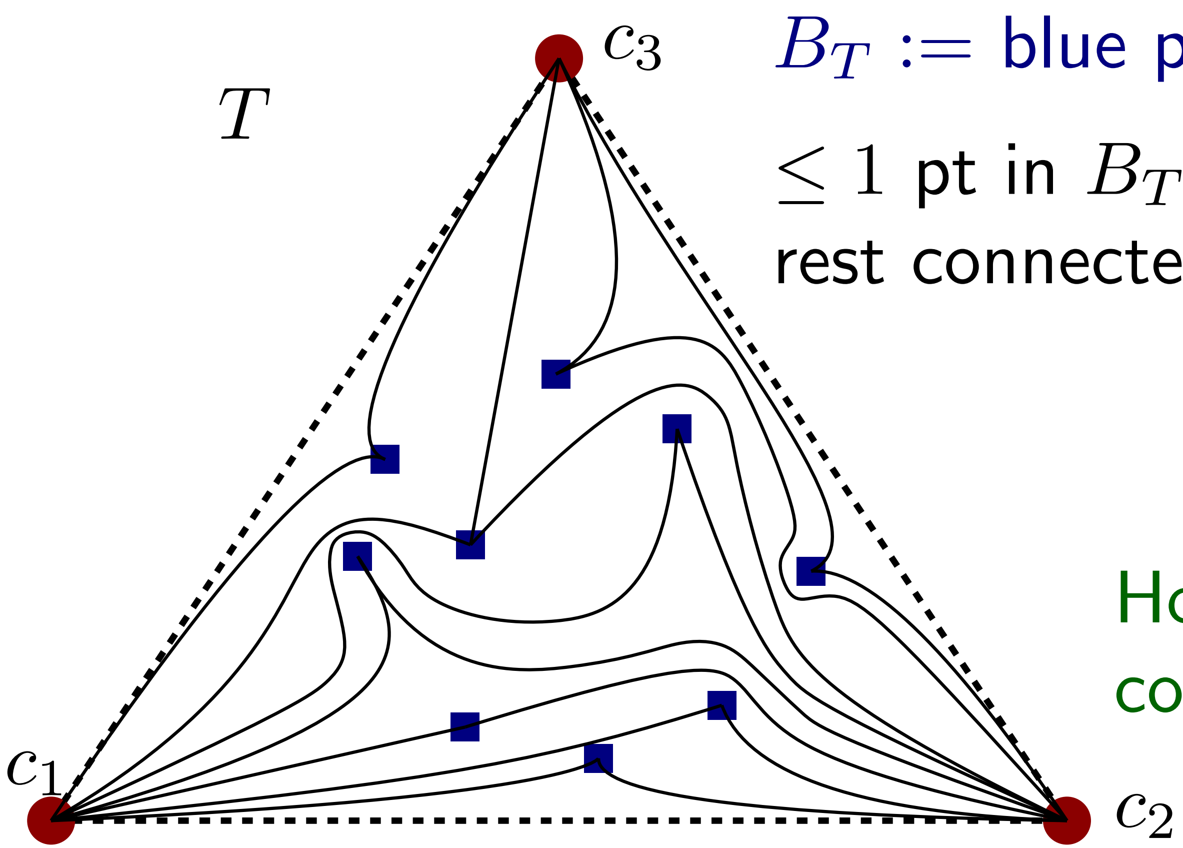


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How to select neighboring corners for each $b \in B_T$?

