Complexity of parameterized problems

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Lecture #4
May 2, 2023
Lower bounds

So far we have seen positive results: basic algorithmic techniques for fixed-parameter tractability.

What kind of negative results we have?

- Can we show that a problem (e.g., CLIQUE) is not FPT?
- Can we show that a problem (e.g., VERTEX COVER) has no algorithm with running time, say, $2^{o(k)} \cdot n^{O(1)}$?
Lower bounds

So far we have seen positive results: basic algorithmic techniques for fixed-parameter tractability.

What kind of negative results we have?

- Can we show that a problem (e.g., \textit{Clique}) is \textbf{not} FPT?
- Can we show that a problem (e.g., \textit{Vertex Cover}) has \textbf{no} algorithm with running time, say, \(2^{o(k)} \cdot n^{O(1)}\)?

This would require showing that \(P \neq NP\): if \(P = NP\), then, e.g., \(k\text{-Clique}\) is polynomial-time solvable, hence FPT.

Can we give some evidence for negative results?
Classical complexity — reminder

NP:
- The class of all languages that can be recognized by a polynomial-time NTM.
- The class of all languages with a witness of polynomial size

Nondeterministic Turing Machine (NTM): single tape, finite alphabet, finite state, head can move left/right only one cell. In each step, the machine can branch into an arbitrary number of directions. Run is successful if at least one branch is successful.

Most problems of interest are in NP: find something that can be checked in polynomial time.
Classical complexity — reminder

**Polynomial-time reduction** from problem $P$ to problem $Q$: a function $\phi$ with the following properties:

- $\phi(x)$ is a yes-instance of $Q$ $\iff$ $x$ is a yes-instance of $P$,
- $\phi(x)$ can be computed in time $|x|^{O(1)}$.

**Fact:** If there is a polynomial-time reduction from $P$ to $Q$ and $Q$ can be solved in polynomial time, then $P$ can be also solved in polynomial time.
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**Definition:** Problem $Q$ is NP-hard if any problem in NP can be reduced to $Q$.

If an NP-hard problem can be solved in polynomial time, then every problem in NP can be solved in polynomial time (i.e., $P = NP$).

**Bottom line:** If $P \neq NP$, then NP-hard problems cannot be solved in polynomial time!
Parameterized complexity

To build a complexity theory for parameterized problems, we need two concepts:

- An appropriate notion of reduction.
- An appropriate hypothesis.

Polynomial-time reductions are not good for our purposes.
Parameterized complexity

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- An appropriate notion of reduction.
- An appropriate hypothesis.

Polynomial-time reductions are not good for our purposes.

**Fact:** Graph $G$ has an independent set $k \Leftrightarrow G$ has a vertex cover of size $n - k$.

**Formally:**

<table>
<thead>
<tr>
<th>Independent Set</th>
<th>Vertex Cover</th>
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<td>$(G, k)$</td>
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- This is a correct polynomial-time reduction.
- However, **Vertex Cover** is FPT, but **Independent Set** is not known to be FPT.
Parameterized reductions

Definition

Parameterized reduction from problem $A$ to problem $B$: a function $\phi$ with the following properties:

- $\phi(x)$ is a yes-instance of $B$ $\iff$ $x$ is a yes-instance of $A$,
- $\phi(x)$ can be computed in time $f(k) \cdot |x|^{O(1)}$, where $k$ is the parameter of $x$,
- If $k$ is the parameter of $x$ and $k'$ is the parameter of $\phi(x)$, then $k' \leq g(k)$ for some function $g$. 
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**Theorem**

If there is a parameterized reduction from problem $A$ to problem $B$ and $B$ is FPT, then $A$ is also FPT.

**Intuitively:** Reduction $A \rightarrow B$ + algorithm for $B$ gives an algorithm for $A$. 
Definition

Parameterized reduction from problem $A$ to problem $B$: a function $\phi$ with the following properties:

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Non-example: Transforming an **Independent Set** instance $(G, k)$ into a **Vertex Cover** instance $(G, n - k)$ is not a parameterized reduction.

Example: Transforming an **Independent Set** instance $(G, k)$ into a **Clique** instance $(\overline{G}, k)$ is a parameterized reduction.
Parameterized reductions

**Theorem**

If there is a parameterized reduction from problem $A$ to problem $B$ and $B$ is FPT, then $A$ is also FPT.

**Proof:** Suppose that

- the reduction has running time $f(k)n^{c_1}$ [we can assume $f$ is monotone],
- the reduction creates an instance with parameter at most $g(k)$, and
- $B$ can be solved in time $h(k)n^{c_2}$.

