Topics in Computational Social Choice Theory

Lecture 01: Introduction on Discrete Fair Division

Hannaneh Akrami

max planck institut informatik

Saarland Informatics Campus

Organization

Seminar: 2+0, 7 CPS

Organized by Kurt Mehlhorn, Nidhi Rathi, and Hannaneh Akrami

When? Every Tuesday 14:15 - 15:45

Requirements: Basic algorithms lecture (Introduction to Algorithms and Data Structures)

Your task:

- Present a paper from the list in 60-85 minutes.
- Write a summary of the paper by August 2nd.
- The presentation needs to be discussed with us at least one week before your scheduled talk.
- Send us your preferred order of the papers by April 30th.

Computational Social Choice Theory

Social Choice Theory: Making a collective desicion from individual preferences.

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Voting

Computational Social Choice Theory

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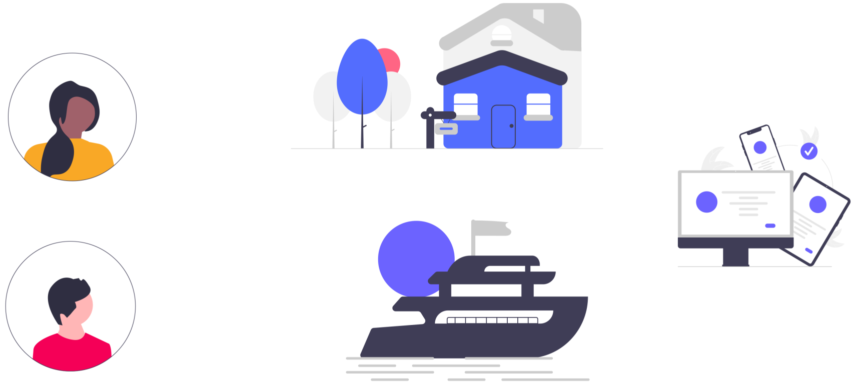
Voting



Resource Allocation

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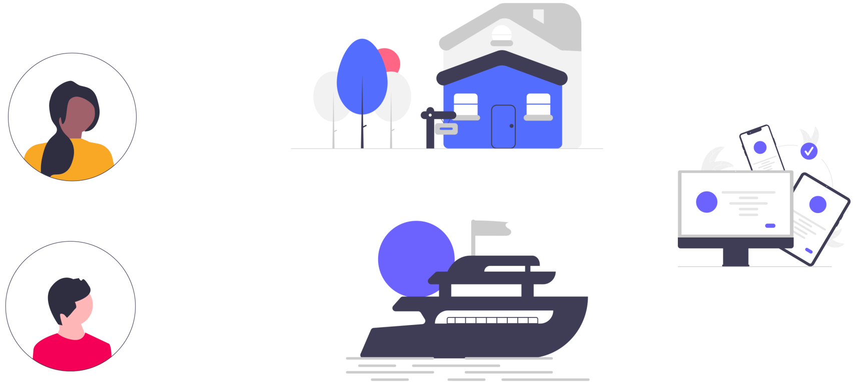

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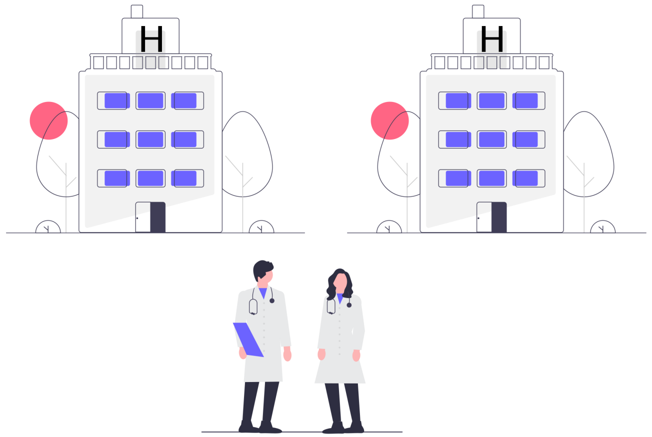

Stable Matchings

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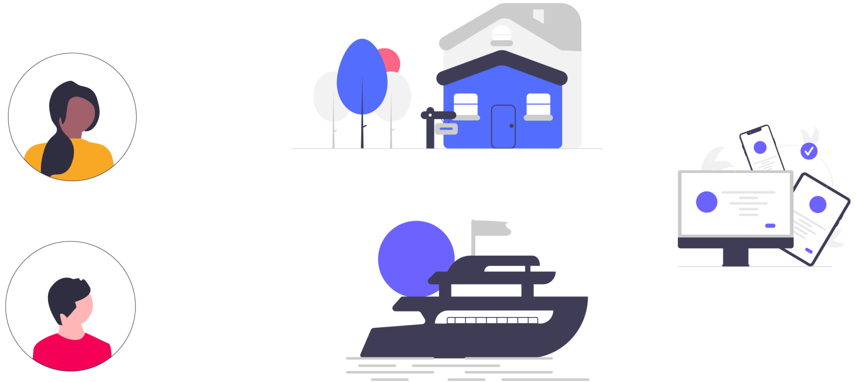

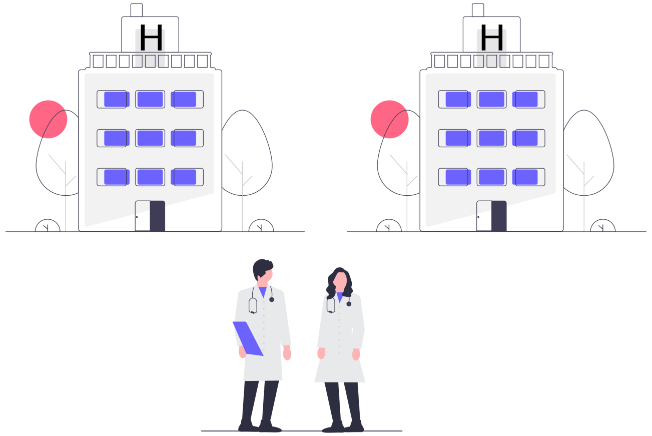

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Economists and Politicians: Does there exists a **social choice** mechanism with the desired economic properties?

