max planck institut informatik

SIC Saarland Informatics Campus

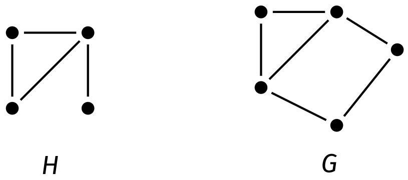
Techniques for Counting Problems, Lecture 8

Limitations of Counting Dichotomies

Philip Wellnitz

Graph Homomorphism

Mapping from graph H to G that preserves edges;
Write $\mathrm{Hom}(H \to G)$ for the set of all graph hom's from H to G.



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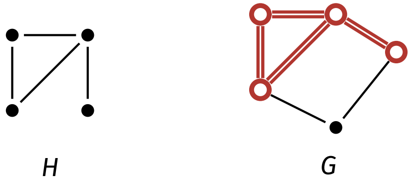

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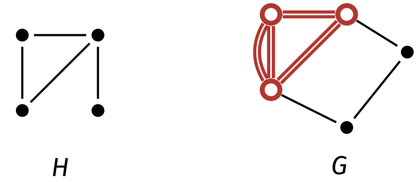

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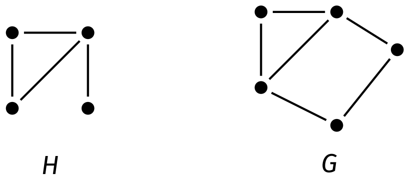

$$\#\mathrm{Hom}(H \to G) = 16$$

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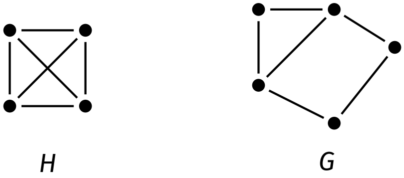

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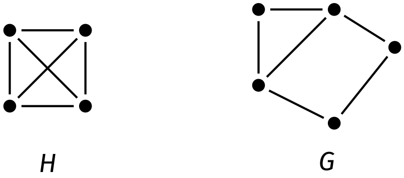

No homomorphisms from H to G.

Graph Homomorphism

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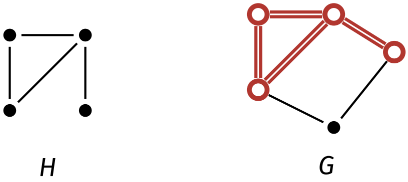

Finding (counting) homomorphisms is important for finding patterns in graphs

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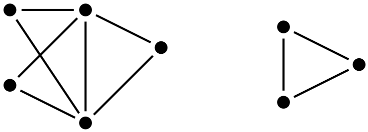

H G

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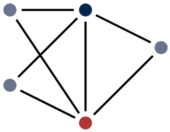
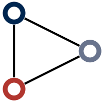


H (G-colored)



G

Finding (counting) homomorphisms generalizes graph coloring problems

Hom($\mathbb{H} \to \mathbb{G}$)

Given graphs $H \in \mathbb{H}$ and $G \in \mathbb{G}$, check if there is a graph hom from H to G.

HOM($\mathbb{H} \to \mathbb{G}$)

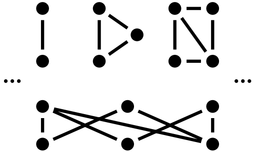
Given graphs $H \in \mathbb{H}$ and $G \in \mathbb{G}$, check if there is a graph hom from H to G.

Graph classes

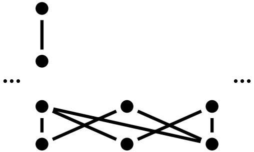
Hom($\mathcal{H} \to \mathcal{G}$)

Given graphs $H \in \mathcal{H}$ and $G \in \mathcal{G}$, check if there is a graph hom from H to G.

