UNIVERSITÄT DES SAARLANDES

max planck institut informatik

Sublinear Algorithms

Lecture 06: Measurements III

European Research Council
Established by the European Commission

Previous lecture

Sparse Recovery from Arbitrary Linear Measurements

This lecture

Sparse Recovery from
Structured Linear Measurements:
Sparse Fourier Transform

Fourier Transform

$$x \in \mathbb{C}^n \leftrightarrow \widehat{x} \in \mathbb{C}^n$$

$$\widehat{x}_f = \frac{1}{\sqrt{n}} \sum_{t \in [n]} x_t \omega^{ft}$$

$$[n] = \{0, \ldots, n-1\} \qquad \omega = e^{-\frac{2\pi i}{n}} = \cos\left(\frac{2\pi}{n}\right) - i \cdot \sin\left(\frac{2\pi}{n}\right)$$

Inverse Fourier Transform:

$$x_t = \frac{1}{\sqrt{n}} \sum_{f \in [n]} \widehat{x}_f \omega^{-ft}$$

Fourier Transform is a Linear Transform

$$\widehat{x} = Fx \qquad F_{f,t} = \frac{1}{\sqrt{n}} \omega^{ft}$$

Fast Fourier Transform

Gauss (1805), Cooley and Tukey (1965)

The DFT of x is computable in $O(n \log n)$ time.

SIAM: "Top 10 Algorithms of the 20th Century"

- simplex algorithm
- Krylov subspace iteration
- matrix decompositions
- Fortran optimizing compiler
- QR algorithm
- Quicksort
- fast Fourier transform
- integer relation detection algorithm
- fast multipole algorithm
- Metropolis algorithm

The Sparse Fourier Transform problem

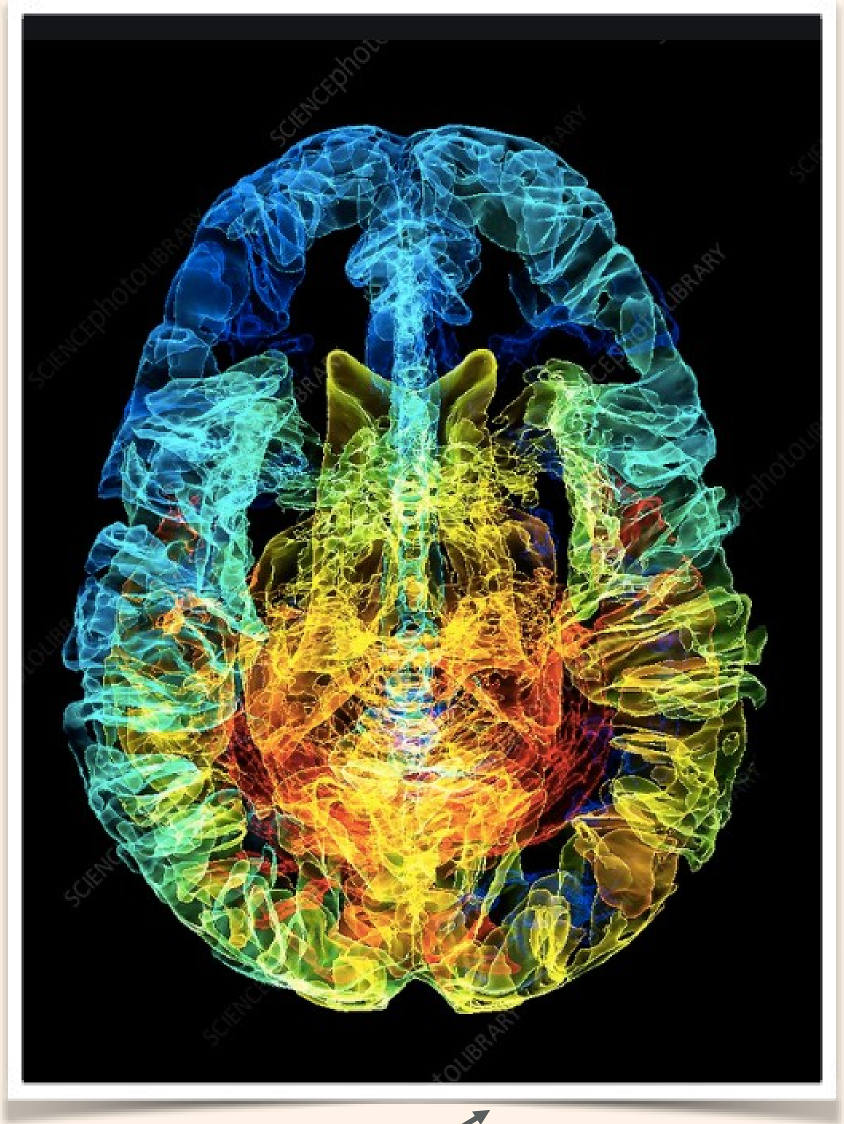
Assume x has an (approximately)
k-sparse Fourier transform.
Can we recover it by not reading n coordinates?