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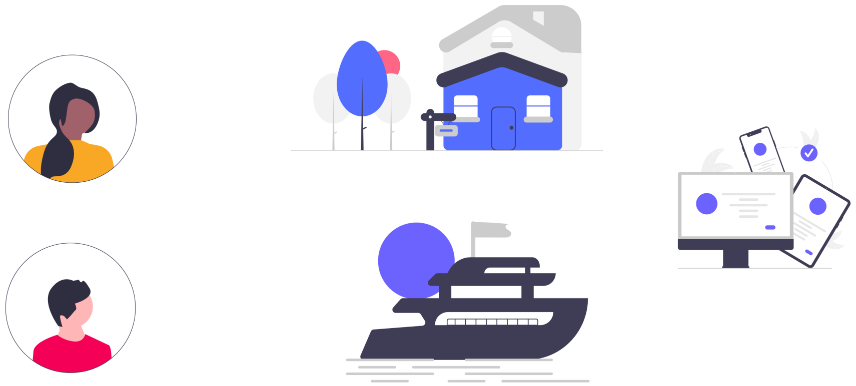

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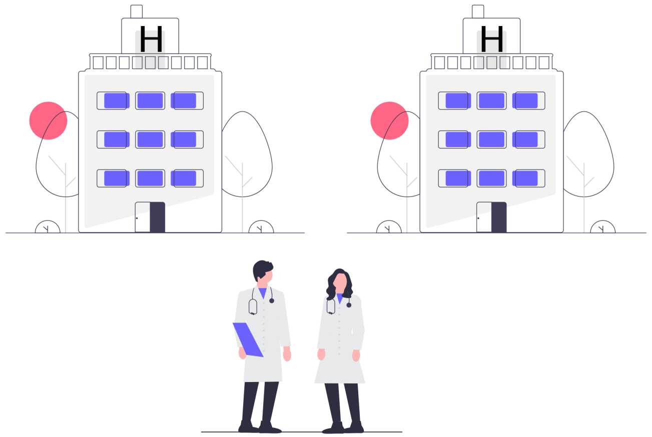

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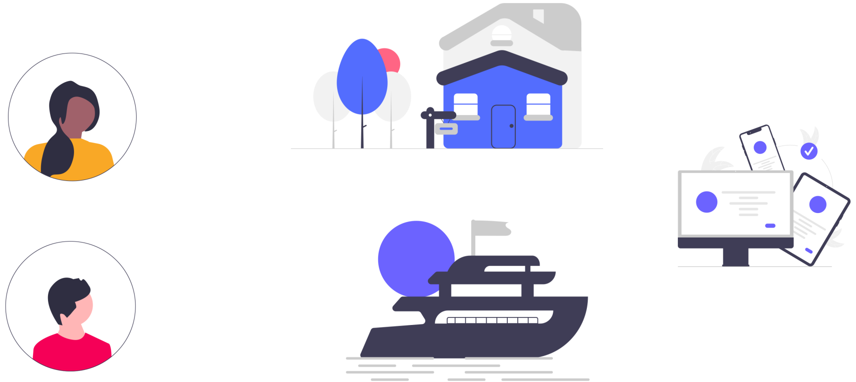

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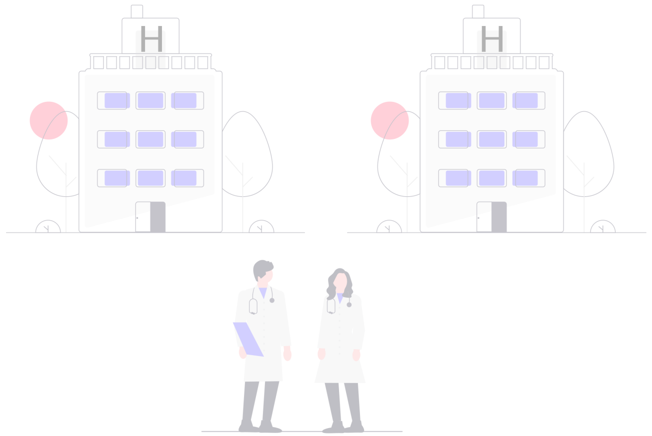

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Fair Division

Divide **items** among **agents** in a **fair** manner.

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Applications:

Partnership dissolution

Divorce settlements

Household chores

Air traffic management

Items

- Desirable
- Undesirable

Items

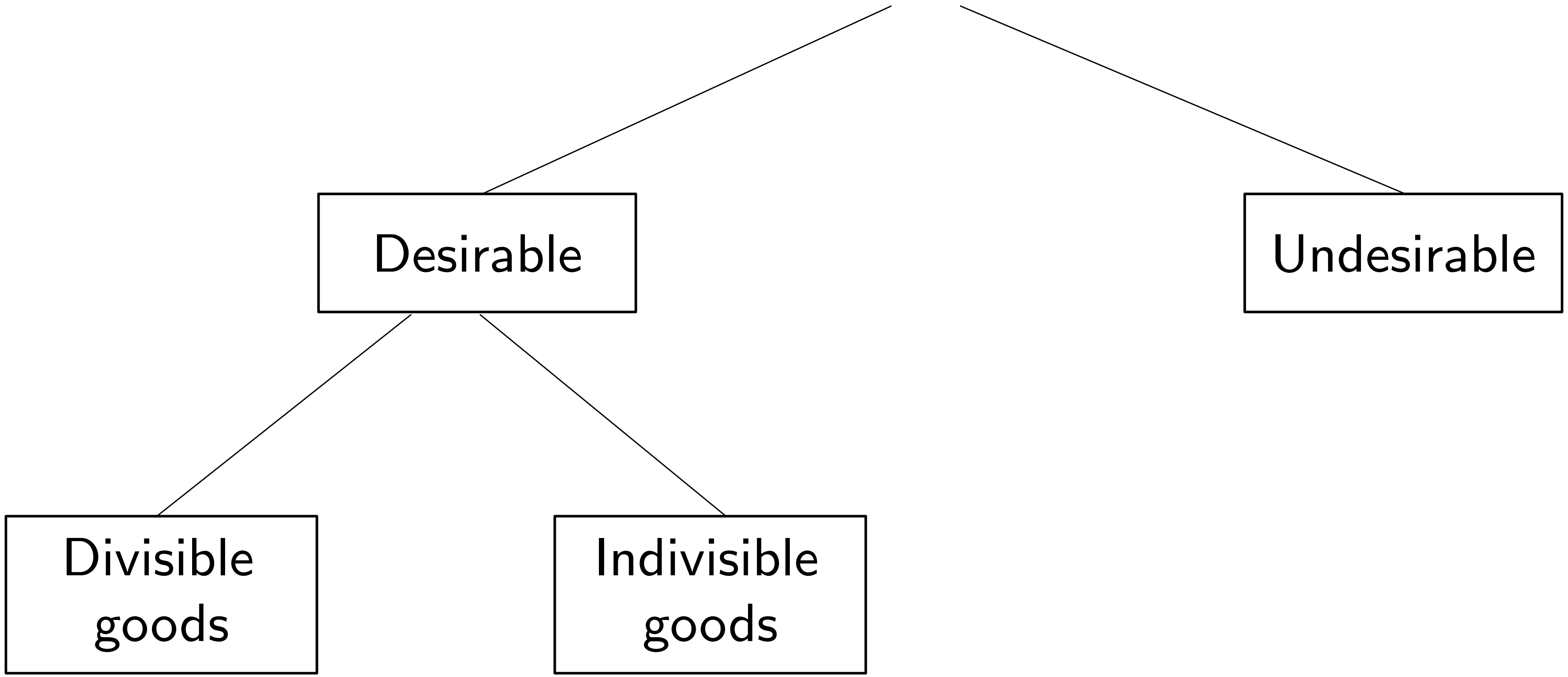
Desirable

Undesirable

Divorce settlements

Household chores

Items


Desirable
Undesirable
Divisible goods
Indivisible goods

Items

Desirable

Undesirable

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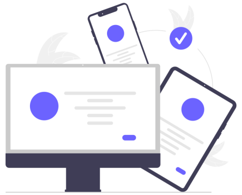

