Configuration Spaces

ground set S with n Elements
finite set C of configurations
two functions $tr, st : C \rightarrow 2^S$ with $tr(c) \cap st(c) = \emptyset$ for all $c \in C$
elements in $tr(c)$ are the *triggers* of c (or "definers" of c) $\tau(c) = |tr(c)|$
elements in $st(c)$ are the *stoppers* of c (or "killers" of c) $\sigma(c) = |st(c)|$

Assume for each $c \in C$ we have $\tau(c) \leq d$, with d a small constant.
We call C *uniform* iff for all $c \in C$ we have $\tau(c) = d$.

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c is *active* for $R \subseteq S$ iff $tr(c) \subseteq R$ and $st(c) \cap R = \emptyset$

$F_0(R)$ is the set of configurations that are active for R
$f_0(R) = |F_0(R)|$ and
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Typical problem: C is given implicitly; determine $F_0(S)$

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Example 1: intervals

S . . . n points on the real line
configurations are all bounded intervals defined by pairs of points in S and all unbounded intervals defined by a point in S

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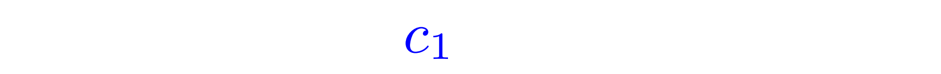
$tr(c_1) = \{e, f\}$ $st(c_1) = \{b, d\}$



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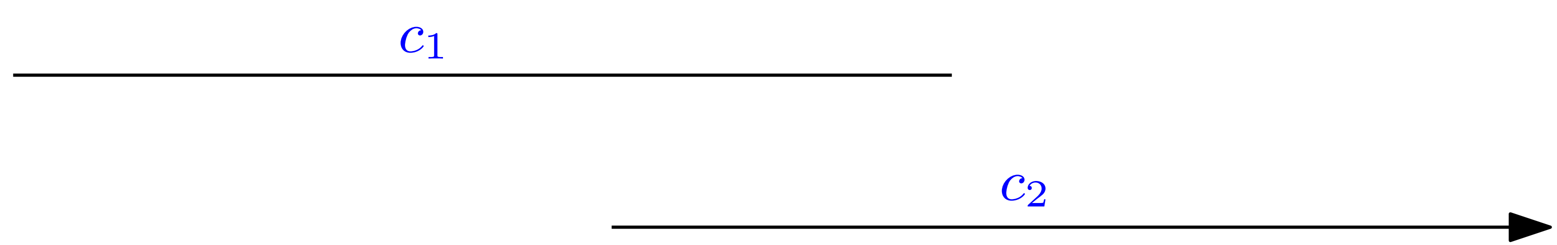

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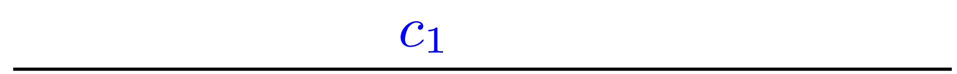
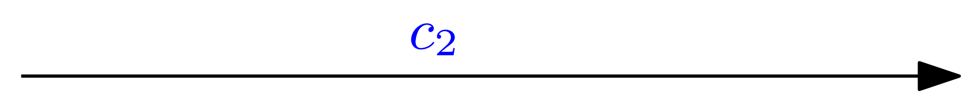
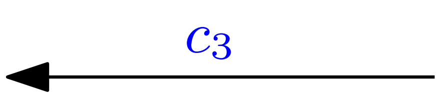


$tr(c_2) = \{b\}$ $st(c_2) = \{c, f\}$





$tr(c_3) = \{a\}$ $st(c_3) = \emptyset$

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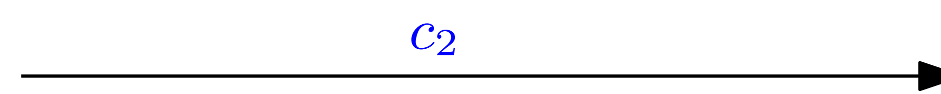

$tr(c_3) = \{a\}$ $st(c_3) = \emptyset$

$tr(c_4) = \{b, c\}$ $st(c_4) = \emptyset$

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For $R \subset S$ consecutive points in R (plus leading and trailing unbounded interval) define the set $F_0(R)$ of active configurations in R

$F_0(R)$ yields the sorted order of R

$F_0(S)$ yields the sorted order of S

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$f_0(R) = |R| + 1$ expected value $f_0(r) = r + 1$

