A Mathematical Background

A.1 Calculus

In this appendix, we state some basic results from calculus used in various places throughout the text. We refer the reader to Ross [1] for a systematic derivation of these results.

Theorem A.1 (Fundamental Theorem of Calculus I, cf. Ross [1] §34.1)**.** *Suppose f is continuous on the interval $[a, b]$ and differentiable on (a, b). If f' is integrable on $[a, b]$, then*

$$\int_a^b f'(x)\, dx = f(b) - f(a).$$

Theorem A.2 (Fundamental Theorem of Calculus II, cf. Ross [1] §34.3)**.** *Suppose f is an integrable function on $[a, b]$. Then for $x \in [a, b]$, define*

$$F(x) = \int_a^x f(t)\, dt.$$

Then F is continuous on $[a, b]$. If f is continuous at $x_0 \in [a, b]$, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

Theorem A.3 (Change of Variables Formula, cf. Ross [1] §34.4)**.** *Suppose u is a continuously differentiable function on an open interval J. Let I be an open interval such that $u(x) \in I$ for all $x \in J$. If f is continuous on I, then $f \circ u$ is continuous on J and*

$$\int_a^b (f \circ u)(x) \cdot u'(x)\, dx = \int_{u(a)}^{u(b)} f(u)\, du.$$

Lemma A.4. *For $k \in \mathbb{N}$, let $\mathcal{F} = \{f_i \mid i \in [k]\}$, where each $f_i \colon [t_0, t_1] \to \mathbb{R}$ is differentiable, and $[t_0, t_1] \subset \mathbb{R}$. Define $F \colon [t_0, t_1] \to \mathbb{R}$ by $F(t) := \max_{i \in [k]} \{f_i(t)\}$. Suppose $\mathcal{F}$ has the property that for every i and t, if $f_i(t) = F(t)$, then $\frac{d}{dt} f_i(t) \leq r$. Then for all $t \in [t_0, t_1]$, we have $F(t) \leq F(t_0) + r(t - t_0)$.*

Proof. We prove the stronger claim that for all a, b satisfying $t_0 \leq a < b \leq t_1$, we have

$$\frac{F(b) - F(a)}{b - a} \leq r. \tag{A.1}$$

To this end, suppose to the contrary that there exist $a_0 < b_0$ satisfying $(F(b_0) - F(a_0))/(b_0 - a_0) \geq r + \varepsilon$ for some $\varepsilon > 0$. We define a sequence of nested intervals $[a_0, b_0] \supset [a_1, b_1] \supset \cdots$ as follows. Given $[a_j, b_j]$, let $c_j = (b_j + a_j)/2$ be the midpoint of a_j and b_j. Observe that

$$\frac{F(b_j) - F(a_j)}{b_j - a_j} = \frac{1}{2}\frac{F(b_j) - F(c_j)}{b_j - c_j} + \frac{1}{2}\frac{F(c_j) - F(a_j)}{c_j - a_j} \geq r + \varepsilon,$$

so that

$$\frac{F(b_j) - F(c_j)}{b_j - c_j} \geq r + \varepsilon \quad \text{or} \quad \frac{F(c_j) - F(a_j)}{c_j - a_j} \geq r + \varepsilon.$$

If the first inequality holds, define $a_{j+1} = c_j$, $b_{j+1} = b_j$, and otherwise define $a_{j+1} = a_j$, $b_j = c_j$. From the construction of the sequence, it is clear that for all j we have

$$\frac{F(b_j) - F(a_j)}{b_j - a_j} \geq r + \varepsilon. \tag{A.2}$$

Observe that the sequences $\{a_j\}_{j=0}^{\infty}$ and $\{b_j\}_{j=0}^{\infty}$ ar both bounded and monotonic, hence convergent. Further, since $b_j - a_j = \frac{1}{2^j}(b_0 - a_0)$, the two sequences share the same limit.

Define

$$c \coloneqq \lim_{j\to\infty} a_j = \lim_{j\to\infty} b_j,$$

and let $f \in \mathcal{F}$ be a function satisfying $f(c) = F(c)$. By the hypothesis of the lemma, we have $f'(c) \leq r$, so that

$$\lim_{h\to 0} \frac{f(c+h) - f(h)}{h} \leq r.$$

Therefore, there exists some $h > 0$ such that for all $t \in [c-h, c+h]$, $t \neq c$, we have

$$\frac{f(t) - f(c)}{t - c} \leq r + \frac{1}{2}\varepsilon.$$

Further, from the definition of c, there exists $N \in \mathbb{N}$ such that for all $j \geq N$, we have $a_j, b_j \in [c-h, c+h]$. In particular this implies that for all sufficiently large j, we have

$$\frac{f(c) - f(a_j)}{c - a_j} \leq r + \frac{1}{2}\varepsilon, \tag{A.3}$$

$$\frac{f(b_j) - f(c)}{b_j - c} \leq r + \frac{1}{2}\varepsilon. \tag{A.4}$$

Since $f(a_j) \leq F(a_j)$ and $f(c) = F(c)$, (A.3) implies that for all $j \geq N$,

$$\frac{F(c) - F(a_j)}{c - a_j} \leq r + \frac{1}{2}\varepsilon.$$

However, this expression combined with with (A.2) implies that for all $j \geq N$

$$\frac{F(b_j) - F(c)}{b_j - c} \geq r + \varepsilon. \tag{A.5}$$

Since $F(c) = f(c)$, the previous expression together with (A.4) implies that for all $j \geq N$ we have $f(b_j) < F(b_j)$.

For each $j \geq N$, let $g_j \in \mathcal{F}$ be a function such that $g_j(b_j) = F(b_j)$. Since $\mathcal{F}$ is finite, there exists some $g \in \mathcal{F}$ such that $g = g_j$ for infinitely many values j. Let $j_0 < j_1 < \cdots$ be the subsequence such that $g = g_{j_k}$ for all $k \in \mathbb{N}$. Then for all j_k, we have $F(b_{j_k}) = g(b_{j_k})$. Further, since F and g are continuous, we have

$$g(c) = \lim_{k\to\infty} g(b_{j_k}) = \lim_{k\to\infty} F(b_{j_k}) = F(c) = f(c).$$

By (A.5), we therefore have that for all k

$$\frac{g(b_{j_k}) - g(c)}{b_{j_k} - c} = \frac{F(b_j) - F(c)}{b_j - c} \geq r + \varepsilon.$$

However, this final expression contradicts the assumption that $g'(c) \leq r$. Therefore, (A.1) holds, as desired. □