Then running the reduction and solving the created instance of $B$ gives an algorithm for $A$ with running time

$$f(k)n^{c_1} + h(g(k)) \cdot (f(k)n^{c_1})^{c_2} \leq f'(k)n^{c_1c_2}$$

for some function $f'$. 
**Multicolored Clique**

A useful variant of **Clique**:

**Multicolored Clique**: The vertices of the input graph $G$ are colored with $k$ colors and we have to find a clique containing one vertex from each color.

(or **Partitioned Clique**)

![Diagram of Multicolored Clique]

**Theorem**

There is a parameterized reduction from **Clique** to **Multicolored Clique**.
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Create $G'$ by replacing each vertex $v$ with $k$ vertices, one in each color class. If $u$ and $v$ are adjacent in the original graph, connect all copies of $u$ with all copies of $v$.

$k$-clique in $G$ $\iff$ multicolored $k$-clique in $G'$.  

![Diagram of clique reduction](image-url)
**Multicolored Clique**

**Theorem**

There is a parameterized reduction from *Clique* to *Multicolored Clique*.

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Similarly: reduction to *Multicolored Independent Set*. 

---

![Diagram](image-url)
Theorem

There is a parameterized reduction from **Multicolored Independent Set** to **Dominating Set**.

**Proof:** Let $G$ be a graph with color classes $V_1, \ldots, V_k$. We construct a graph $H$ such that $G$ has a multicolored $k$-independent set iff $H$ has a dominating set of size $k$.

The dominating set has to contain one vertex from each of the $k$ cliques $V_1, \ldots, V_k$ to dominate every $x_i$ and $y_i$. 
There is a parameterized reduction from \textsc{Multicolored Independent Set} to \textsc{Dominating Set}.

\textbf{Proof:} Let $G$ be a graph with color classes $V_1, \ldots, V_k$. We construct a graph $H$ such that $G$ has a multicolored $k$-independent set iff $H$ has a dominating set of size $k$.

- The dominating set has to contain one vertex from each of the $k$ cliques $V_1, \ldots, V_k$ to dominate every $x_i$ and $y_i$.
- For every edge $e = uv$, an additional vertex $w_e$ ensures that these selections describe an independent set.
Variants of Dominating Set

- **Dominating Set**: Given a graph, find $k$ vertices that dominate every vertex.
- **Red-Blue Dominating Set**: Given a bipartite graph, find $k$ vertices on the red side that dominate the blue side.
- **Set Cover**: Given a set system, find $k$ sets whose union covers the universe.
- **Hitting Set**: Given a set system, find $k$ elements that intersect every set in the system.

All of these problems are equivalent under parameterized reductions, hence at least as hard as Clique.
Basic hypotheses

It seems that parameterized complexity theory cannot be built on assuming $P \neq NP$ – we have to assume something stronger.

**Engineers’ Hypothesis**

$k$-CLIQUE cannot be solved in time $f(k) \cdot n^{O(1)}$. 
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**Theorists’ Hypothesis**

$k$-STEP HALTING PROBLEM (is there a path of the given NTM that stops in $k$ steps?) cannot be solved in time $f(k) \cdot n^{O(1)}$.
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Which hypothesis is the most plausible?
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**Exponential Time Hypothesis (ETH)**

$n$-variable 3SAT cannot be solved in time $2^{o(n)}$.

Which hypothesis is the most plausible?
Summary

- **Independent Set** and **k-Step Halting Problem** can be reduced to each other $\Rightarrow$ Engineers’ Hypothesis and Theorists’ Hypothesis are equivalent! (proof omitted)

- **Independent Set** and **k-Step Halting Problem** can be reduced to **Dominating Set**.
Summary

- **Independent Set** and **k-Step Halting Problem** can be reduced to each other $\Rightarrow$ Engineers’ Hypothesis and Theorists’ Hypothesis are equivalent! (proof omitted)

- **Independent Set** and **k-Step Halting Problem** can be reduced to **Dominating Set**.

- Is there a parameterized reduction from **Dominating Set** to **Independent Set**?

  Probably not. Unlike in NP-completeness, where most problems are equivalent, here we have a hierarchy of hard problems.

  - **Independent Set** is $W[1]$-complete.
  - **Dominating Set** is $W[2]$-complete.

- Does not matter if we only care about whether a problem is FPT or not!
Boolean circuit

A **Boolean circuit** consists of input gates, negation gates, AND gates, OR gates, and a single output gate.

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**Circuit Satisfiability**: Given a Boolean circuit $C$, decide if there is an assignment on the inputs of $C$ making the output true.