$d = 2$

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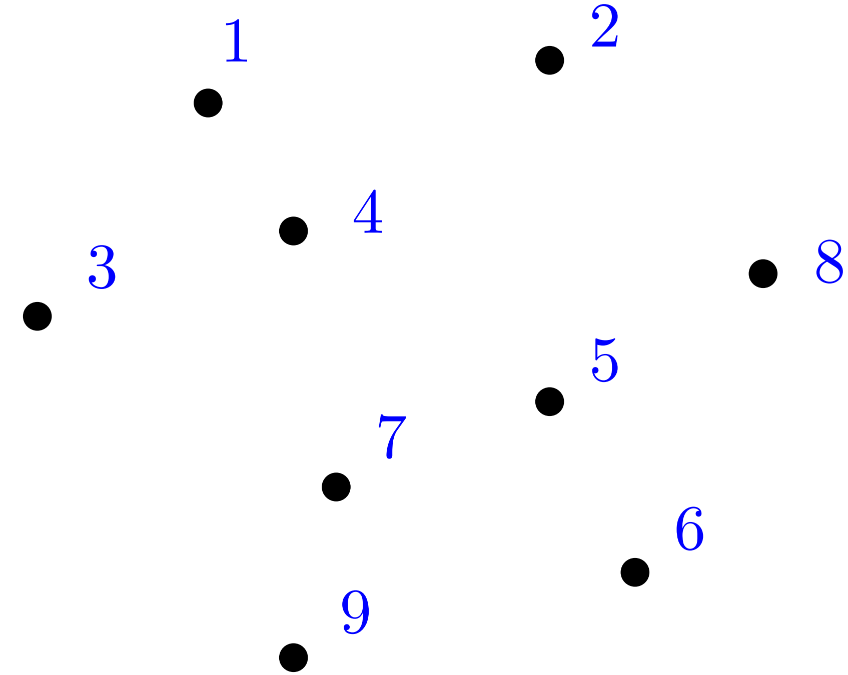
Example 2: halfplanes

S . . . n points in the real plane in non-degenerate position
configurations are all closed halfplanes bounded by lines that are defined by pairs of points in S



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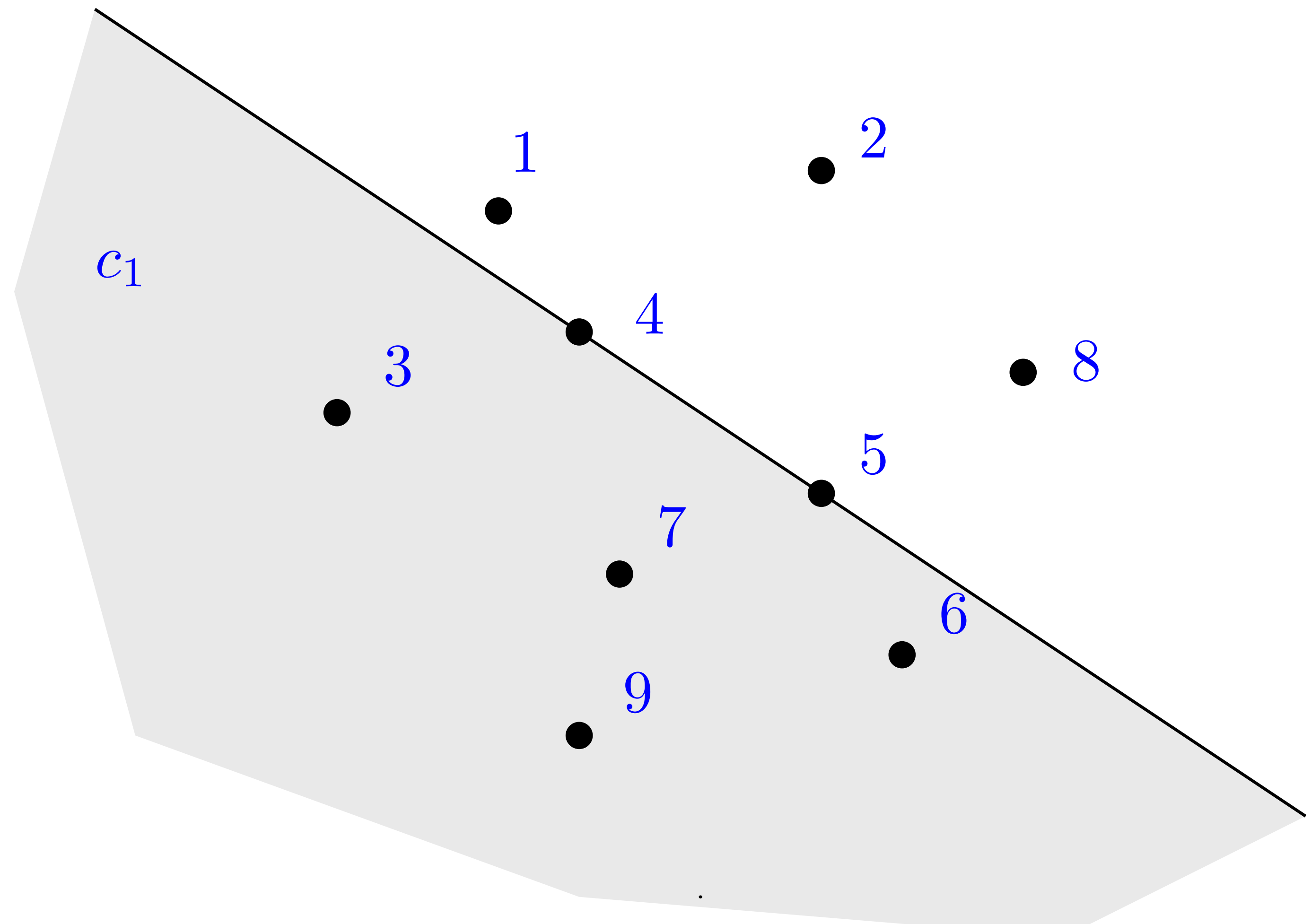
$tr(c_1) = \{4, 5\}$
$st(c_1) = \{3, 6, 7, 9\}$