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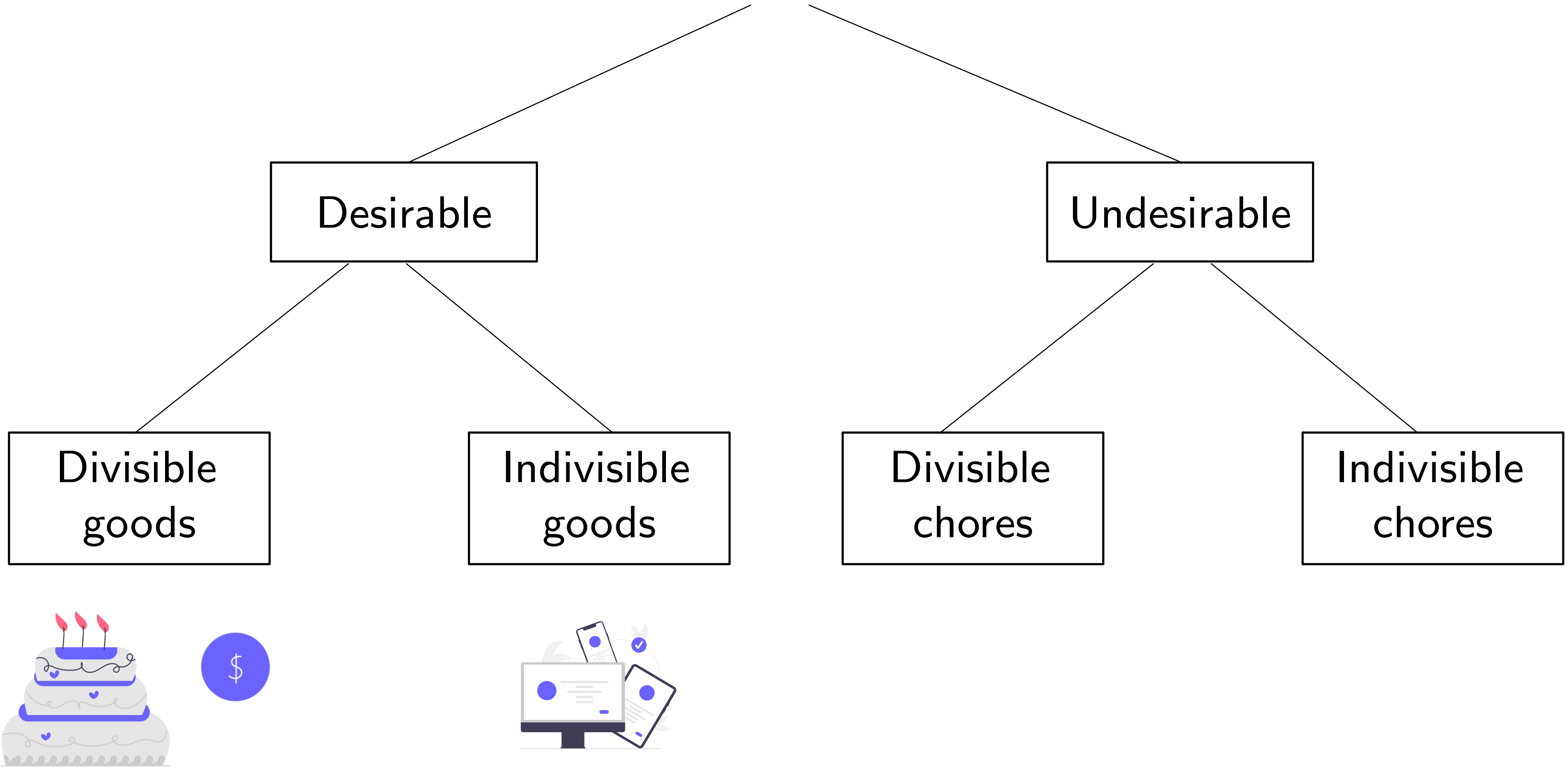
Desirable
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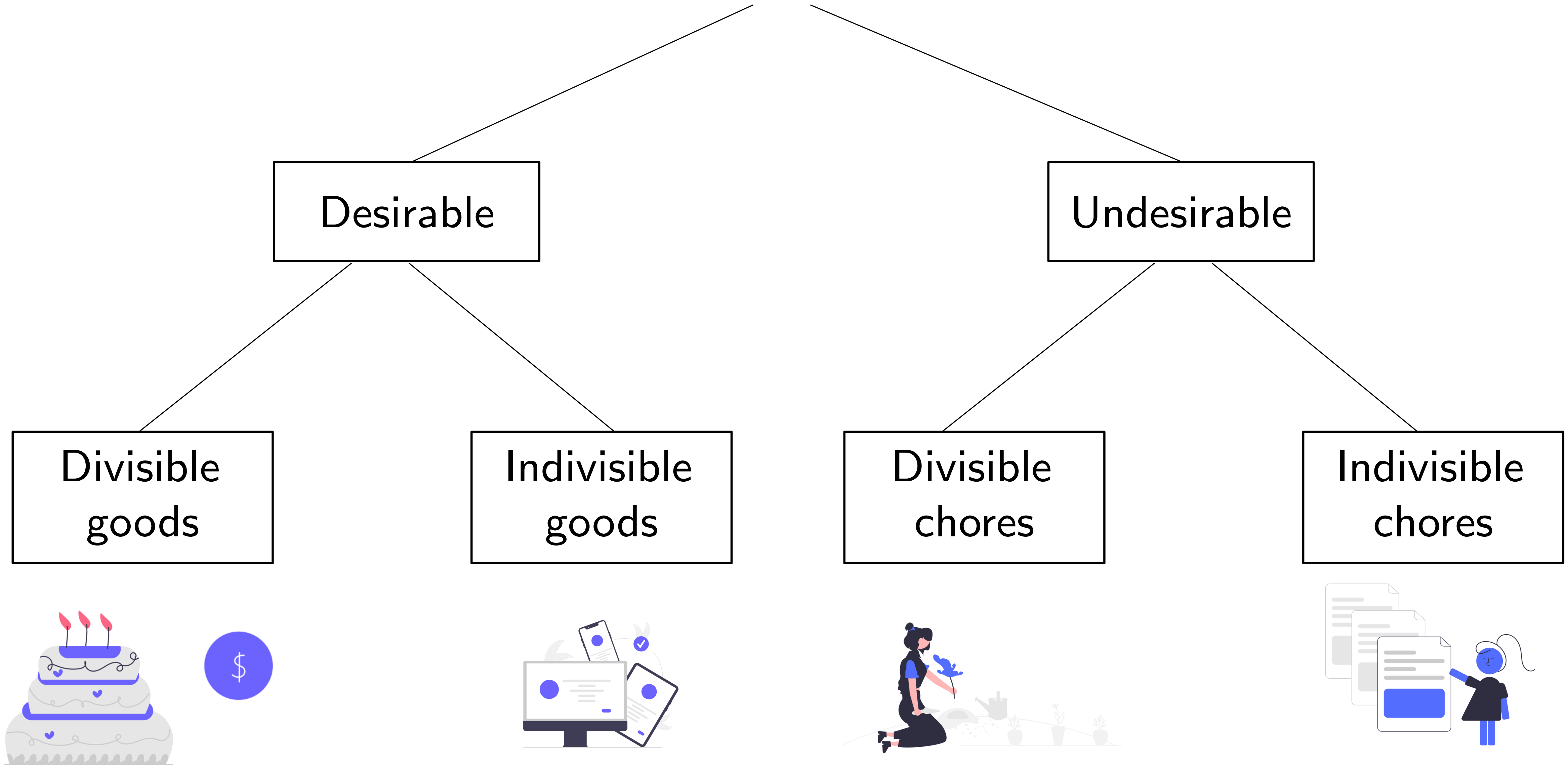
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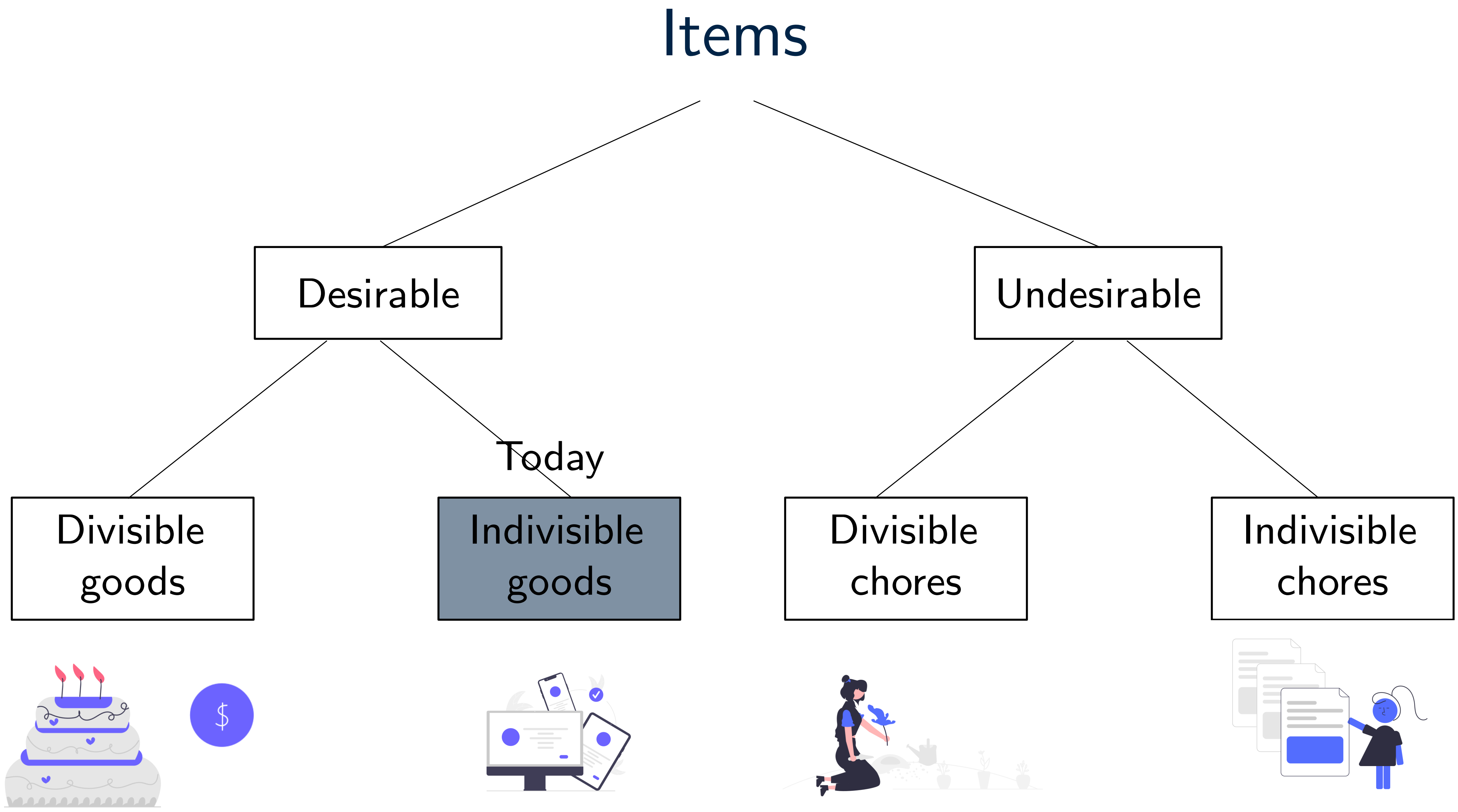
Items
Desirable
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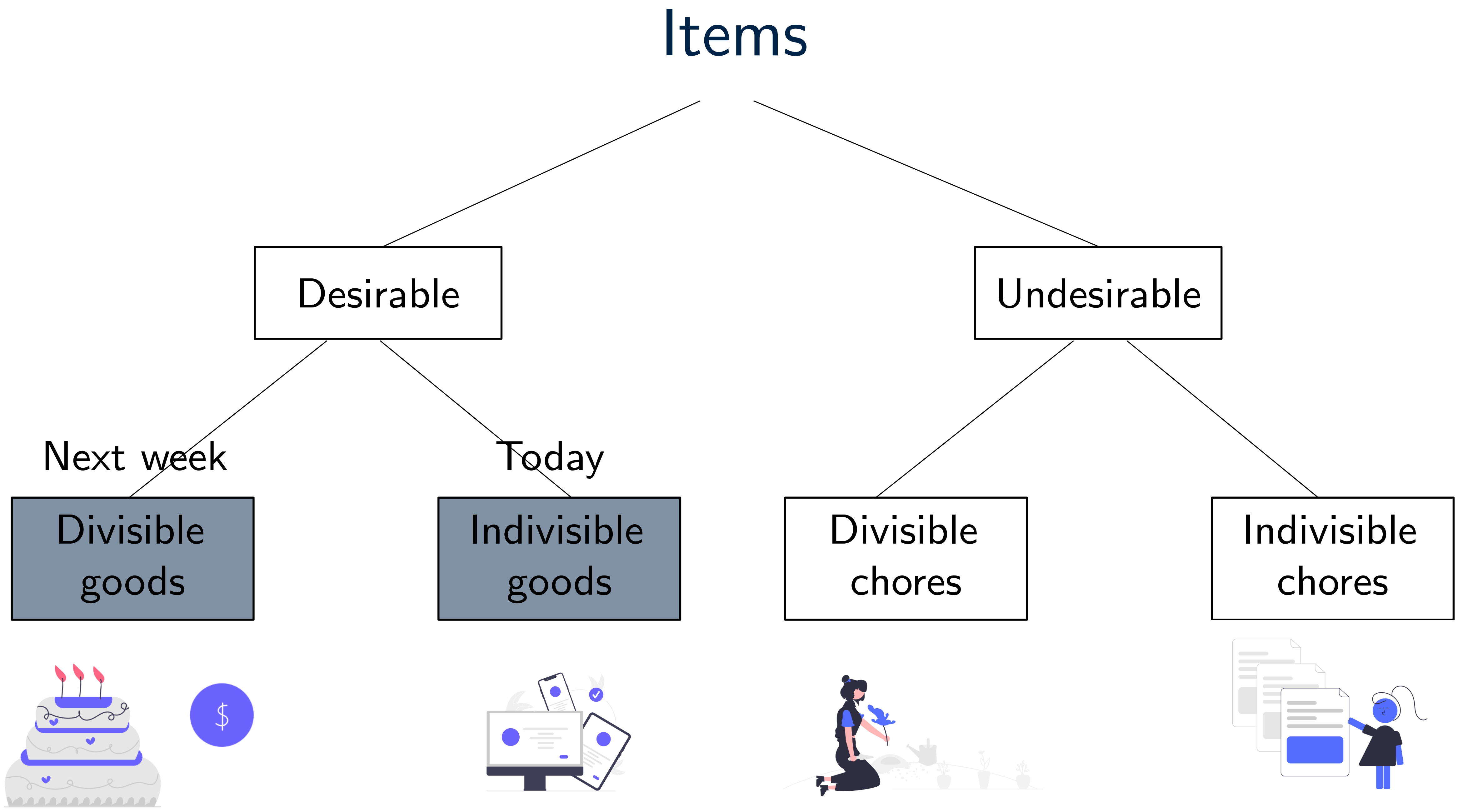
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Undesirable
Next week
Today
Divisible goods
Indivisible goods
Divisible chores
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Discrete Fair Division