If there is a corner $c$ s.t. there is no halfspace $h \subset \mathbb{R}^3$ containing only $b, c$ among $B_T \cup \{c_1, c_2, c_3, o\}$, then connect $b'$ to other two corners

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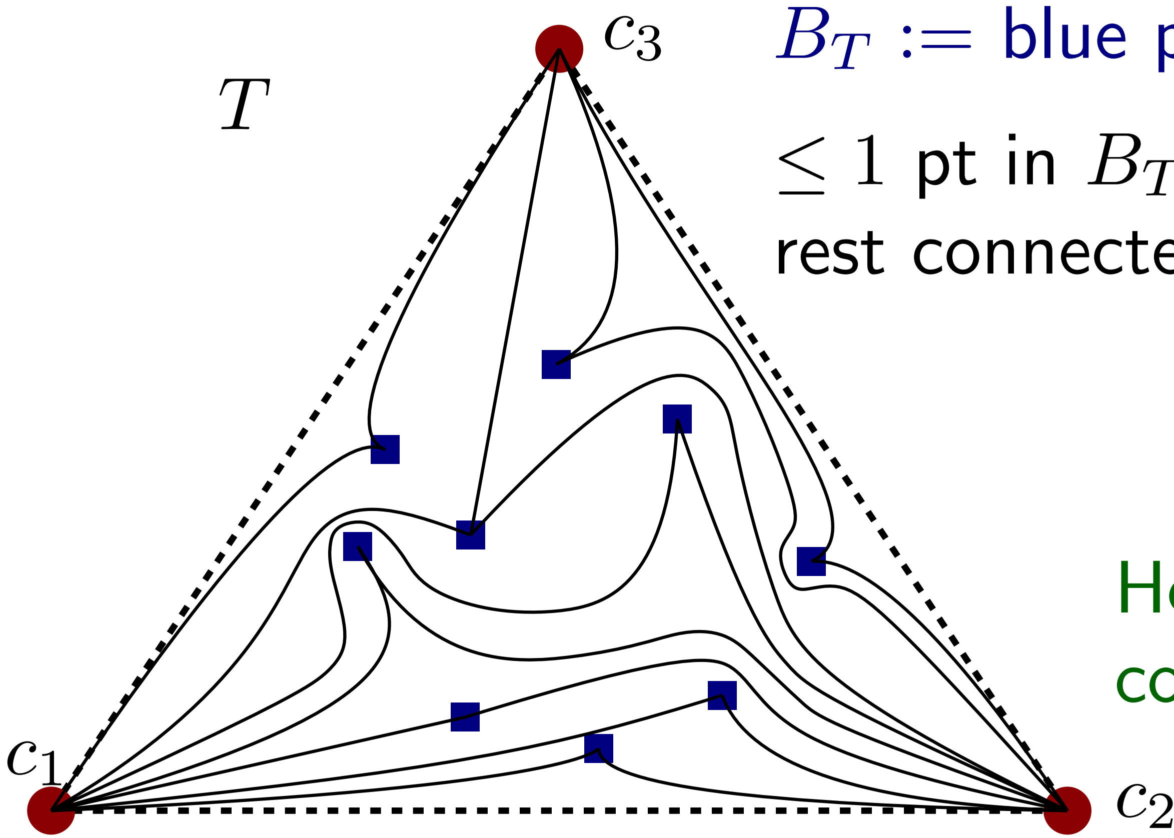


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**Claim.** For all but one $b \in B_T$ there is such a corner.

# Corner connections via Radon's thm

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*Proof.* by contradiction:
assume $b_1, b_2 \in B_T$ have no good corner.
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$\Rightarrow$ there is no halfspace containg exactly $b_1, c_1$ $\lightning$

$\square$

# $G$ construction correctness

**Lemma.** Halfspaces have the locality property, and it is witnessed by $G$.

$G$ is planar bipartite. ✓

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    If $D'$ contains $c, c'$, then at least one connects to $b$.
    If $D'$ contains just $c$, then $bc \in E(G)$ by def of $G$. □

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Exact has matching lower bound.

**Theorem.** Hitting set of size $k$ for halfspaces with points in $\mathbb{R}^3$ can be computed in time $n^{O(\sqrt{k})}$.

In $\mathbb{R}^{\geq 4}$, there is $n^{\Omega(k)}$ lower bound.