**Weight of an assignment**: number of true values.

**Weighted Circuit Satisfiability**: Given a Boolean circuit $C$ and an integer $k$, decide if there is an assignment of weight $k$ making the output true.
**Weighted Circuit Satisfiability**

**Independent Set** can be reduced to **Weighted Circuit Satisfiability**:

![Diagram for Independent Set]

**Dominating Set** can be reduced to **Weighted Circuit Satisfiability**:

![Diagram for Dominating Set]
**Weighted Circuit Satisfiability**

**Independent Set** can be reduced to **Weighted Circuit Satisfiability**:

\[
\begin{align*}
&x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_6 \quad x_7 \\
&\text{\small \textbullet} \quad \text{\small \textbullet} \quad \text{\small \textbullet} \quad \text{\small \textbullet} \quad \text{\small \textbullet} \quad \text{\small \textbullet} \quad \text{\small \textbullet}
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To express **Dominating Set**, we need more complicated circuits.
Depth and weft

The **depth** of a circuit is the maximum length of a path from an input to the output. A gate is **large** if it has more than 2 inputs. The **weft** of a circuit is the maximum number of large gates on a path from an input to the output.

**Independent Set:** weft 1, depth 3

**Dominating Set:** weft 2, depth 2
The W-hierarchy

Let $C[t, d]$ be the set of all circuits having width at most $t$ and depth at most $d$.

**Definition**

A problem $P$ is in the class $W[t]$ if there is a constant $d$ and a parameterized reduction from $P$ to Weighted Circuit Satisfiability of $C[t, d]$.

We have seen that Independent Set is in $W[1]$ and Dominating Set is in $W[2]$.

**Fact:** Independent Set is $W[1]$-complete.

**Fact:** Dominating Set is $W[2]$-complete.
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**Fact:** **Independent Set** is $W[1]$-complete.

**Fact:** **Dominating Set** is $W[2]$-complete.

If any $W[1]$-complete problem is FPT, then $FPT = W[1]$ and every problem in $W[1]$ is FPT.


$\Rightarrow$ If there is a parameterized reduction from **Dominating Set** to **Independent Set**, then $W[1] = W[2]$. 
Weft is a term related to weaving cloth: it is the thread that runs from side to side in the fabric.
Parameterized reductions

Typical NP-hardness proofs: reduction from e.g., **Clique** or **3SAT**, representing each vertex/edge-variable/clause with a gadget.

![Graph diagram]

Usually doesn’t work for parameterized reduction: cannot afford the parameter increase.
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```
  v_1  v_2  v_3  v_4  v_5  v_6
```