Divide **indivisible items** among **agents** in a **fair** manner.

Input: $\mathcal{I} = (N, M, V)$

- N: set of n agents
- M: set of m indivisible goods
- Valuation functions $v_i : 2^M \to \mathbb{R}_{\geq 0}$

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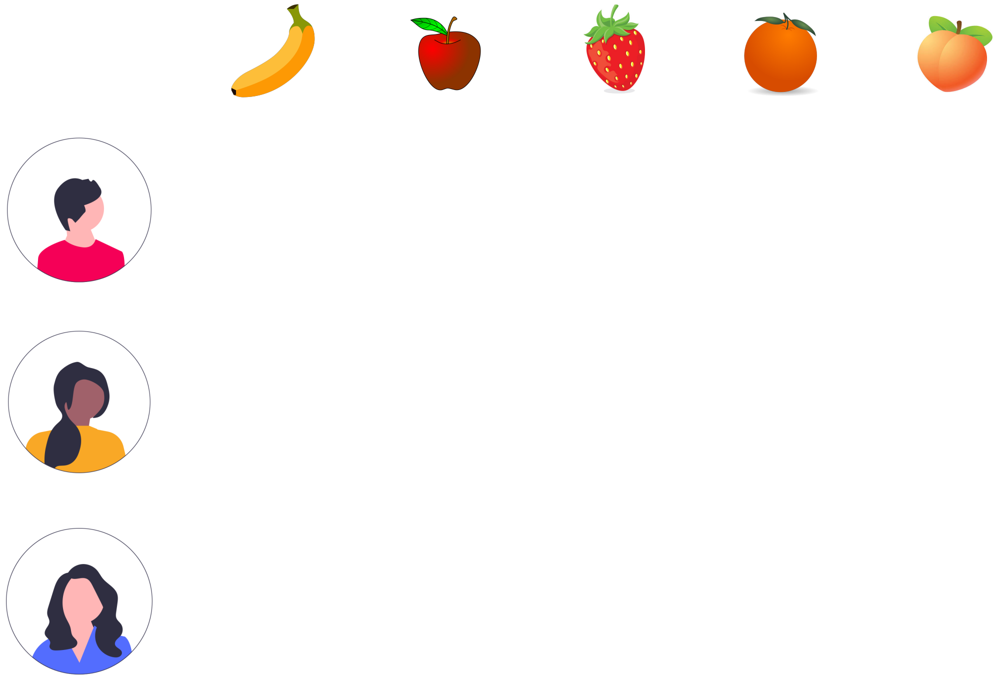