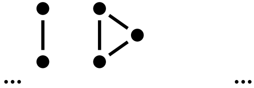
Graph classes



All Graphs ($\top$)



All Bipartite Graphs



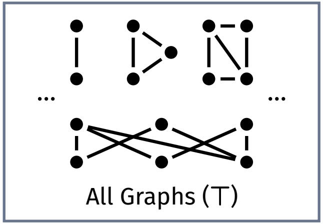
All Cliques

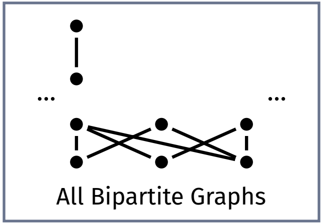
Summary: Counting Graph Homomorphisms

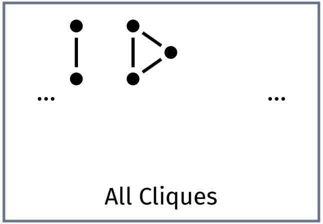
Hom($\mathbb{H} \to \mathbb{G}$)

Given graphs $H \in \mathbb{H}$ and $G \in \mathbb{G}$, check if there is a graph hom from H to G.

Graph classes ← set of graphs







Hom($\mathbb{H} \to \mathbb{G}$)

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Hom($\mathbb{H} \to \mathbb{G}$)

Given graphs $H \in \mathbb{H}$ and $G \in \mathbb{G}$, check if there is a graph hom from H to G.

NP-complete

Hom($\top \to \top$)

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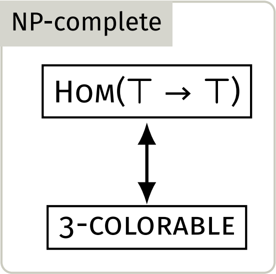
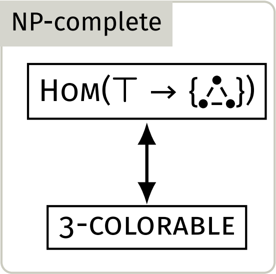


Summary: Known Results

Hom($\mathbb{H} \to \mathbb{G}$)

Given graphs $H \in \mathbb{H}$ and $G \in \mathbb{G}$, check if there is a graph hom from H to G.



$\textsc{Hom}(\mathbb{H} \to \mathbb{G})$

Given graphs $H \in \mathbb{H}$ and $G \in \mathbb{G}$, check if there is a graph hom from H to G.

Are there fast algorithms for special cases of $\textsc{Hom}(\top \to \top)$?

Hom($\mathbb{H} \to \mathbb{G}$)

Given graphs $H \in \mathbb{H}$ and $G \in \mathbb{G}$, check if there is a graph hom from H to G.

What makes Hom($\top \to \top$) hard?

$\text{Hom}(\mathbb{H} \to \mathbb{G})$

Given graphs $H \in \mathbb{H}$ and $G \in \mathbb{G}$, check if there is a graph hom from H to G.

	poly-time solvable	NP-complete
$\text{Hom}(\top \to \mathbb{G})$	$\mathbb{G}$ contains only bipartite graphs [Hell, Nešetřil '90]	$\mathbb{G}$ contains a non-bipartite graph [Hell, Nešetřil '90]

#Hom($\mathbb{H} \to \mathbb{G}$)

Given graphs $H \in \mathbb{H}$ and $G \in \mathbb{G}$, count all graph homomorphisms from H to G.

	poly-time solvable	#P-complete
#Hom($\top \to \mathbb{G}$)	(explicit criterion exists) [Dyer, Greenhill '00]	(explicit criterion exists) [Dyer, Greenhill '00]

Hom($\mathbb{H} \to \mathbb{G}$)

Given graphs $H \in \mathbb{H}$ and $G \in \mathbb{G}$, check if there is a graph hom from H to G.

What about *the other side*, Hom($\mathbb{H} \to \top$)?

Hom($\mathbb{H} \to \mathbb{G}$)

Given graphs $H \in \mathbb{H}$ and $G \in \mathbb{G}$, check if there is a graph hom from H to G.

