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Geometric algorithms with limited resources, Exercise Sheet 2

https://www.mpi-inf.mpg.de/departments/algorithms-complexity/teaching/summer21/geometric-algorithms-with-limited-resources

Total Points: 40 Due: 8am, Thursday, **May 13th**, 2021

You are allowed to collaborate on the exercise sheets, but you have to write down a solution on your own, ***using your own words****. Please indicate the names of your collaborators for each exercise you solve. Further, cite all external sources that you use (books, websites, research papers, etc.). You need to collect at least 50% of all points on exercise sheets to be admitted to the exam.*

Exercise 1 — 4 points

Let $P_1, \ldots, P_k$ be streaming algorithms that require $s_i(n)$ space and $t_i(n)$ time ($i = 1, \ldots, k$). Show that the pipeline $P_k(P_{k-1}(\ldots(P_1(x))\ldots))$ can be realized as a streaming algorithm in $O(\sum_{i=1}^k s_i(n_i))$ space and $O(\sum_{i=1}^k t_i(n_i))$ time, where n_i is the output length of $P_{i-1}(P_{i-2}(\ldots(P_1(x))\ldots))$. (Note that the pipeline is a streaming algortihm that has to write the output in sequence and cannot read it.)

Exercise 2 — 5+5 points

Consider the pipeline $SampleMerge_k(SampleMerge_k(\ldots((PartSort_k(S)))$ on an unsorted array S that was discussed in Lecture 3. Let $a_1 \ldots, a_k$ be the output of the pipeline.

a) Suppose that S has length $2^j k$ and $SampleMerge_k$ is applied j times. Let $L_j(i)$ and $M_j(i)$ be the least and most possible elements in S strictly smaller than a_i. Show that they satisfy the following recursion:

$$L_j(i) = \min_{\substack{p+q=i\\ p>0,\ q\geq 0}} (L_{j-1}(2p) + L_{j-1}(2q) + 1) \text{ and } M_j(i) = \max_{\substack{p+q=i\\ p>0,\ q\geq 0}} (M_{j-1}(2p) + M_{j-1}(2q+2)),$$

where we set $L_j(0) = -1$.

b) Compute $L_j(i)$ and $M_j(i)$ explicitly, and prove that with $j = O(\log \frac{n}{k})$ applications of sample merge, the rank of a_i and a_{i+1} differ by at most $O(\frac{n}{k} \log n)$.

Exercise 3 — 6 points

Give a sublinear space algorithm (with arbitrarily many passes) that decides if a given 3-dimensional LP instance is unbounded or not.

Exercise 4 — 5+2 points

Let S denote a set of geometric objects in $\mathbb{R}^d$ and suppose that there is an algorithm A with running time $T(k)$ that determines if a set S of k such objects is *disjoint* (i.e the k objects are pairwise disjoint). Furthermore, we say that S is ϵ-far from being *disjoint* if there is no set $Q \subseteq S$ with $|Q| < \epsilon n$ such that $S \setminus Q$ is *disjoint*.

a) Show that there exists an 1-sided error tester for DISJOINTNESS that rejects an ϵ-far set S with probability at least 2/3 using at most $8\sqrt{n/\epsilon}$ samples. What is the running time of the tester?
Hint: Note that the lemma in slide 7 of lecture 4 might be helpful.

b) How many samples would be sufficient to achieve a success probability of 9/10 in the soundness case?

Exercise 5 — 5+8 points

a) Let Ω be an arbitrary set of n elements and k, ℓ, s be arbitrary integers. Let $W_1, \ldots W_k$ be disjoint subsets of Ω each of size ℓ and S be a set containing s elements chosen independently and uniformly at random from Ω. For any $p \in (0,1)$, if $s < (n-(\ell-1)) \cdot (p/k)^{1/\ell}$, then

$$\Pr[\exists j \in [k] : W_j \subseteq S] \leq p$$

b) Show that there is no ϵ-tester for DISJOINTNESS with sample complexity $o(\sqrt{n/\epsilon})$.
Hint: Consider the following set S^* as a counterexample: S^* consists of $n - \epsilon n$ pairwise disjoint objects and an additional ϵn pairwise disjoint ones, each intersecting with exactly one of the former ones.