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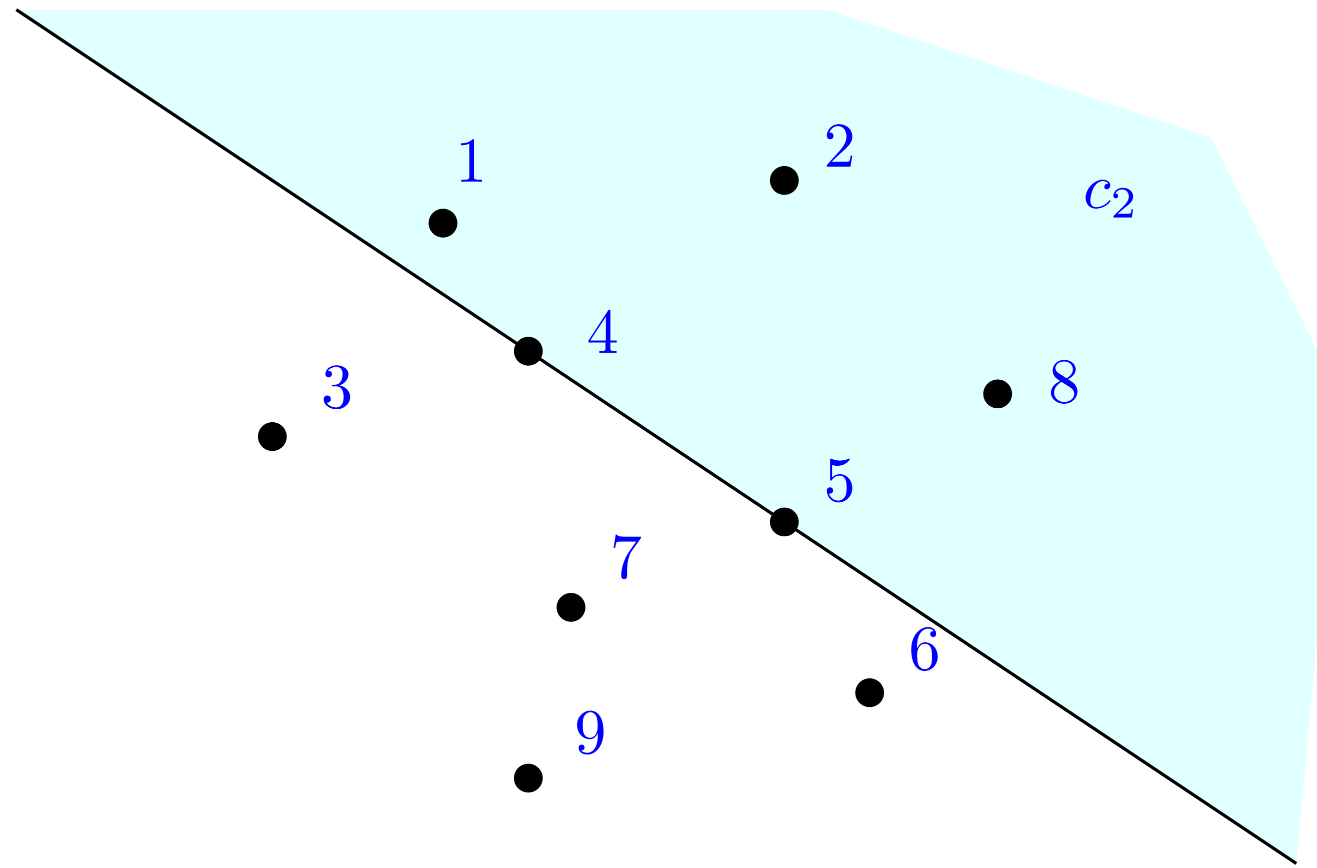
$tr(c_2) = \{4, 5\}$
$st(c_2) = \{1, 2, 8\}$