When is Hom($\mathbb{H} \to \top$) easy?

Hom($\mathbb{H} \to \mathbb{G}$)

Given graphs $H \in \mathbb{H}$ and $G \in \mathbb{G}$, check if there is a graph hom from H to G.

When is Hom($\mathbb{H} \to \top$) easy?

Always in time $O(|V(G)|^{|V(H)|})$ (brute-force)

(fast if $|V(H)|$ bounded for all $H \in \mathbb{H}$, this is the boring case)

Hom($\mathbb{H} \to \mathbb{G}$) **Parameter: $|V(H)|$**

Given graphs $H \in \mathbb{H}$ and $G \in \mathbb{G}$, check if there is a graph hom from H to G.

When is Hom($\mathbb{H} \to \top$) fixed-parameter tractable?

(in $O(f(|V(H)|) \cdot poly(|V(G)|))$ time)

Hom($\mathbb{H} \to \mathbb{G}$) — Parameter: $|V(H)|$

Given graphs $H \in \mathbb{H}$ and $G \in \mathbb{G}$, check if there is a graph hom from H to G.

	FPT ($f(\|V(H)\|) \cdot poly(\|V(G)\|)$ time)	W[1]-hard (not (much) faster than brute-force)
Hom($\mathbb{H} \to \top$)	"$\mathbb{H}$ contains only graphs with small treewidth" [Grohe '03]	"$\mathbb{H}$ contains graphs with arbitrary large tw" [Grohe '03]

#HOM($\mathbb{H} \to \mathbb{G}$) Parameter: $|V(H)|$

Given graphs $H \in \mathbb{H}$ and $G \in \mathbb{G}$, count all graph homomorphisms from H to G.

	FPT ($f(\lvert V(H)\rvert) \cdot poly(\lvert V(G)\rvert)$ time)	#W[1]-hard (not (much) faster than brute-force)
#HOM($\mathbb{H} \to \top$)	"$\mathbb{H}$ contains only graphs with small treewidth" [Dalmau, Jonsson '04]	"$\mathbb{H}$ contains a graph with large treewidth" [Dalmau, Jonsson '04]

#Hom($\mathbb{H} \to \mathbb{G}$) Parameter: $|V(H)|$

Given graphs $H \in \mathbb{H}$ and $G \in \mathbb{G}$, count all graph homomorphisms from H to G.

Complexity dichotomies when restricting either $\mathbb{G}$ or $\mathbb{H}$.

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What if we restrict *both sides*?

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What if we restrict *both sides*?

This lecture.

$\#\textsc{Hom}(\mathbb{H} \to \mathbb{G})$ Parameter: $|V(H)|$

Given graphs $H \in \mathbb{H}$ and $G \in \mathbb{G}$, count all graph homomorphisms from H to G.

Theorem

For any problem P in #W[1] (or W[1]), there are graph classes $\mathbb{H}_P$ and $\mathbb{G}_P$ such that P is equivalent to $\#\textsc{Hom}(\mathbb{H}_P \to \mathbb{G}_P)$ (or $\textsc{Hom}(\mathbb{H}_P \to \mathbb{G}_P)$).

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- Cannot hope for clear categorization into FPT/W[1]-hard for all pairs $(\mathcal{H}, \mathcal{G})$

  (recall Ladner's Theorem: If P ≠NP, there are NP-intermediate problems;
  similar results by Downey and Fellows for FPT/W[1])

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Recall: #HOM($\mathbb{H} \to \top$) is #W[1]-hard if $\mathbb{H}$ has "unbounded treewidth" [DalJon'04]