```
  C_1  C_2  C_3  C_4
```

Usually doesn’t work for parameterized reduction: cannot afford the parameter increase.

Types of parameterized reductions:

- Reductions keeping the structure of the graph.
  - CLIQUE ⇒ INDEPENDENT SET

- Reductions with vertex representations.
  - MULTICOLORED INDEPENDENT SET ⇒ DOMINATING SET

- Reductions with vertex and edge representations.
**Odd Set**

**Odd Set**: Given a set system $\mathcal{F}$ over a universe $U$ and an integer $k$, find a set $S$ of at most $k$ elements such that $|S \cap F|$ is odd for every $F \in \mathcal{F}$.

**Theorem**

**Odd Set** is $W[1]$-hard parameterized by $k$. 
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**First try:** Reduction from *Multicolored Independent Set*. Let \(U = V_1 \cup \ldots \cup V_k\) and introduce each set \(V_i\) into \(\mathcal{F}\).

⇒ The solution has to contain exactly one element from each \(V_i\).

If \(xy \in E(G)\), how can we express that \(x \in V_i\) and \(y \in V_j\) cannot be selected simultaneously?
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If \( xy \in E(G) \), how can we express that \( x \in V_i \) and \( y \in V_j \) cannot be selected simultaneously? Seems difficult:

- introducing \( \{x, y\} \) into \( \mathcal{F} \) forces that exactly one of \( x \) and \( y \) appears in the solution,
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If $xy \in E(G)$, how can we express that $x \in V_i$ and $y \in V_j$ cannot be selected simultaneously? Seems difficult:

- introducing $\{x, y\}$ into $F$ forces that **exactly one** of $x$ and $y$ appears in the solution,
- introducing $\{x\} \cup (V_j \setminus \{y\})$ into $F$ forces that either **both** $x$ and $y$ or **none** of $x$ and $y$ appear in the solution.
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Odd Set

Reduction from **Multicolored Clique**.

- \( U := \bigcup_{i=1}^{k} V_i \cup \bigcup_{1 \leq i < j \leq k} E_{i,j} \).
- \( k' := k + \binom{k}{2} \).
- Let \( \mathcal{F} \) contain \( V_i \ (1 \leq i \leq k) \) and \( E_{i,j} \ (1 \leq i < j \leq k) \).
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- For every \( v \in V_i \) and \( x \neq i \), we introduce the sets:
  - \( (V_i \setminus \{v\}) \cup \{\text{every edge from } E_{i,x} \text{ with endpoint } v\} \)
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- $v \in V_i$ selected $\iff$ edges with endpoint $v$ are selected from $E_{i,x}$ and $E_{x,i}$
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- \( v_i \in V_i \) selected \( \iff \) edge \( v_i v_j \) is selected in \( E_{i,x} \)
Vertex and edge representation

Key idea

- Represent the vertices of the clique by \( k \) gadgets.
- Represent the edges of the clique by \( \binom{k}{2} \) gadgets.
- Connect edge gadget \( E_{i,j} \) to vertex gadgets \( V_i \) and \( V_j \) such that if \( E_{i,j} \) represents the edge between \( x \in V_i \) and \( y \in V_j \), then it forces \( V_i \) to \( x \) and \( V_j \) to \( y \).
Variants of Hitting Set

The following problems are $W[1]$-hard, with very similar proofs:

- **Odd Set**
- **Exact Odd Set** (find a set of size exactly $k$ . . .)
- **Exact Even Set**
- **Unique Hitting Set**
  (at most $k$ elements that hit each set exactly once)
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A problem that is also W[1]-hard, but requires very different techniques:

- **Even Set**: Given a set system $\mathcal{F}$ and an integer $k$, find a nonempty set $S$ of at most $k$ elements such that $|F \cap S|$ is even for every $F \in \mathcal{F}$. 


Summary

- By parameterized reductions, we can show that lots of parameterized problems are at least as hard as \textit{Clique}, hence unlikely to be fixed-parameter tractable.
- Connection with Turing machines gives some supporting evidence for hardness (only of theoretical interest).
- The \textit{W}-hierarchy classifies the problems according to hardness (only of theoretical interest).
- Important trick in \textit{W[1]}-hardness proofs: vertex and edge representations.
Shift of focus

FPT or W[1]-hard?
Shift of focus

What is the best possible multiplier $f(k)$ in the running time $f(k) \cdot n^{O(1)}$?

- $2^k$
- $1.0001^k$
- $2^{\sqrt{k}}$

What is the best possible exponent $g(k)$ in the running time $f(k) \cdot n^{g(k)}$?

- $n^{O(k)}$
- $n^{\log k}$
- $n^{\log \log k}$
Better algorithms for Vertex Cover

- We have seen a $2^k \cdot n^{O(1)}$ time algorithm.
- Easy to improve to, e.g., $1.4656^k \cdot n^{O(1)}$.
- Current best $f(k)$: $1.2738^k \cdot n^{O(1)}$.
- Lower bounds?
  - Is, say, $1.001^k \cdot n^{O(1)}$ time possible?
  - Is $2^{k/\log k} \cdot n^{O(1)}$ time possible?
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- Lower bounds?
  - Is, say, $1.001^k \cdot n^{O(1)}$ time possible?
  - Is $2^k/\log k \cdot n^{O(1)}$ time possible?

Of course, for all we know, it is possible that $P = NP$ and Vertex Cover is polynomial-time solvable.

⇒ We can hope only for conditional lower bounds.
Exponential Time Hypothesis (ETH)

3CNF: $\phi$ is a conjunction of clauses, where each clause is a disjunction of at most 3 literals (= a variable or its negation), e.g., $(x_1 \lor x_3 \lor \overline{x}_4) \land (\overline{x}_2 \lor \overline{x}_3) \lor (x_1 \lor x_2 \lor x_4)$.

3SAT: given a 3CNF formula $\phi$ with $n$ variables and $m$ clauses, decide whether $\phi$ is satisfiable.

- Current best algorithm is $1.30704^n$ [Hertli 2011].
- Can we do significantly better, e.g., $2^{O(n/\log n)}$?
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Hypothesis introduced by Impagliazzo, Paturi, and Zane in 2001:

Exponential Time Hypothesis (ETH) [consequence of]

There is no $2^{o(n)}$-time algorithm for $n$-variable 3SAT.
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- Can we do significantly better, e.g., \( 2^{O(n/\log n)} \)?

Hypothesis introduced by Impagliazzo, Paturi, and Zane in 2001:

Exponential Time Hypothesis (ETH) [real statement]

There is a constant \( \delta > 0 \) such that there is no \( O(2^{\delta n}) \) time algorithm for 3SAT.
Sparsification

Exponential Time Hypothesis (ETH) [consequence of]

There is no $2^{o(n)}$-time algorithm for $n$-variable 3SAT.

Observe: an $n$-variable 3SAT formula can have $m = \Omega(n^3)$ clauses.

Are there algorithms that are subexponential in the size $n + m$ of the 3SAT formula?
### Sparsification

**Exponential Time Hypothesis (ETH) [consequence of]**

There is no $2^{o(n)}$-time algorithm for $n$-variable 3SAT.

**Observe:** an $n$-variable 3SAT formula can have $m = \Omega(n^3)$ clauses.

Are there algorithms that are subexponential in the size $n + m$ of the 3SAT formula?

**Sparsification Lemma**

<table>
<thead>
<tr>
<th>$2^{o(n)}$-time algorithm for $n$-variable 3SAT.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\uparrow$</td>
</tr>
<tr>
<td>There is a $2^{o(n+m)}$-time algorithm for $n$-variable $m$-clause 3SAT.</td>
</tr>
</tbody>
</table>

**Intuitively:** When considering a hard 3SAT instance, we can assume that it has $m = O(n)$ clauses.
Lower bounds based on ETH

**Exponential Time Hypothesis (ETH) + Sparsification Lemma**

There is no $2^{o(n+m)}$-time algorithm for $n$-variable $m$-clause 3SAT.

The textbook reduction from 3SAT to **Vertex Cover**:

```
  x₁  x₂  x₃  x₄

  x₁  x₂  x₃  x₄

  x₁  x₂  x₃  x₄
```

```
  △  △  △  △  △  △
```
Lower bounds based on ETH

Exponential Time Hypothesis (ETH) + Sparsification Lemma

There is no $2^{o(n+m)}$-time algorithm for $n$-variable $m$-clause 3SAT.

The textbook reduction from 3SAT to Vertex Cover:

formula is satisfiable $\iff$ there is a vertex cover of size $n + 2m$
Lower bounds based on ETH

**Exponential Time Hypothesis (ETH) + Sparsification Lemma**

There is no $2^{o(n+m)}$-time algorithm for $n$-variable $m$-clause 3SAT.

The textbook reduction from 3SAT to **Vertex Cover**:

<table>
<thead>
<tr>
<th>3SAT formula $\phi$</th>
<th>Graph $G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$ variables</td>
<td>$O(n + m)$ vertices</td>
</tr>
<tr>
<td>$m$ clauses</td>
<td>$O(n + m)$ edges</td>
</tr>
</tbody>
</table>

![Graph representation](image)
Lower bounds based on ETH

**Exponential Time Hypothesis (ETH) + Sparsification Lemma**

There is no $2^{o(n+m)}$-time algorithm for $n$-variable $m$-clause $3SAT$.

The textbook reduction from $3SAT$ to $VERTEX$ $COVER$:

- **3SAT formula $\phi$**
  - $n$ variables
  - $m$ clauses

  $\Rightarrow$

- **Graph $G$**
  - $O(n + m)$ vertices
  - $O(n + m)$ edges

**Corollary**

Assuming ETH, there is no $2^{o(n)}$ algorithm for $VERTEX$ $COVER$ on an $n$-vertex graph.
Lower bounds based on ETH

Exponential Time Hypothesis (ETH) + Sparsification Lemma

There is no $2^{o(n+m)}$-time algorithm for $n$-variable $m$-clause 3SAT.

The textbook reduction from 3SAT to Vertex Cover:

3SAT formula $\phi$
- $n$ variables
- $m$ clauses

$\Rightarrow$
Graph $G$
- $O(n + m)$ vertices
- $O(n + m)$ edges

Corollary
Assuming ETH, there is no $2^{o(k)} \cdot n^{O(1)}$ algorithm for Vertex Cover.
Other problems

There are polytime reductions from 3SAT to many problems such that the reduction creates a graph with $O(n + m)$ vertices/edges.

Consequence: Assuming ETH, the following problems cannot be solved in time $2^{o(n)}$ and hence in time $2^{o(k)} \cdot n^{O(1)}$ (but $2^{O(k)} \cdot n^{O(1)}$ time algorithms are known):

- **Vertex Cover**
- **Longest Cycle**
- **Feedback Vertex Set**
- **Multiway Cut**
- **Odd Cycle Transversal**
- **Steiner Tree**
- ...

Seems to be the natural behavior of FPT problems?
The race for better FPT algorithms

- Single exponential
- Subexponential
- Double exponential
- "Slightly super-exponential"
- Tower of exponentials

- $2^k$
- $2^{k^{1.2}}$
- $2^{k^{1.6}}$
- $2^{k^{2}}$
- $2^{k^{O(1)}}$
- $2^{O(k)}$
- $2^{O(k^{1/3})}$
- $2^{O(k^{1/2})}$
- $2^{O(k^{1})}$
- $2^{O(k \log k)}$
- $2^{O(k \log k \log \log k)}$
- $f(k)$
**Edge Clique Cover**

**Edge Clique Cover**: Given a graph $G$ and an integer $k$, cover the edges of $G$ with at most $k$ cliques. (the cliques need not be edge disjoint)

**Equivalently**: can $G$ be represented as an intersection graph over a $k$ element universe?
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(the cliques need not be edge disjoint)

**Simple algorithm (sketch)**

- If two adjacent vertices have the same neighborhood ("twins"), then remove one of them.
- If there are no twins and isolated vertices, then $|V(G)| > 2^k$ implies that there is no solution.
- Use brute force.

Running time: $2^{2^{O(k)}} \cdot n^{O(1)}$ — double exponential dependence on $k$!
**Edge Clique Cover**

**Problem:** Given a graph $G$ and an integer $k$, cover the edges of $G$ with at most $k$ cliques.

(the cliques need not be edge disjoint)

Double-exponential dependence on $k$ cannot be avoided!