Equivalently: Pick a sublinear
subset of rows of the (I)DFT matrix
which suffice for sparse recovery.

Goals: Minimize
number of measurements
+
Running time

We will assume that n is prime

Magnetic Resonance Imaging (MRI)



Observes Fourier Transform of

A baby example

Recover vector x such that:

$$\widehat{x}_{f^\star} \neq 0$$

$$\widehat{x}_f = 0, f \neq f^\star$$

Perform 2 measurements:

$$x_0 = \frac{1}{\sqrt{n}} \sum_{t \in [n]} \widehat{x}_f = \frac{1}{\sqrt{n}} \widehat{x}_{f^\star}$$

and

$$x_1 = \frac{1}{\sqrt{n}} \sum_{t \in [n]} \widehat{x}_f \omega^{-f} = \frac{1}{\sqrt{n}} \widehat{x}_{f^\star} \omega^{-f^\star}$$

$$\frac{x_1}{x_0} = \frac{\widehat{x}_{f^\star} \omega^{-f^\star}}{\widehat{x}_{f^\star}} = \omega^{-f^\star}$$

Infer frequency $f^\star$

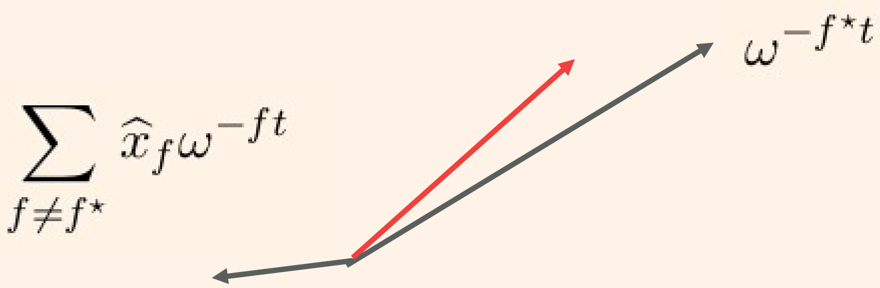
1-sparse Fourier Transform

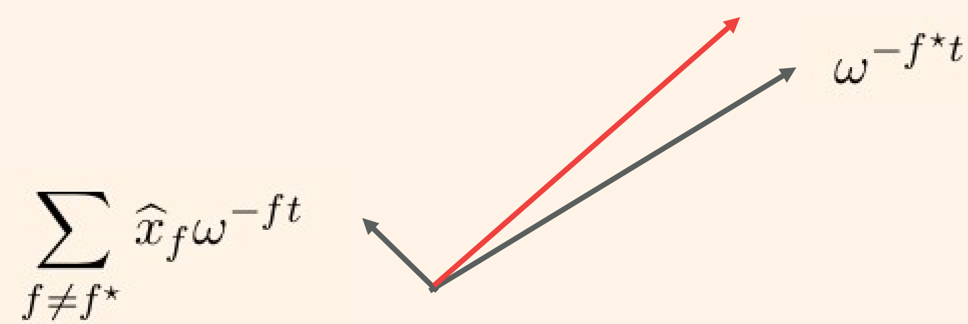
Recover vector x such that:

$$\widehat{x}_{f^\star} = 1$$

$$|\widehat{x}_f| \le \frac{1}{100n}, f \neq f^\star$$

$$x_t = \frac{1}{\sqrt{n}} \cdot \left(\omega^{-f^\star t} + \underbrace{\sum_{f \neq f^\star} \widehat{x}_f \omega^{-ft}}_{\text{magnitude } \le \frac{1}{100}} \right)$$