#Hom($\mathcal{H} \to \mathcal{G}$) Parameter: $|V(H)|$

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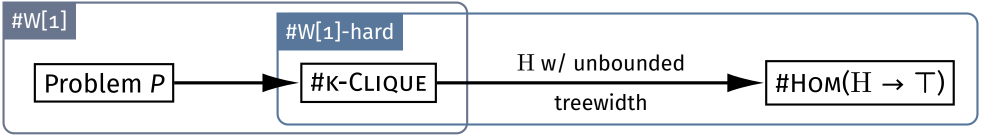

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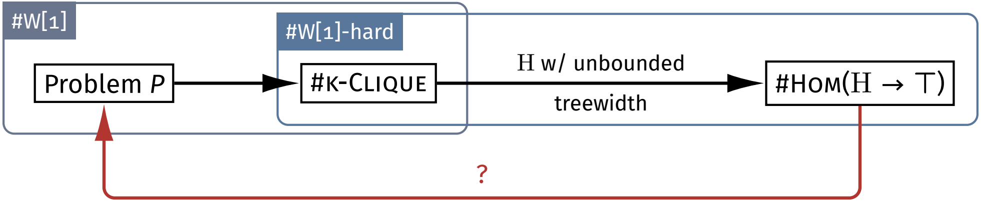

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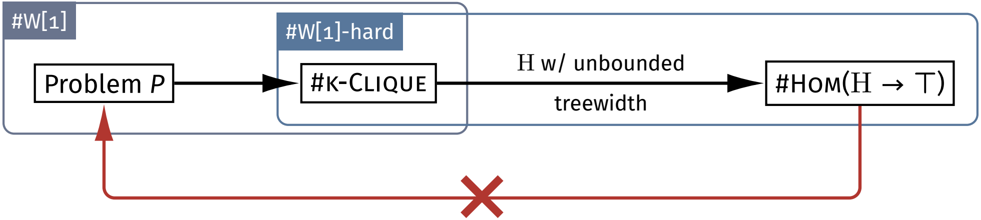

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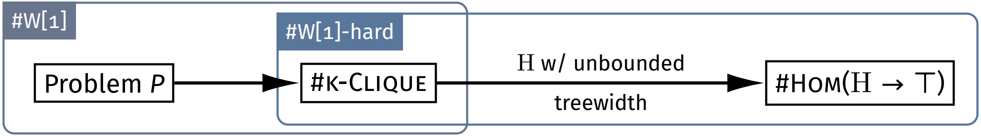

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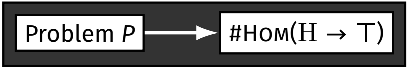

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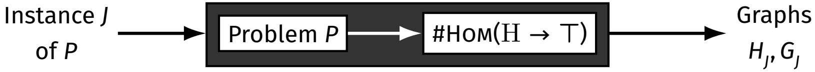

Proof Ideas

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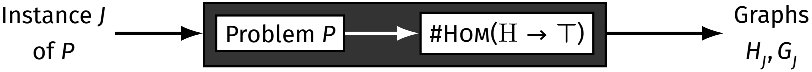
For any P in #W[1], there are H_P, G_P such that P is equivalent to #HOM($\mathrm{H}_P \to \mathrm{G}_P$).



Approach:

$$\mathrm{H}_P := \{H_J \mid \text{instance } J \text{ of } P\}$$
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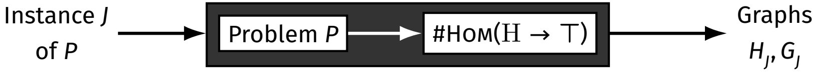

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$\mathrm{P} \leqslant \#\textsc{Hom}(\mathrm{H}_P \to \mathrm{G}_P)$ ✓

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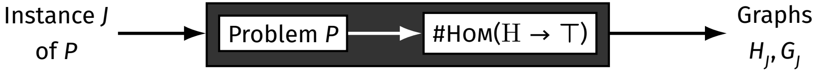

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#Hom($\mathrm{H}_P \to \mathrm{G}_P$) $\overset{?}{\leq}$ P