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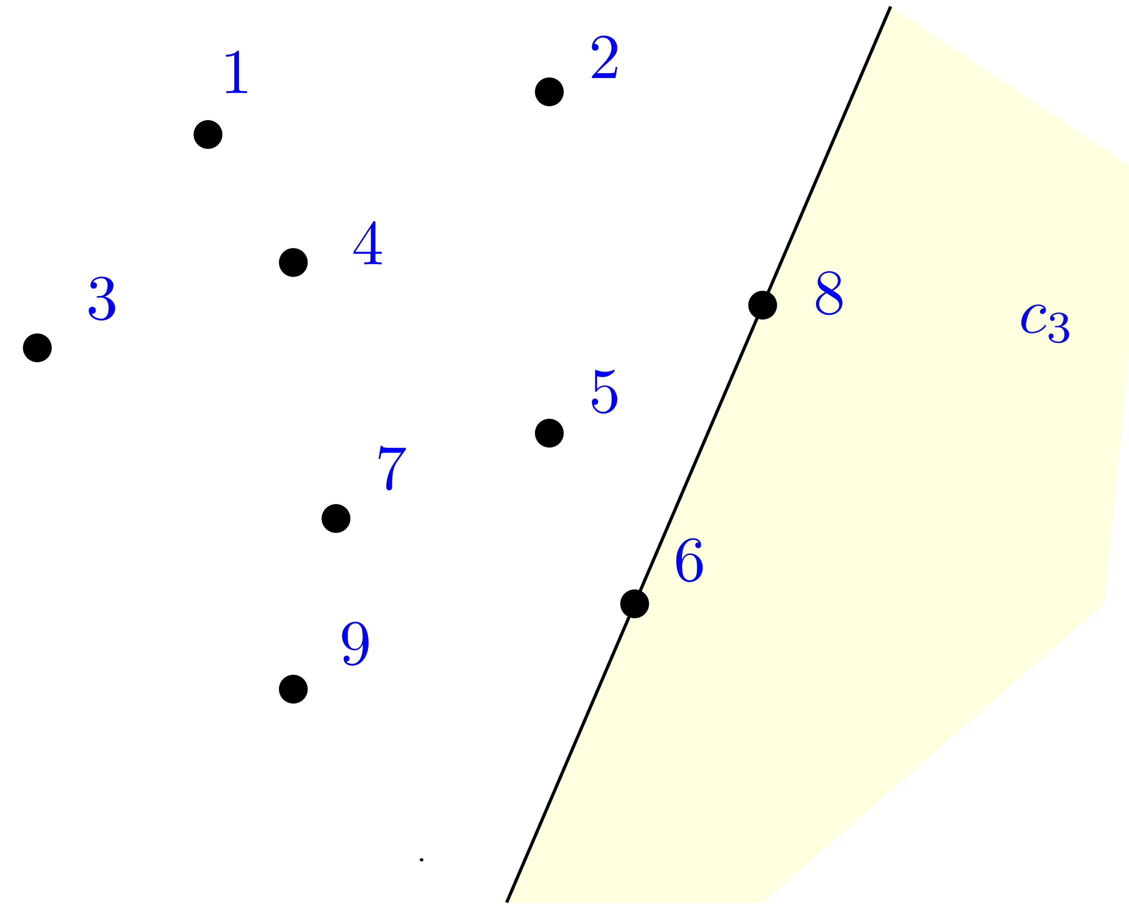

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Example 2: halfplanes

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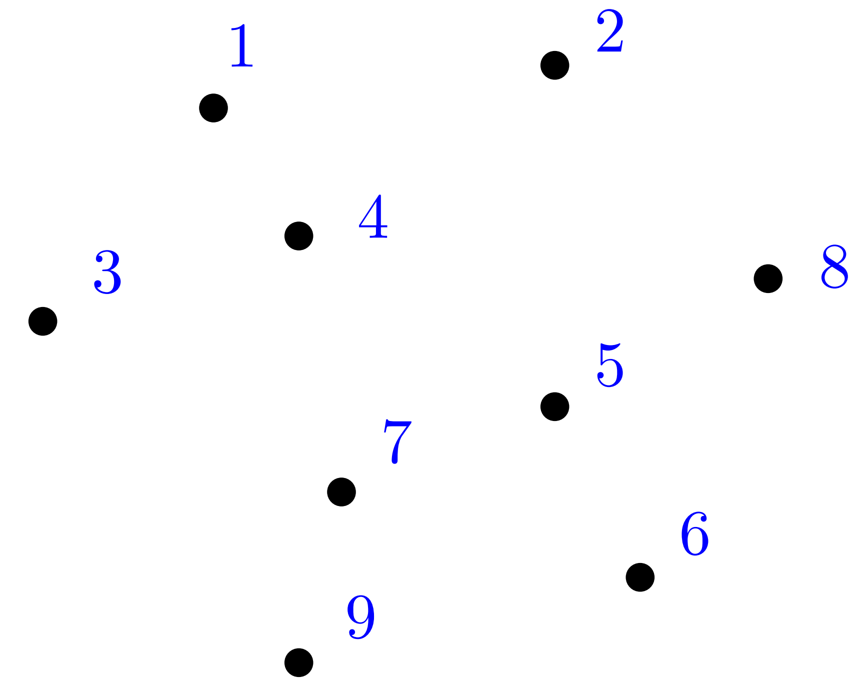
For $R \subset S$ consecutive points around the convex hull of R define the set $F_0(R)$ of active configurations in R