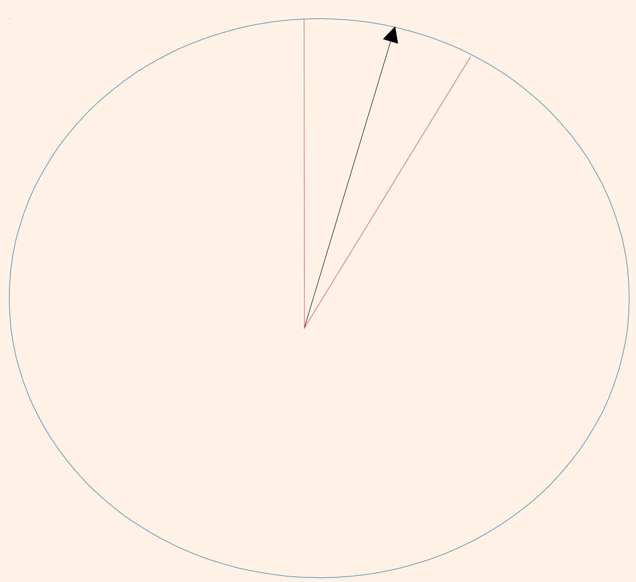


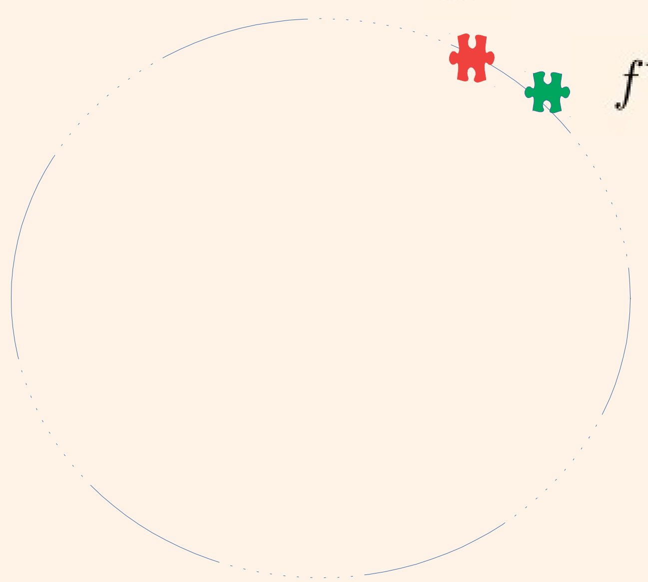


Can learn the angle $2\pi \frac{f^\star t}{n}$ up to O(1) error

Can learn an interval $I_t \subseteq \mathbb{Z}_n$ of length n/8: $f^\star \cdot t \in I_t$

1-sparse Fourier Transform





Indistinguishable from another frequency f : $ft \approx_{n/16} f^\star t$

Random *t:* $\Pr\left\{|f^\star t - ft|_o \leq \frac{n}{16}\right\} =$

$$\Pr\left\{(f^\star - f)t \bmod n \in \left[\frac{n}{16}\right] \cup \left([n] \setminus \left[\frac{15n}{16}\right]\right)\right\} = \frac{\frac{n}{8}}{n} = \frac{1}{8}$$

Uniform over [n]

O(log n) random time points for a union-bound over all frequencies!

Estimation for Sparse Fourier Transform

Moral: Rules of unity can simulate
pairwise-independent random signs

Goal: Appoximate every element of the DFT of x,
using a few samples to x

To estimate Fourier coefficient at f,
let's look at →

$$x_t \omega^{ft} = \frac{1}{\sqrt{n}} \left(\sum_{f' \in [n]} \widehat{x}_{f'} \omega^{-f't} \right) \omega^{ft} =$$

$$\frac{1}{\sqrt{n}} \left(\widehat{x}_f + \sum_{f' \neq f} \widehat{x}_{f'} \omega^{(f-f')t} \right)$$

pick a random point in
time domain

$$\mathrm{E}\left\{ \omega^{(f-f')t} \right\} = \frac{1}{n} \cdot \left(1 + \omega + \omega^2 + \ldots \omega^{n-1}\right) = \frac{1}{n} \frac{\omega^n - 1}{\omega - 1} = 0, f \neq f'$$

because the following quantity is uniform over [n]:

$$(f - f')t \bmod n, f \neq f'$$

Estimation for Sparse Fourier Transform

$$\sqrt{n} \cdot x_t \omega^{ft} - \widehat{x}_f = \underbrace{\sum_{f' \neq f} \widehat{x}_f \omega^{(f-f')t}}_{\text{point}-\text{wise error}}$$

$$\left| \sum_{f' \neq f} \widehat{x}_{f'} \omega^{(f-f')t} \right|^2 = \left(\sum_{f' \neq f} \widehat{x}_{f'} \omega^{(f-f')t} \right) \cdot \left(\overline{\sum_{f' \neq f} \widehat{x}_{f'} \omega^{(f-f')t}} \right) =$$

$$\sum_{f', f'' \neq f} \widehat{x}_{f'} \cdot \overline{\widehat{x}_{f''}} \omega^{(f''-f')t} =$$

$$\underbrace{\sum_{f' \neq f} |\widehat{x}_f|^2}_{\text{diagonal terms}} + \underbrace{\sum_{f', f'' \neq f : f' \neq f''} \widehat{x}_f \cdot \overline{\widehat{x}_{f'}} \cdot \omega^{(f''-f')t}}_{\text{diagonal terms}}$$ ← *0* on expectation

With 2/3 probability, squared point-wise error is at most 3 times

Estimation for Sparse Fourier Transform

Pick O(B) **random** points in time domain, and **average** the corresponding estimators to obtain

$$\text{est}_f$$

such that with constant probability

$$|\widehat{x}_f - \text{est}_f| \leq \sqrt{\frac{1}{B}\sum_{f' \neq f} |\widehat{x}_{f'}|^2} \leq \frac{1}{\sqrt{B}}\|\widehat{x}\|_2$$

Looks like an F_2 analog of the CountMin guarantee, right?

Can repeat **log(n/δ)** times and take the median, to get point-wise approximations **for all** frequencies simultaneously with probability **1-δ**

Morals

One the surface, Fourier Transform has
expressibility comparable
to linear sketching with arbitrary measurements

Point queries/estimates
Binary splitting
Cancellation as in F_2 estimation

Going deeper: much more complicated,
connections with some scary areas of math

Thank you