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Techniques for Counting Problems, Exercise Sheet 2

www.mpi-inf.mpg.de/departments/algorithms-complexity/teaching/summer23/counting

Total Points: **100** Due: Thursday, **May 11**, 2023

You are allowed to collaborate on the exercise sheets. Justify your answers. Cite all external sources that you use (books, websites, research papers, etc.). You need to collect at least 50% of all points on exercise sheets to be admitted to the exam. Please hand in your solutions before the lecture on the day of the deadline.

Exercise 1 — **25** points

Let G be an undirected graph with an even number of vertices. Show that the number of directed even cycle covers is $\#\mathsf{PM}(G)^2$.

Exercise 2 — **10** + **10** points

An octahedron is a three-dimensional object (with eight faces) consisting of two pyramids that are "glued together" at the base face consisting of four edges.

a Give a Pfaffian orientation of a planar embedding of an octahedron.

b Use this Pfaffian orientation to compute the number of perfect matchings of an octahedron.

Hint: Computers can compute determinants.

Exercise 3 — **15** points

Prove that it can be decided in polynomial time whether a bipartite graph has an even number of perfect matchings.

Exercise 4 — **5** + **15** points

The graph $K_{3,3}$ (i.e., the complete bipartite graph with 3 vertices in each part of the bipartition) is not planar.

a Given some orientation of the edges of $K_{3,3}$, let $\overrightarrow{A}$ be the corresponding oriented adjacency matrix, and let $\overrightarrow{B}$ be the corresponding oriented biadjacency matrix. Show that $\det(\overrightarrow{A}) = \det(\overrightarrow{B})^2$

b Show that there is no orientation of the edges of $K_{3,3}$ such that $\#\mathsf{PM}(K_{3,3})^2 = \det(\overrightarrow{A})$.

Exercise 5 — **20** points

Prove that computing the number of proper 3-colorings of a bipartite graph is #P-complete using a reduction from #BIS, the problem of counting independent sets in a bipartite graph (which can be assumed to be #P-complete).