$F_0(R)$ yields the convex hull of R

$F_0(S)$ yields the convex hull of S

$f_0(R) \leq |R|$
expected value $f_0(r) \leq r$

$d = 2$

Example 2': halfspaces

S ... n points in 3-space in non-degenerate position
configurations are all closed halfspaces bounded by planes that are defined by triples of points in S

For $R \subset S$ triples of points that span facets of the convex hull of R define the set $F_0(R)$ of active configurations in R

$F_0(R)$ yields the convex hull of R

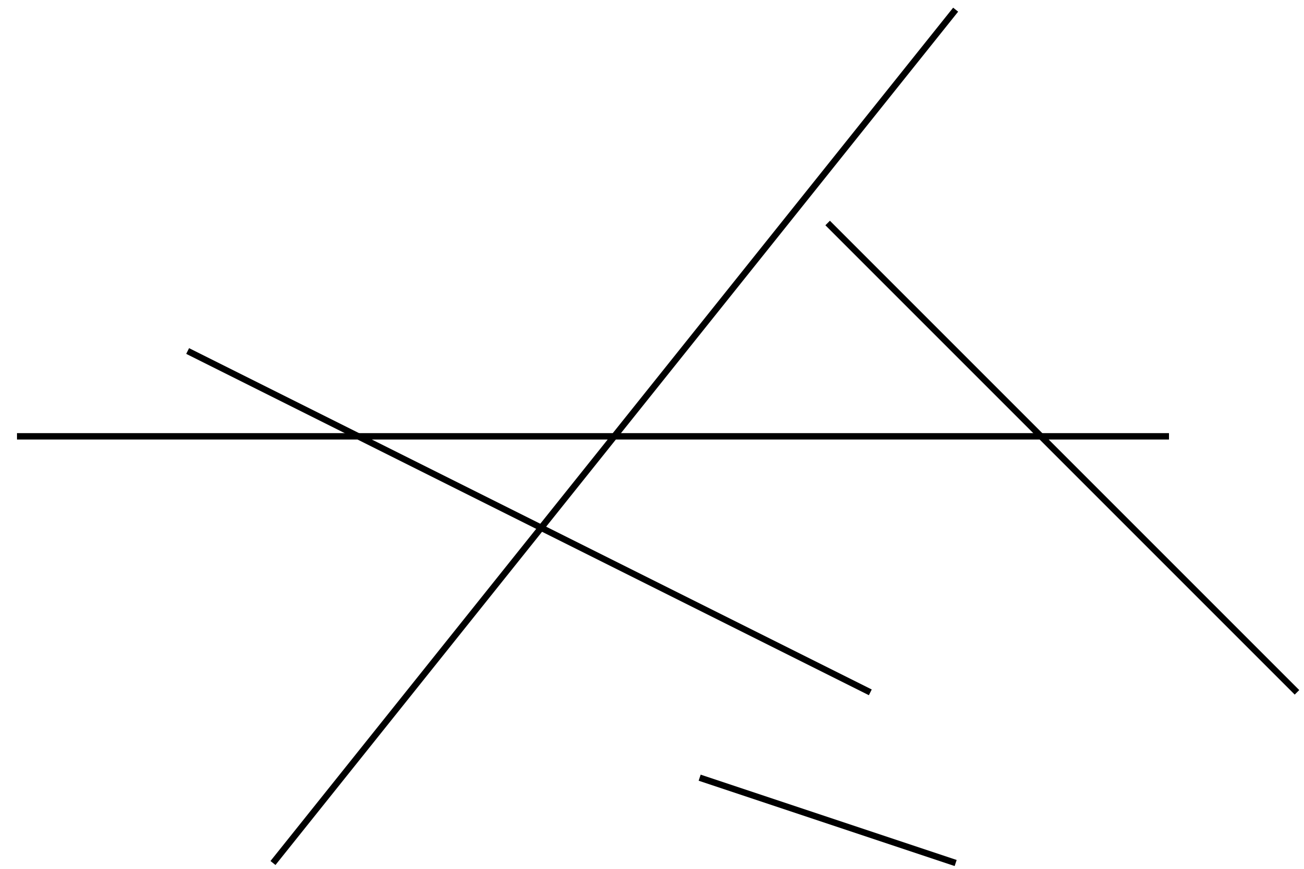
$F_0(S)$ yields the convex hull of S

