Ray shooting and volume approximation

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Geometric algorithms with limited resources
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Overview

• Testing if convex polytopes intersect without preprocessing – wrap-up
• Ray shooting, nearest neighbor
• Volume approximation
**Theorem** (Chazelle, Liu, Magen ’06)
Given convex polyhedra \( P \) and \( Q \) by DCEL, and stored in a way that we can sample an edge from either, we can decide if \( P \) and \( Q \) intersect in \( O(\sqrt{n}) \) time.

Recall:
- sample from both of size \( r = \sqrt{n} \)
- find separating plane \( H \) (if not, return intersection)
- \( p \in H \cap V(P) \) has neighbor \( p_1 \) on other side, find it by resampling
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Todo: prove $E(|C_p| + |C_q|) = O(n/r)$. 
Sublinear intersection of convex polytopes without preprocessing

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Last time:
Ground set $S$, (sample) set $R \subset S$ of size $r$. $\varphi : 2^S \to \mathbb{R}$ Let

$$V(R) := \{ s \in S \setminus R \mid \varphi(R \cup \{s\}) \neq \varphi(R) \}$$

$$X(R) := \{ s \in R \mid \varphi(R \setminus \{s\}) \neq \varphi(R) \}$$

Set $v_r := \mathbb{E}(V(R))$ and $x_r := \mathbb{E}(X(R))$.

**Sampling Lemma** (Gärtner, Welzl ’01)
For $0 \leq r < n$, we have:

$$\frac{v_r}{n-r} = \frac{x_r+1}{r+1}.$$
Perturbing and tweaking the sampling distribution

$M$: multiset of vertices of $P \cup Q$, where $p$ has $\deg(p)$ copies

$M'$: perturb $M$ by moving infinitesimally randomly towards edge midpoints

$D_3$: Choose $R_p \cup R_q$ by selecting each vertex of $M'$ indep. with prob. $r/n$

$D_2$: Choose $R_p \cup R_q$ by selecting each vertex of $M$ indep. with prob. $r/n$
Ray shooting, Voronoi pt location

**Theorem**
Given a convex polytope (as DCEL) of $n$ vertices and a directed line, their intersection can be computed in $O(\sqrt{n})$ time.

**Theorem**
Given a Delaunay triangulation or a Voronoi diagram as DCEL, we can compute point location (i.e., identify the cell a given query point falls into) in $O(\sqrt{n})$ time.

\[ p = (p_x, p_y) \rightarrow H_p : z = 2p_x x + 2p_y y - (p_x^2 + p_y^2) \]
Nearest point of a polytope

\( n_P(q) \): nearest point of \( P \) to \( q \)
\( \xi_P(\ell) \): point of largest \( \ell \)-coordinate in \( P \)
\( \xi_P(H, \ell) \): point of largest \( \ell \)-coordinate in \( P \cap H \)

**Theorem**
Given a convex polytope \( P \) (as DCEL) of \( n \) vertices, a point \( q \) and a directed line \( \ell \), we can compute \( n_P(q), \xi_P(\ell), \xi_P(H, \ell) \) in \( O(\sqrt{n}) \) time.
Volume approximation

Theorem
Given $\varepsilon > 0$ and a convex polyope $P$ on $n$ vertices, we can compute a $(1 + \varepsilon)$-approximation of its volume in $O(\sqrt{n}/\varepsilon)$ time.
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Stage 1. Reshaping into ball-like polytope

Stage 2. Coreset-like approximation with $O(1/\varepsilon)$ size polytope $Q$ s.t. $P \subset Q \subset P_{\varepsilon}$ by projecting $(1/\sqrt{\varepsilon})$-net of sphere.
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Stage 1 will use:

**Theorem.** Any compact convex object $K \subset \mathbb{R}^d$ has a unique maximum volume ellipsoid $E \subseteq K$.

**Theorem (John 1948).** For any compact convex $K \subset \mathbb{R}^d$ with $E$ centered at the origin, $E \subseteq K \subseteq dE$. 