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	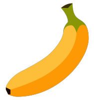	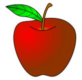			
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	1	0	5	1	1
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Input: $\mathcal{I} = (N, M, V)$

- $N = \{a_1, a_2, a_3\}$
- $M = \{g_1, g_2, g_3, g_4, g_5\}$
- $X_1 = \{g_1\}$, $X_2 = \{g_2, g_5\}$, $X_3 = \{g_3\}$, $P = \{g_4\}$
- $v_1(X_1) = 4$, $v_1(X_2) = 3$

	g_1	g_2	g_3	g_4	g_5
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assuming v_1 is additive: for all $S \subseteq M$, $v_1(S) = \sum_{g \in S} v_i(\{g\})$

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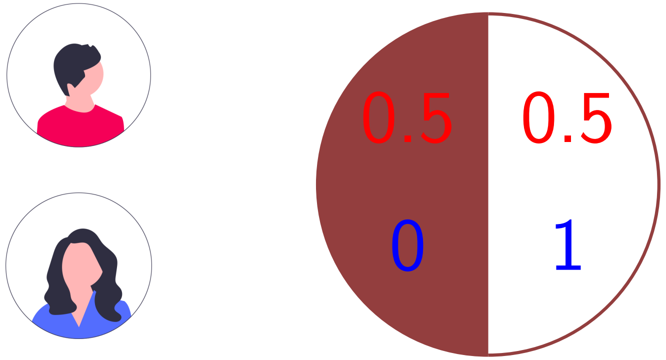
An allocation is **complete**, if $P = \emptyset$ and **partial** otherwise.

Fairness



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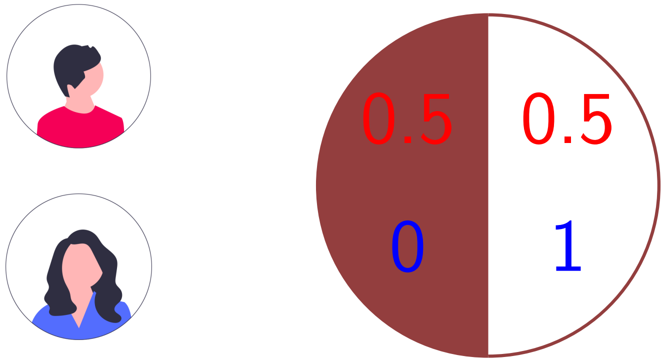
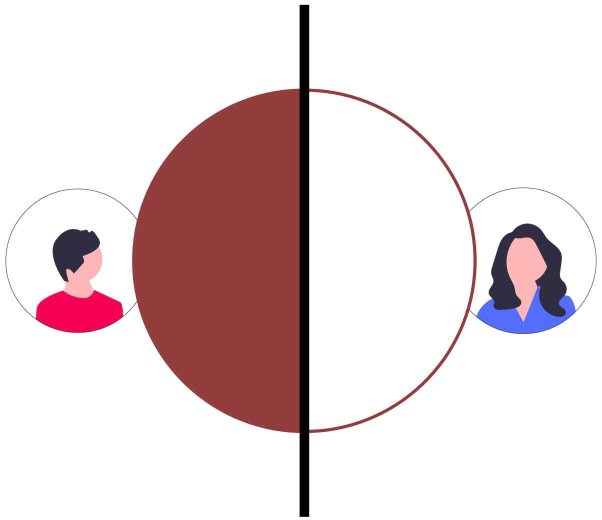
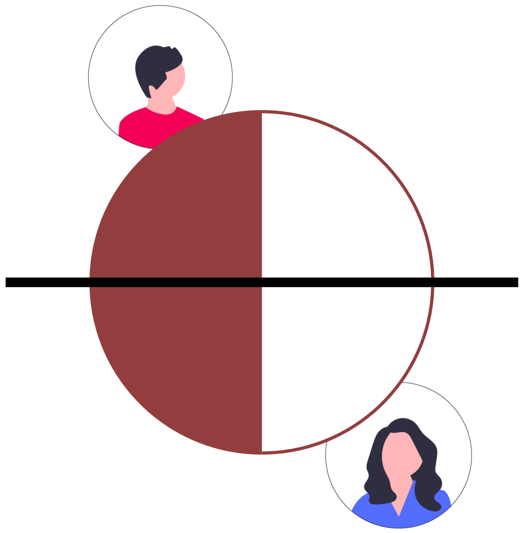
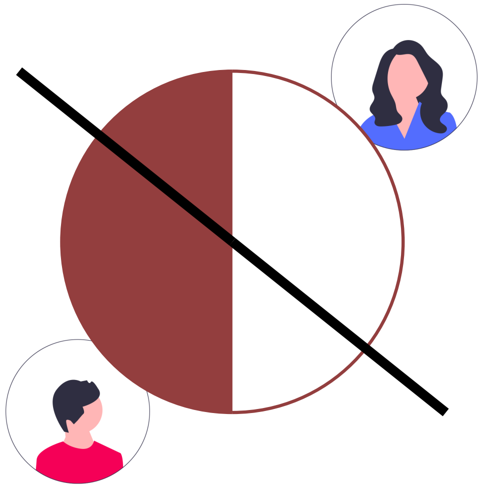


Which allocation is fair?

1. 

2. 

3. 