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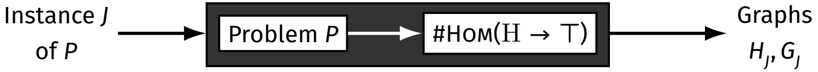

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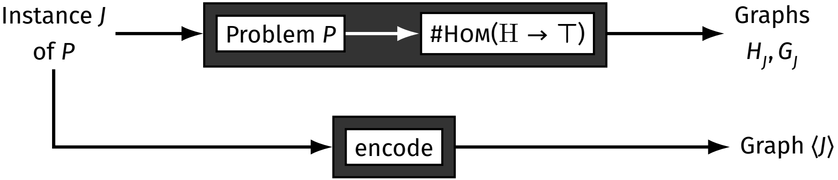
P $\leqslant$ #Hom($\mathcal{H}_P \to \mathcal{G}_P$) ✓

#Hom($\mathcal{H}_P \to \mathcal{G}_P$) $\not\leqslant$ P

How do we obtain instance J from (H_J, G_J)?

Theorem

For any P in #W[1], there are H_P, G_P such that P is equivalent to #HOM($\mathrm{H}_P \to \mathrm{G}_P$).



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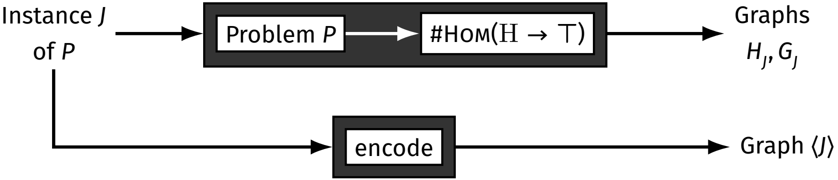

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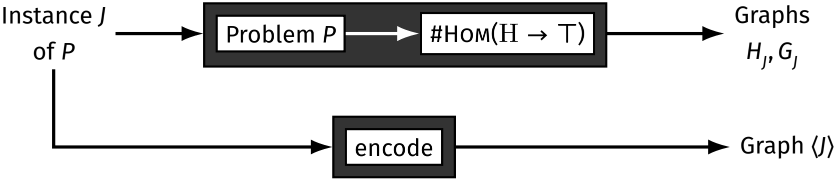

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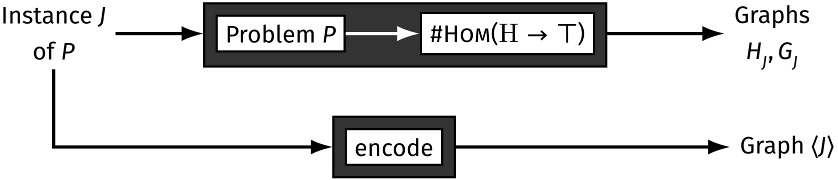
P $\leqslant$ #Hom($\mathrm{H}_P \to \mathrm{G}_P$) ✓

(ensure #Hom($H_J \to \langle J\rangle$) = 0)

Theorem

For any P in #W[1], there are $\mathcal{H}_P$, $\mathcal{G}_P$ such that P is equivalent to $\#\text{Hom}(\mathcal{H}_P \to \mathcal{G}_P)$.



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$P \leqslant \#\text{Hom}(\mathcal{H}_P \to \mathcal{G}_P)$ ✓
(ensure $\#\text{Hom}(H_J \to \langle J\rangle) = 0$)

$\#\text{Hom}(\mathcal{H}_P \to \mathcal{G}_P) \overset{?}{\leqslant} P$

Proof Ideas

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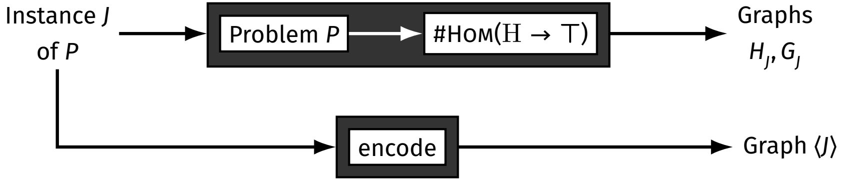