$f_0(R) \leq 2 \cdot |R| - 4$
expected value $f_0(r) \leq O(r)$

$d = 3$

Example 3: trapezoidations of segment arrangements

n segments
K intersection points



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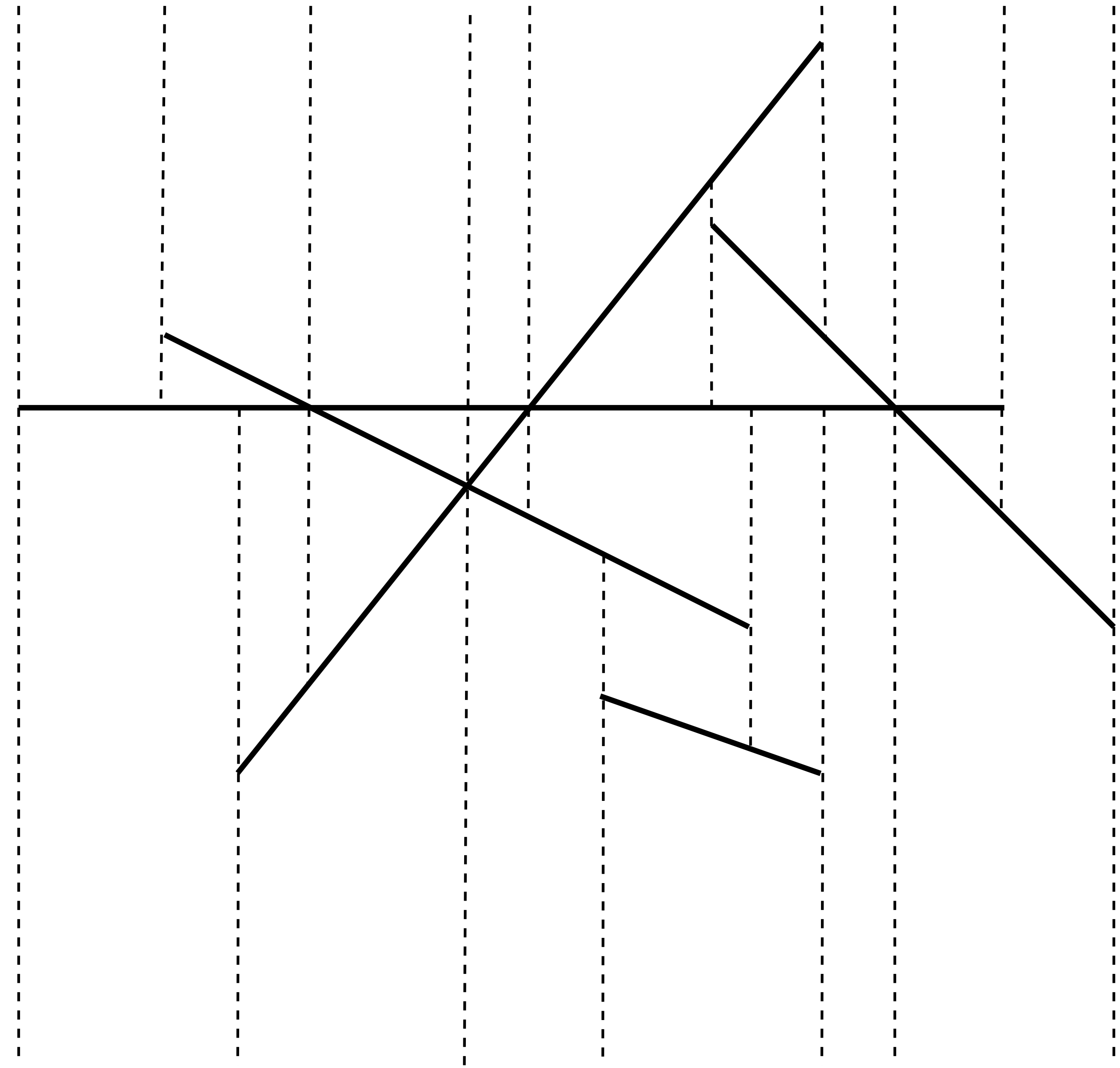
at most $3(n + K) + 1$ trapezoids



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Example 3: trapezoidations of segment arrangements

S ... n segments in the plane in non-degenerate position with K intersection points
configurations are all trapezoids that appear in a trapezoidation of some subset of S