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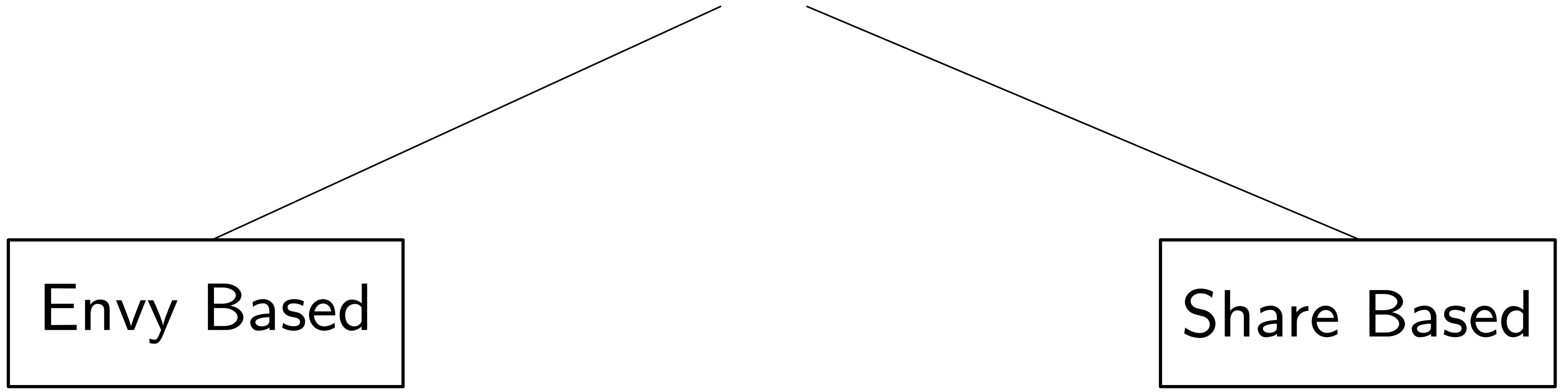

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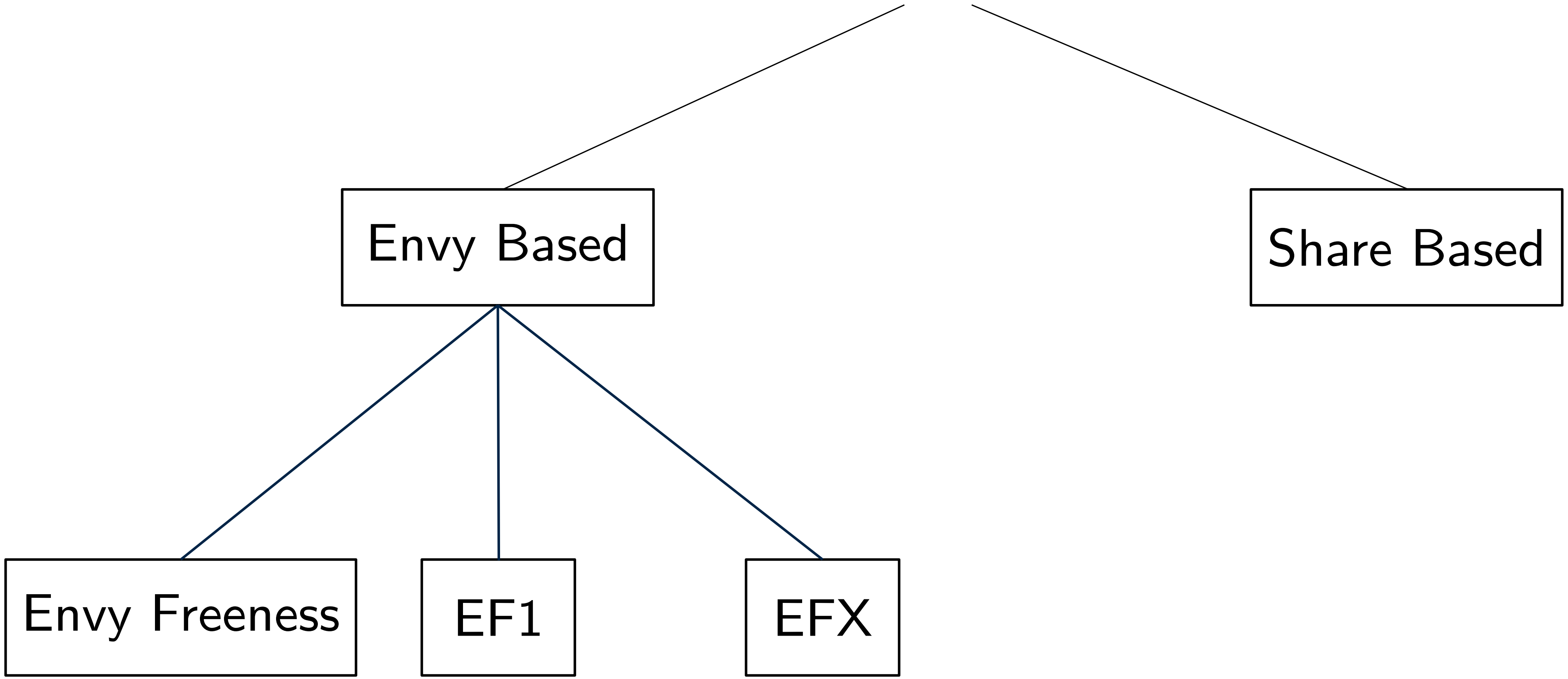