Approach:

$$\mathcal{H}_P := \{H_J \mid \text{instance } J \text{ of } P\}$$
$$\mathcal{G}_P := \{G_J \cup \langle J\rangle \mid \text{instance } J \text{ of } P\}$$

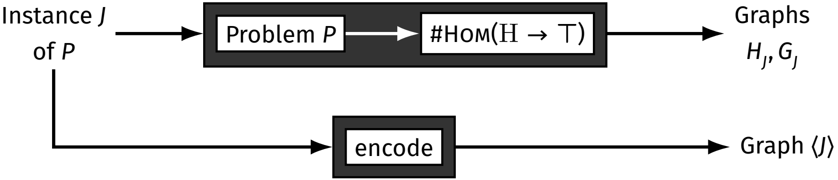
P $\leqslant$ #Hom($\mathcal{H}_P \to \mathcal{G}_P$) ✓
(ensure #Hom($H_J \to \langle J\rangle$) = 0)

#Hom($\mathcal{H}_P \to \mathcal{G}_P$) $\overset{?}{\leqslant}$ P
How do we handle malformed input (H_J, G_L)?

Proof Ideas

Theorem

For any P in #W[1], there are $\mathcal{H}_P$, $\mathcal{G}_P$ such that P is equivalent to #Hom($\mathcal{H}_P \to \mathcal{G}_P$).



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$\#\mathrm{Hom}(\mathcal{H}_P \to \mathcal{G}_P) \overset{?}{\leqslant} \mathrm{P}$

How do we ensure $\#\mathrm{Hom}(H_J \to G_L \cup \langle L \rangle) = 0$?

Theorem

For any P in #W[1], there are H_P, G_P such that P is equivalent to #HOM($\mathrm{H}_P \to \mathrm{G}_P$).

$P \preccurlyeq$ #HOM($\mathrm{H}_P \to \mathrm{G}_P$)	#HOM($\mathrm{H}_P \to \mathrm{G}_P$) $\preccurlyeq P$
Can solve instance J with ($H_J, G_J \cup \langle J \rangle$) by computing #Hom($H_J \to G_J \cup \langle J \rangle$)	Can extract instance J from pair ($H_J, G_J \cup \langle J \rangle$)
(ensuring #Hom($H_J \to \langle J \rangle$) = 0)	How do we ensure #Hom($H_J \to G_L \cup \langle L \rangle$) = 0?

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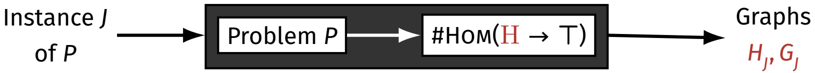

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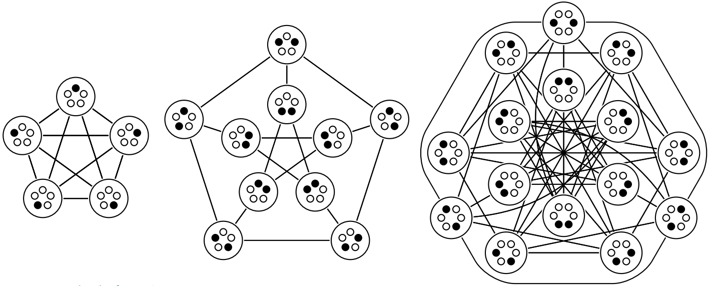