**Theorem**

Assuming ETH, there is no $2^{o(k)} \cdot n^{O(1)}$ time algorithm for **Edge Clique Cover**.

**Proof:**

3SAT

$n$ variables

$\Rightarrow$

**Edge Clique Cover**

$k = O(\log n)$
The race for better FPT algorithms

Single exponential

"Slightly super-exponential"

Subexponential

Tower of exponentials

Double exponential
Slightly superexponential algorithms

Running time of the form $2^{O(k \log k)} \cdot n^{O(1)}$ appear naturally in parameterized algorithms usually because of one of two reasons:

1. Branching into $k$ directions at most $k$ times explores a search tree of size $k^k = 2^{O(k \log k)}$.
   
   **Example:** Feedback Vertex Set in the first lecture.

2. Trying $k! = 2^{O(k \log k)}$ permutations of $k$ elements (or partitions, matchings, ...)

Can we avoid these steps and obtain $2^{O(k)} \cdot n^{O(1)}$ time algorithms?
Given strings $s_1, \ldots, s_k$ of length $L$ over alphabet $\Sigma$, and an integer $d$, find a string $s$ (of length $L$) such that Hamming distance $d(s, s_i) \leq d$ for every $1 \leq i \leq k$.

(Hamming distance: number of differing positions)

\[
\begin{array}{cccccccc}
  s_1 & C & B & D & C & C & A & C & B & B \\
  s_2 & A & B & D & B & C & A & B & D & B \\
  s_3 & C & D & D & B & A & C & C & B & D \\
  s_4 & D & D & A & B & A & C & C & B & D \\
  s_5 & A & C & D & B & D & D & C & B & C \\
\end{array}
\]
Closest String

Given strings $s_1, \ldots, s_k$ of length $L$ over alphabet $\Sigma$, and an integer $d$, find a string $s$ (of length $L$) such that Hamming distance $d(s, s_i) \leq d$ for every $1 \leq i \leq k$.

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Different parameters:
- Number $k$ of strings.
- Length $L$ of strings
- Maximum distance $d$.
- Alphabet size $|\Sigma|$.
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(Hamming distance: number of differing positions)

Different parameters:
- Number $k$ of strings.
- Length $L$ of strings
- Maximum distance $d$.
- Alphabet size $|\Sigma|$.

We can ask for running time for example

- $f(d)n^{O(1)}$: FPT parameterized by $d$
- $f(k, |\Sigma|)n^{O(1)}$: FPT with combined parameters $k$ and $|\Sigma|$
Closest String

**Theorem**

**Closest String** can be solved in time \(2^{O(d \log d)} n^{O(1)}\).

- **Main idea:** Given a string \(y\) at Hamming distance \(\ell\) from some solution, we use branching to find a string at distance at most \(\ell - 1\) from some solution.
- Initially, \(y = x_1\) is at distance at most \(d\) from some solution.
Theorem

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- **Main idea:** Given a string $y$ at Hamming distance $\ell$ from some solution, we use branching to find a string at distance at most $\ell - 1$ from some solution.
- Initially, $y = x_1$ is at distance at most $d$ from some solution.
- If $y$ is not a solution, then there is an $x_i$ with $d(y, x_i) \geq d + 1$.
  - Look at the first $d + 1$ positions $p$ where $x_i[p] \neq y[p]$. For every solution $z$, it is true for one such $p$ that $x_i[p] = z[p]$.
  - Branch on choosing one of these $d + 1$ positions and replace $y[p]$ with $x_i[p]$:
    - distance of $y$ from solution $z$ decreases to $\ell - 1$.
- Running time $(d + 1)^d \cdot n^{O(1)} = 2^{O(d \log d)} n^{O(1)}$. 
Theorem

Assuming ETH, **Closest String** has no $2^{o(d \log d)} n^{O(1)}$ algorithm.

Proof:

3SAT
$O(d \log d)$ variables

$\Rightarrow$

**Closest String**
distance $d$
Shift of focus

**FPT or W[1]-hard?**

**Qualitative question**

**Quantitative question**

What is the best possible multiplier \( f(k) \) in the running time \( f(k) \cdot n^{O(1)} \)?

