Volume approximation

Sándor Kisfaludi-Bak

Geometric algorithms with limited resources
Summer semester 2021

max planck institut
informatik

Recap

Theorem
Given a convex polytope (as DCEL) of n vertices and a directed line, their intersection can be computed in $O(\sqrt{n})$ time.

Theorem
Given a Delaunay triangulation or a Voronoi diagram as DCEL, we can compute point location (i.e., identify the cell a given query point falls into) in $O(\sqrt{n})$ time.

$n_P(q)$: nearest point of P to q
$\xi_P(\ell)$: point of largest ℓ-coordinate in P
$\xi_P(H, \ell)$: point of largest ℓ-coordinate in $P \cap H$

Theorem
Given a convex polytope P (as DCEL) of n vertices, a point q and a directed line ℓ, we can compute $n_P(q), \xi_P(\ell), \xi_P(H, \ell)$ in $O(\sqrt{n})$ time.

Volume approximation

Theorem
Given $\varepsilon > 0$ and a convex polyope P on n vertices with a DCEL, we can compute a $(1+\varepsilon)$-approximation of its volume in $O(\sqrt{n}/\varepsilon)$ time.

Volume approximation

Theorem
Given $\varepsilon > 0$ and a convex polyope P on n vertices with a DCEL, we can compute a $(1+\varepsilon)$-approximation of its volume in $O(\sqrt{n}/\varepsilon)$ time.

Stage 1. Reshaping into ball-like polytope

Stage 2. Coreset-like approximation with $O(1/\varepsilon)$ size polytope Q s.t. $P \subset Q \subset P_\varepsilon$ by projecting $\sqrt{\varepsilon}$-net of sphere

Volume approximation

Theorem
Given $\varepsilon > 0$ and a convex polyope P on n vertices with a DCEL, we can compute a $(1+\varepsilon)$-approximation of its volume in $O(\sqrt{n}/\varepsilon)$ time.

Stage 1. Reshaping into ball-like polytope

Stage 2. Coreset-like approximation with $O(1/\varepsilon)$ size polytope Q s.t. $P \subset Q \subset P_\varepsilon$ by projecting $\sqrt{\varepsilon}$-net of sphere

Stage 1 will use:

Theorem. Any compact convex object $K \subset \mathbb{R}^d$ has a unique maximum volume ellipsoid $\mathcal{E} \subseteq K$.

Theorem (John 1948). For any compact convex $K \subset \mathbb{R}^d$ with $\mathcal{E}$ centered at the origin, $\mathcal{E} \subseteq K \subseteq d\mathcal{E}$.

Volume approximation

Theorem
Given $\varepsilon > 0$ and a convex polyope P on n vertices with a DCEL, we can compute a $(1+\varepsilon)$-approximation of its volume in $O(\sqrt{n}/\varepsilon)$ time.

Stage 1. Reshaping into ball-like polytope

Stage 2. Coreset-like approximation with $O(1/\varepsilon)$ size polytope Q s.t. $P \subset Q \subset P_\varepsilon$ by projecting $\sqrt{\varepsilon}$-net of sphere

Stage 1 will use:

Theorem. Any compact convex object $K \subset \mathbb{R}^d$ has a unique maximum volume ellipsoid $\mathcal{E} \subseteq K$.

Theorem (John 1948). For any compact convex $K \subset \mathbb{R}^d$ with $\mathcal{E}$ centered at the origin, $\mathcal{E} \subseteq K \subseteq d\mathcal{E}$.

If K has $O(1)$ vertices, then constant-approximation $\mathcal{E}'$ of $\mathcal{E}$ can be computed in $O_d(1)$ time s.t. $\mathcal{E}' \subset P \subset c_d\mathcal{E}'$.

Stage 1: Reshaping into ball-like object

Compute $\xi_P(.)$ for $\pm x, \pm y, \pm z$, let $w_1 w_2$ be the most distant pair.

Claim

$$\frac{\text{diam}(P)}{\sqrt{3}} \leq \text{dist}(w_1, w_2)$$

Stage 1: Reshaping into ball-like object

Compute $\xi_P(.)$ for $\pm x, \pm y, \pm z$, let $w_1 w_2$ be the most distant pair.

Claim

$$\frac{\mathrm{diam}(P)}{\sqrt{3}} \leq \mathrm{dist}(w_1, w_2)$$

P_w: intersection of P with plane perpendicular to $w_1 w_2$ going through w.
Let $w_0 =$ midpoint of $w_1 w_2$.

Claim

Suppose $S \subset P_{w_0}$ is such that $\mathrm{Area}(\mathrm{conv}(S)) \geq c_1 \mathrm{Area}(P_{w_0})$. Then

$$\mathrm{Vol}(\mathrm{conv}(S \cup \{w_1, w_2\})) \geq c_2 \mathrm{Vol}(P).$$

Reshaping to ball-like object: linear transfomration

Ellipsoid: $x^T A^T A x \leq 1$ for some PSD matrix A.
Let $T = A/\det(A)$.

Reshaping to ball-like object: linear transfomration

Ellipsoid: $x^T A^T Ax \leq 1$ for some PSD matrix A.
Let $T = A/\det(A)$.

Assume wlog. that $B_c \subseteq P \subseteq B_1$ for some constsant $c > 0$.

Stage 2: approximate polytope

Goal: compute convex polytope Q such that $P \subseteq Q \subseteq P_\varepsilon$.

Construction of Q:

Stage 2: approximate polytope

Goal: compute convex polytope Q such that $P \subseteq Q \subseteq P_\varepsilon$.

Construction of Q:

Theorem[Dudley '76]

$$P \subset Q \subset P_\varepsilon$$

Constructing Q in sublinear time

Compute $\sqrt{\varepsilon}$-net of the unit sphere

Volume approximation wrap-up