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Geometric algorithms with limited resources, Exercise Sheet 4

https://www.mpi-inf.mpg.de/departments/algorithms-complexity/teaching/summer21/geometric-algorithms-with-limited-resources

Total Points: 40 Due: 8am, Thursday, **June 10th**, 2021

You are allowed to collaborate on the exercise sheets, but you have to write down a solution on your own, ***using your own words****. Please indicate the names of your collaborators for each exercise you solve. Further, cite all external sources that you use (books, websites, research papers, etc.). You need to collect at least 50% of all points on exercise sheets to be admitted to the exam.*

Exercise 1 — **6+7** points

Let P and Q be two convex polytopes in $\mathbb{R}^3$ given by their vertices, and suppose that they are in general position, i.e., no four vertices in $P \cup Q$ are coplanar. We are looking for a separating plane of P and Q.

a) Give an $O(n)$ algorithm that finds a separating plane maximizing the variable 'a' which contains exactly three vertices of $P \cup Q$ (if a separating plane exists).
Hint: Transform $P \cup Q$ so that the plane with equation $ax + by + cz = 1$ and the discussed separation LP with the objective of maximizing a works.

b) Suppose that the separation LP is infeasible, and it returns four constraints that cannot be simultaneously satisfied. Give an $O(n)$ algorithm that identifies an intersection point of P and Q.

Exercise 2 — **10** points

Let P be an n-vertex polytope in $\mathbb{R}^3$ given by a DCEL, and let v be a fixed vertex, whose k incident edges form the cyclic sequence $S = e_1, \ldots, e_k$. Let E be a uniform random sample of $\sqrt{n}$ edges of P, and let $E' = e_{i_1}, \ldots, e_{i_\ell}$ be the cyclic subsequence of edges in $E \cap S$. Show that the first two elements of E' have expected distance at most $O(\sqrt{n})$, or more precisely, that

$$\mathbf{E}\Big(\mathbf{I}[|E'| < 2] \cdot k + \mathbf{I}[|E'| \geq 2] \cdot (i_2 - i_1)\Big) = O(\sqrt{n}),$$

where $\mathbf{I}[x]$ is the indicator of event x.

Exercise 3 — **3+5** points

Consider the algorithm on monotonicity testing of images presented in the lecture, and the set of samples $S_1 = S_1^1 \cup S_1^2$ such that $S_1^1 \cap S_1^2 = \emptyset$, $|S_1^1| = \Theta(g_1/\epsilon)$ and $|S_1^2| = \Theta(g_1 \log n/\epsilon^2)$, where $g_1 = \frac{n^{2/3}}{w(M)^{1/3}}$.

(a) Prove that for any fixed choice of $X \subseteq S_1^1$, if the fraction of 1-pixels that do not belong to any submatrix in $\mathcal{M}(X)$ is at least $\epsilon/16$, then with high probability, at least one such 1-pixel is selected in S_1^2.

(b) Show that unless there exists a violating pair in S_1^1, the set of non-heavy submatrices $\mathcal{M}(S_1^1)$ contains at least an $1 - \epsilon/8$ fraction of the 1-pixels in the image M.

Exercise 4 — **9** points

Design an algorithm to obtain an estimate of the number of 1-pixels in an image m, denoted by $w(M)$. More specifically, the algorithm should output, with high constant probability, $w(M)/c \leq \hat{w} \leq c \cdot w(M)$ for some constant c. The algorithm should use $\tilde{O}\left(\min\left\{\sqrt{w(M)}, \frac{n^2}{w(M)}\right\}\right)$ samples or queries (according to the sparse image testing model) in expectation.