Δ some trapezoid in trapezoidation for some $U \subset S$.
$tr(\Delta)$ are all segments in U that intersect the boundary of Δ
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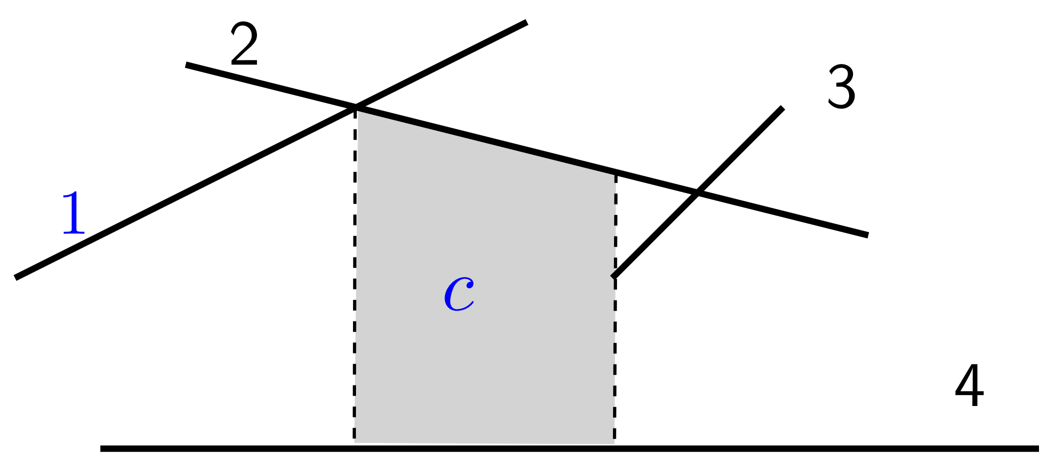

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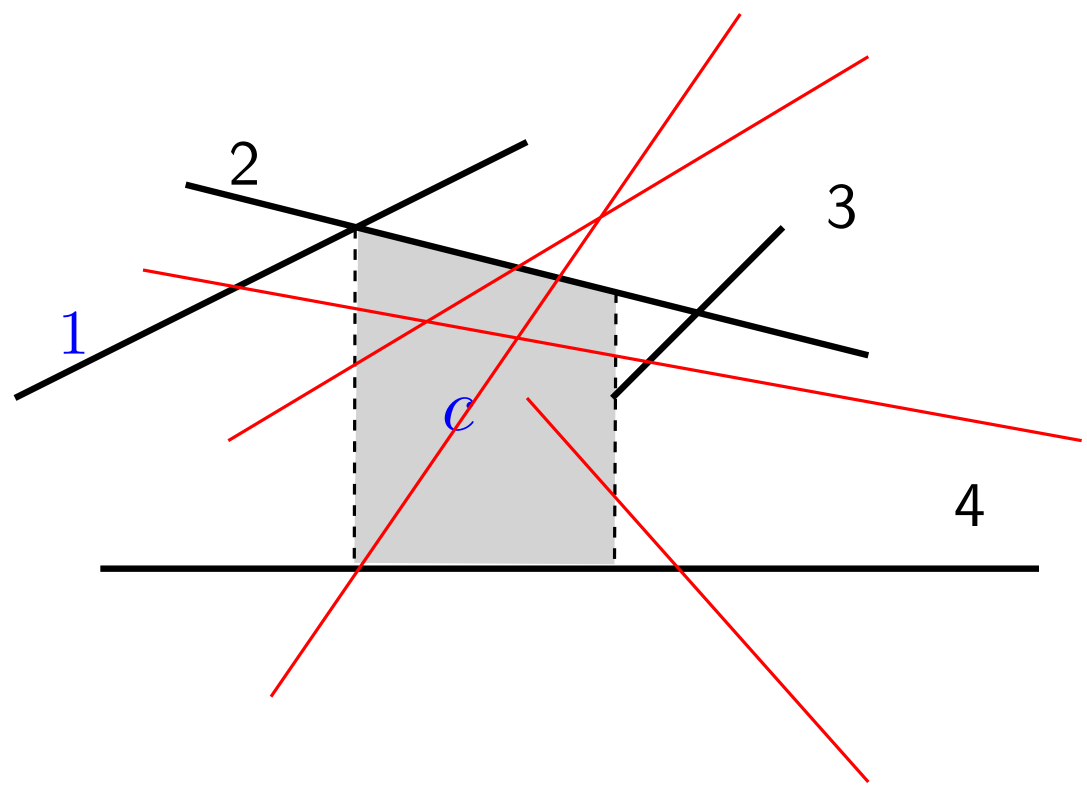

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$F_0(R)$ yields the trapezoidation of R

$F_0(S)$ yields trapezoidation of S

$f_0(R) \leq 3 \cdot (|R| + K_R) + 1$ and the expected value $f_0(r) \leq O(r + \frac{r(r-1)}{n(n-1)} K)$

$d = 4$

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Randomized Incremental Construction (RIC)

Configuration Spaces: Randomized Incremental Construction

Put S in random order $s_1, \ldots, s_n$.
Let $S_r = \{s_1, \ldots, s_r\}$.

for r **from** 1 **to** n **do**
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Typical additional bookkeeping:
associate each $s \notin S_r$ with some $c \in F_0(S_r)$ with $s \in st(c)$
associate each $c \in F_0(S_r)$ with non-empty $st(c)$ with one element in that set

RIC: strange quicksort

RIC: $2d$ convex hulls

RIC: $3d$ convex hulls

RIC: trapezoidations of segments

RIC: expected running time analysis

c becomes active during an enumeration of S if it is active for some "prefix" S_r.
i small integer

$$X_i = \sum_{\substack{c \in C \text{ s.t. } c \text{ becomes active} \\ \text{during random enumeration of } S}} (\tau(c) + \sigma(c))^{\underline{i}}.$$