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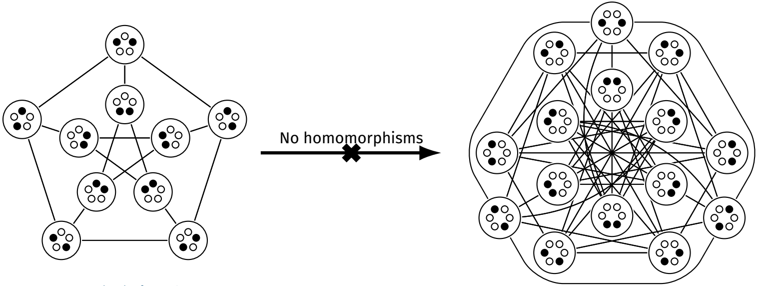

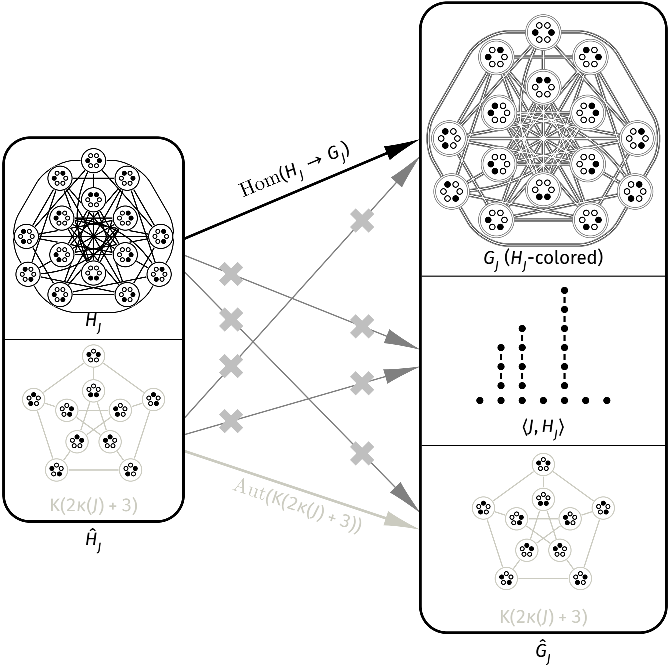

Hom($H_J \to G_J$)
H_J
K(2κ(J) + 3)
$\hat{H}_J$
Aut(K(2κ(J) + 3))
G_J (H_J-colored)
⟨J, H_J⟩
K(2κ(J) + 3)
$\hat{G}_J$

#Hom($\mathbb{H} \to \mathbb{G}$) Parameter: $|V(H)|$

Given graphs $H \in \mathbb{H}$ and $G \in \mathbb{G}$, count all graph homomorphisms from H to G.

Theorem ✓

For any problem P in #W[1] (or W[1]), there are graph classes $\mathbb{H}_P$ and $\mathbb{G}_P$ such that P is equivalent to #Hom($\mathbb{H}_P \to \mathbb{G}_P$) (or Hom($\mathbb{H}_P \to \mathbb{G}_P$)).

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⇝ Need to look at specific pairs of graph classes

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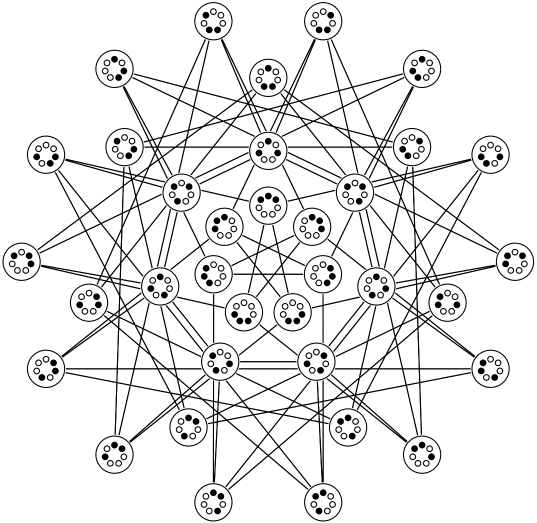

Thank you!

TikZ code for Kneser graphs available on GitHub

`github.com/PH111P/tikz-kneser`

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