What is the best possible exponent \( g(k) \) in the running time \( f(k) \cdot n^{g(k)} \)?

\( 2^k \), \( 1.0001^k \), \( 2^{\sqrt{k}} \), \( n^{O(k)} \), \( n^{\log k} \), \( n^{\log \log k} \)
Better algorithms for \( W[1] \)-hard problems

- \( O(n^k) \) algorithm for \( k\text{-}\text{CLIQUE} \) by brute force.
- \( O(n^{0.79k}) \) algorithms using fast matrix multiplication.
- \( W[1] \)-hardness of \( k\text{-}\text{CLIQUE} \) gives evidence that there is no \( f(k) \cdot n^{O(1)} \) time algorithm.
- But what about improvements of the exponent \( O(k) \)?
Better algorithms for W[1]-hard problems

- \(O(n^k)\) algorithm for \(k\)-CLIQUE by brute force.
- \(O(n^{0.79k})\) algorithms using fast matrix multiplication.
- W[1]-hardness of \(k\)-CLIQUE gives evidence that there is no \(f(k) \cdot n^{O(1)}\) time algorithm.
- But what about improvements of the exponent \(O(k)\)?

Theorem

Assuming ETH, \(k\)-CLIQUE has no \(f(k) \cdot n^{o(k)}\) algorithm for any computable function \(f\).

In particular, ETH implies that \(k\)-CLIQUE is not FPT.
## Basic hypotheses

**Engineers’ Hypothesis**

$k$-CLIQUE cannot be solved in time $f(k) \cdot n^{O(1)}$.

**Theorists’ Hypothesis**

$k$-STEP HALTING PROBLEM (is there a path of the given NTM that stops in $k$ steps?) cannot be solved in time $f(k) \cdot n^{O(1)}$.

**Exponential Time Hypothesis (ETH)**

$n$-variable 3SAT cannot be solved in time $2^{o(n)}$. 
Lower bound for $k$-CLIQUE

Theorem
Assuming ETH, $k$-CLIQUE has no $f(k) \cdot N^{o(k)}$ algorithm for any computable function $f$.

Proof:
Textbook reduction from 3SAT to 3-COLORING shows that, assuming ETH, there is no $2^{o(n)}$ time algorithm for 3-COLORING on an $n$-vertex graph. Then

$$N^{o(k)} \text{ algorithm for } \text{CLIQUE} \Rightarrow (3^{n/k})^{o(k)} = 3^{o(n)} = 2^{o(n)} \text{ algorithm for } 3\text{-COLORING}$$
Lower bound for $k$-CLIQUE

**Theorem**

Assuming ETH, $k$-CLIQUE has no $f(k) \cdot N^{o(k)}$ algorithm for any computable function $f$.

**Proof:**

Create a vertex per each consistent coloring of each group.
Lower bound for $k$-CLIQUE

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Assuming ETH, $k$-CLIQUE has no $f(k) \cdot N^{o(k)}$ algorithm for any computable function $f$.

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Lower bound for \textit{k-Clique}

**Theorem**

Assuming ETH, \textit{k-Clique} has no $f(k) \cdot N^{o(k)}$ algorithm for any computable function $f$.

**Proof:**

Create a vertex per each consistent coloring of each group.
Lower bound for $k$-CLIQUE

**Theorem**
Assuming ETH, $k$-CLIQUE has no $f(k) \cdot N^{o(k)}$ algorithm for any computable function $f$.

**Proof:**

Connect two vertices if they represent colorings that are consistent together.

$\leq 3^{n/k}$
Lower bound for $k$-CLIQUE

**Theorem**

Assuming ETH, $k$-CLIQUE has no $f(k) \cdot N^{o(k)}$ algorithm for any computable function $f$.

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Lower bound for $k$-CLIQUE

**Theorem**

Assuming ETH, $k$-CLIQUE has no $f(k) \cdot N^{o(k)}$ algorithm for any computable function $f$.

**Proof:**

Left graph has a 3-coloring $\iff$ Right graph contains a $k$-clique

$\leq 3^{n/k}$
Theorem

Assuming ETH, \textit{k-Clique} has no \( f(k) \cdot N^{o(k)} \) algorithm for any computable function \( f \).

Proof:

- We have constructed a new graph with \( N = k \cdot 3^{n/k} \) vertices that has a \( k \)-clique if and only if the original graph is 3-colorable.