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RIC Theorem:

$$A_i = O\Big(d^i n^i \sum_{0 \le r \le n} f_0(r)/r^{i+1}\Big)$$

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convex hull in the plane or in 3-space: : $f_0(r) = O(r) \implies A_1 = O(n \log n)$

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trapezoidations of segments: $f_0(r) = O(r + \frac{r^2}{n^2} K) \implies A_1 = O(K + n \log n)$

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RIC Lemma:

$$A_i \le d^{\underline{i+1}} n^{\underline{i}} \sum_{0 \le r \le n} f_0(r)/r^{\underline{i+1}}$$

with equality if the configuration space is uniform.

Ingredients for proof of RIC Lemma

configuration c: $b = \tau(c)$ and $k = \sigma(c)$

$p_r(c) = \Pr(c$ is active in random subset of size $r)$

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Pr(c is active at stage r)=Pr(c becomes active)$\cdot \frac{\binom{r}{b}\binom{n-r}{k}}{\binom{n}{b+k}}$

$f_0(r) = \sum_{c \in C} p_r(c)$

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$$\binom{A}{B}\binom{B}{C}=\binom{A}{C}\binom{A-C}{A-B}=\binom{A}{C}\binom{A-C}{B-C} \qquad \sum_r \binom{r-A-1}{B-A-1}\binom{N-r}{C}=\binom{N-A}{B+C-A}$$

Sampling Theorem

For integer $i \geq 0$ and $R \subseteq S$ define

$$B_i(R) = \sum_{c \text{ active for } R} \sigma(c)^{\underline{i}}$$

and let $B_i(r)$ be the expectation of $B_i(R)$ with R chosen uniformly at random from $\binom{S}{r}$.

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Theorem:

$$B_i(r) = O\left(\left(\frac{n-r}{r}\right)^i \overline{f_0}(r)\right) \qquad \text{where } \overline{f_0}(r) = \frac{1}{r+1} \sum_{0 \leq j \leq r} f_0(j)$$

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Lemma:

$$B_i(r) \leq \frac{(d+1)^{\overline{i+1}}}{(r+1)^{\overline{i+1}}}(n-r)^{\underline{i}} \sum_{0\leq j\leq r} f_0(j)$$

Ingredients for the proof of the sampling Lemma

Same ingredients as for the RIC Lemma

it all reduces to showing that

$$\binom{r+i+1}{b+i+1}\binom{n-r-i}{k-i} \leq \sum_{0 \leq j \leq r} \binom{j}{b}\binom{n-j}{k}$$

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Same ingredients as for the RIC Lemma

it all reduces to showing that

$$\binom{r+i+1}{b+i+1}\binom{n-r-i}{k-i} \leq \sum_{0\leq j\leq r} \binom{j}{b}\binom{n-j}{k}$$

you argue this inequality by considering all binary strings of length $n+1$ with exactly $b+k+1$ digits '1' of which exactly $b+i+1$ are in the first $r+i+1$ positions

Sampling concentration Lemma

Lemma: Assuming that the number of configurations is $O(n^d)$ the following holds:
If R is a random subset of S of size r then with probability at most $1/2$ for each c that is active for R the number of stoppers $\sigma(c)$ is $O(\frac{n}{r} \log r)$.

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Proof sketch: consider configuration c with $\tau(c) = d$ and $\sigma(c) = k$.

The probability that c is active for R is roughly

$$\left(\frac{r}{n}\right)^d \left(1 - \frac{r}{n}\right)^k$$

Sampling concentration Lemma

Lemma: Assuming that the number of configurations is $O(n^d)$ the following holds:
If R is a random subset of S of size r then with probability at least $1/2$ for each c that is active for R the number of stoppers $\sigma(c)$ is $O(\frac{n}{r}\log r)$.

Proof sketch: consider configuration c with $\tau(c) = d$ and $\sigma(c) = k$.

The probability that c is active for R is roughly

$$\left(\frac{r}{n}\right)^d \left(1 - \frac{r}{n}\right)^k$$

which is $\leq \left(\frac{r}{n}\right)^d e^{-\frac{rk}{n}}$

By having $k > \alpha \frac{n}{r} \log r$ for sufficiently large α this is $O(1/n^d)$ and for $O(n^d)$ configurations this sums to less than $1/2$.

Cuttings for lines

Cutting Lemma:
Let S be a set of n lines in the plane in non-degenerate position and let r be some number less than n.
In $O(nr)$ expected time you can find a partition of the plane into $O(r^2)$ trapezoids so that each trapezoid is intersected by at most n/r of the lines in S.

Triangle range searching in the plane

Preprocess a set S of n points in the plane so that for any query triangle T you can quickly determine the points of S that are contained in T.