Exercise 1

Let $G$ be an undirected graph with an even number of vertices. Show that the number of directed even cycle covers is $\#PM(G)^2$.

Exercise 2

An octahedron is a three-dimensional object (with eight faces) consisting of two pyramids that are “glued together” at the base face consisting of four edges.

a. Give a Pfaffian orientation of a planar embedding of an octahedron.

b. Use this Pfaffian orientation to compute the number of perfect matchings of an octahedron.

Hint: Computers can compute determinants.

Exercise 3

Prove that it can be decided in polynomial time whether a bipartite graph has an even number of perfect matchings.

Exercise 4

The graph $K_{3,3}$ (i.e., the complete bipartite graph with 3 vertices in each part of the bipartition) is not planar.

a. Given some orientation of the edges of $K_{3,3}$, let $A$ be the corresponding oriented adjacency matrix, and let $B$ be the corresponding oriented biadjacency matrix. Show that $\det(A) = \det(B)^2$.

b. Show that there is no orientation of the edges of $K_{3,3}$ such that $\#PM(K_{3,3})^2 = \det(A)$.

Exercise 5

Prove that computing the number of proper 3-colorings of a bipartite graph is $\#P$-complete using a reduction from $\#BIS$, the problem of counting independent sets in a bipartite graph (which can be assumed to be $\#P$-complete).