- Suppose that \textit{k-Clique} has a \( 2^k \cdot N^{o(k)} \) time algorithm.
- Doing the reduction with \( k := \log n \) gives us an algorithm for \textit{3-Coloring} with running time

\[
2^k \cdot N^{o(k)} = n \cdot (\log n)^{o(\log n)} \cdot 3^{n \cdot o(\log n) / \log n} = 2^{o(n)}.
\]
Lower bound for $k$-CLIQUE

**Theorem**

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$$2^k \cdot N^{o(k)} = n \cdot (\log n)^{o(\log n)} \cdot 3^{n \cdot o(\log n) / \log n} = 2^{o(n)}.$$

- Choosing $k := \log \log n$ would rule out a $2^{2^k} \cdot N^{o(k)}$ algorithm etc.
- In general, we need to choose roughly $k := f^{-1}(n)$ groups (technicalities omitted).
Tight bounds

**Theorem**
Assuming ETH, \( k\text{-CLIQUE} \) has no \( f(k) \cdot n^{o(k)} \) algorithm for any computable function \( f \).

**Transferring to other problems:**

\[
\begin{align*}
\text{\( k\text{-CLIQUE} \)} & \quad \Rightarrow \\
(x, k) & \quad \Rightarrow \\
\text{Problem A} & \\
(x', O(k)) & \\
\end{align*}
\]

\[
\begin{align*}
\text{\( f(k) \cdot n^{o(k)} \) algorithm} & \quad \Leftarrow \\
\text{\( f(k) \cdot n^{o(k)} \) algorithm} & \\
\end{align*}
\]

Bottom line: To rule out \( f(k) \cdot n^{o(k)} \) algorithms, we need a parameterized reduction that blows up the parameter at most linearly. To rule out \( f(k) \cdot n^{o(\sqrt{k})} \) algorithms, we need a parameterized reduction that blows up the parameter at most quadratically.
Tight bounds

**Theorem**

Assuming ETH, *k*-CLIQUE has no $f(k) \cdot n^{o(k)}$ algorithm for any computable function $f$.

**Transferring to other problems:**

\[
\begin{align*}
\text{k-CLIQUE} & \quad (x, k) \\
f(k) \cdot n^{o(k)} & \quad \text{algorithm} \\
\Rightarrow & \\
\text{Problem A} & \quad (x', k^2) \\
\leftarrow & \\
f(k) \cdot n^{o(\sqrt{k})} & \quad \text{algorithm}
\end{align*}
\]
Tight bounds

Theorem

Assuming ETH, \( k\text{-CLIQUE} \) has no \( f(k) \cdot n^{o(k)} \) algorithm for any computable function \( f \).

Transferring to other problems:

\[
\begin{align*}
\text{\( k\text{-CLIQUE} \)} & \quad \Rightarrow \\
(x, k) & \quad \Rightarrow \\
\text{Problem } A & \\
(x', g(k)) & \\
\text{\( f(k) \cdot n^{o(k)} \)} & \\
\text{algorithm} & \\
\Leftrightarrow \\
\Rightarrow & \\
\text{\( f(k) \cdot n^{o(g^{-1}(k))} \)} & \\
\text{algorithm}
\end{align*}
\]
Tight bounds

Theorem

Assuming ETH, $k$-\textsc{CLIQUE} has no $f(k) \cdot n^{o(k)}$ algorithm for any computable function $f$.

Transfering to other problems:

\[
\begin{align*}
\text{k-CLIQUE} & \quad \Rightarrow \quad \text{Problem A} \\
(x, k) & \quad \Rightarrow \quad (x', g(k)) \\
f(k) \cdot n^{o(k)} & \quad \iff \quad f(k) \cdot n^{o(g^{-1}(k))}
\end{align*}
\]

Bottom line:
- To rule out $f(k) \cdot n^{o(k)}$ algorithms, we need a parameterized reduction that blows up the parameter at most \emph{linearly}.
- To rule out $f(k) \cdot n^{o(\sqrt{k})}$ algorithms, we need a parameterized reduction that blows up the parameter at most \emph{quadratically}.

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Tight bounds

Assuming ETH, there is no $f(k)n^{o(k)}$ time algorithms for

- **Set Cover**
- **Hitting Set**
- **Connected Dominating Set**
- **Independent Dominating Set**
- **Partial Vertex Cover**
- **Dominating Set** in bipartite graphs
- ...
Parameterized reductions from \textsc{Clique} or \textsc{Independent Set} can give evidence that a problem is not FPT.

ETH can give tight bounds on the $f(k)$ for FPT problems.

ETH can give tight bounds on the exponent of $n$ for W[1]-hard problems.