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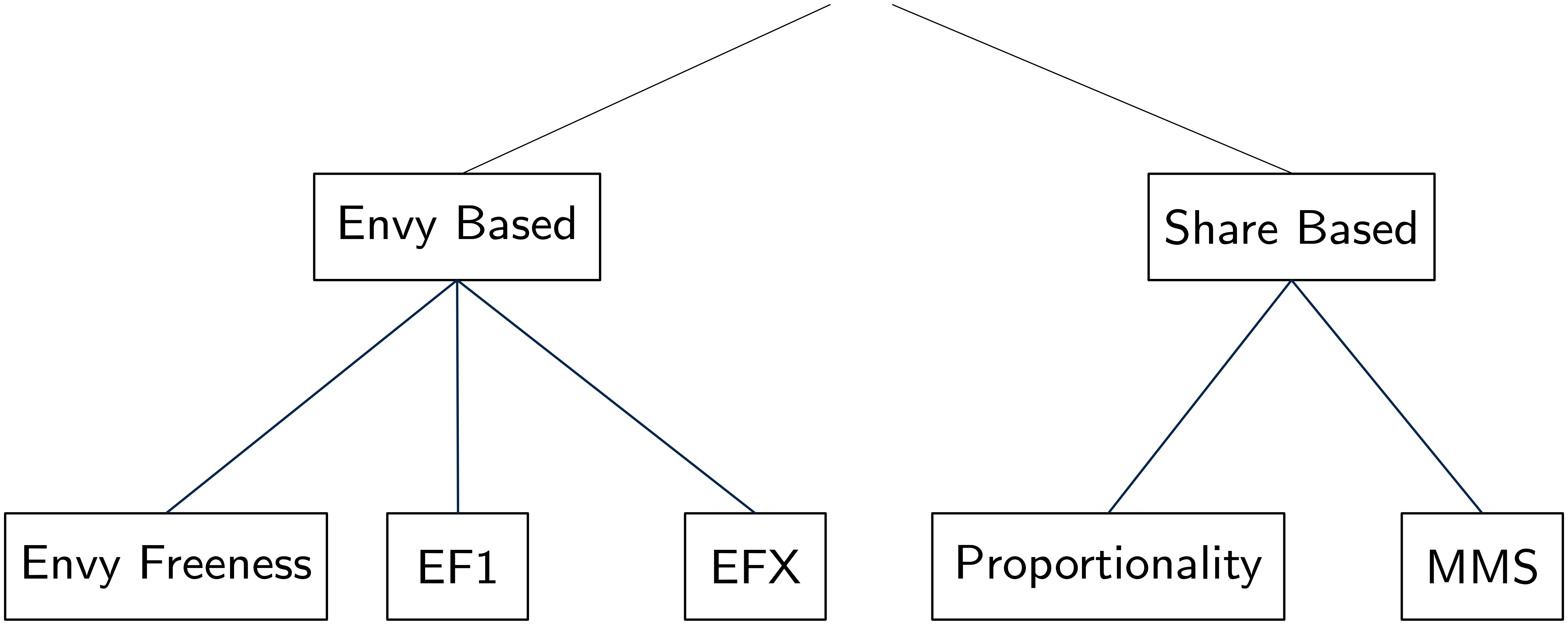

Envy Based
Share Based
Envy Freeness
EF1
EFX
Proportionality
MMS

Envy Freeness

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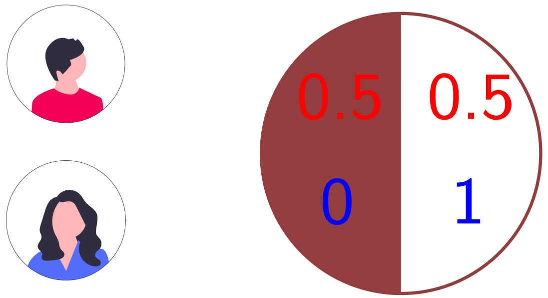
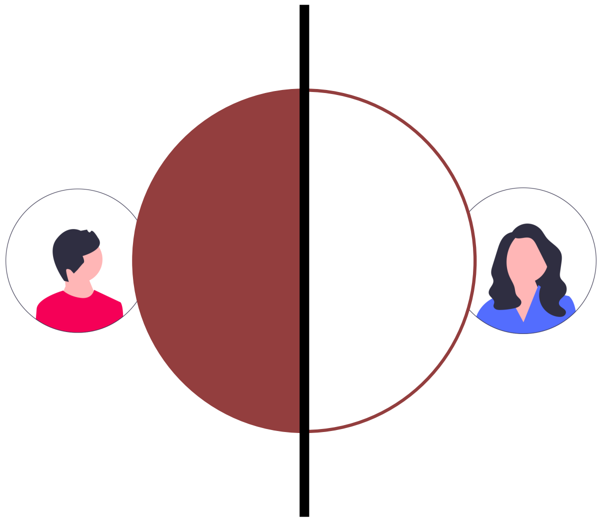
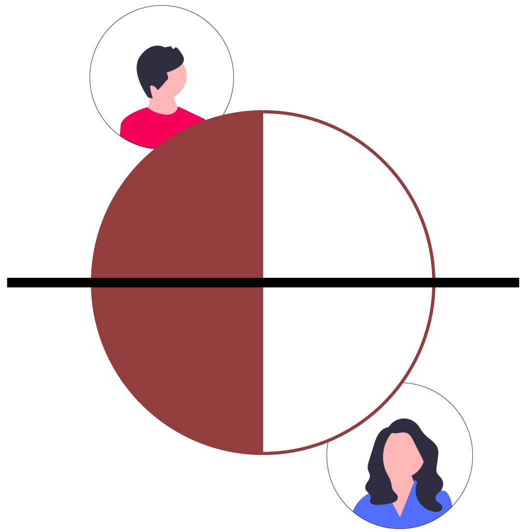
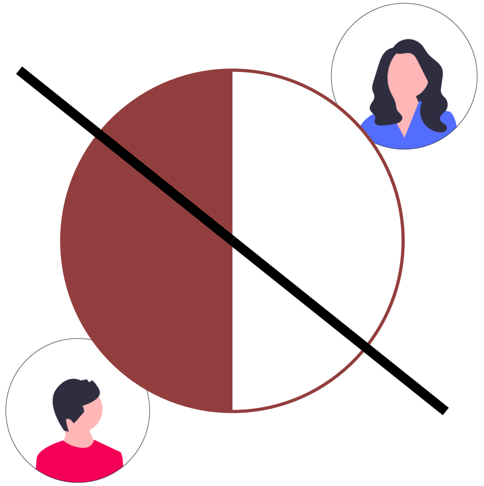


Which allocation is envy free?

1. 

2. 

3. 

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Do complete envy free allocations always exist?

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EF1

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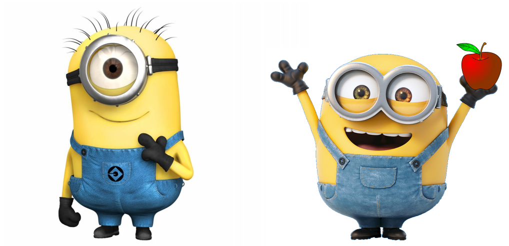

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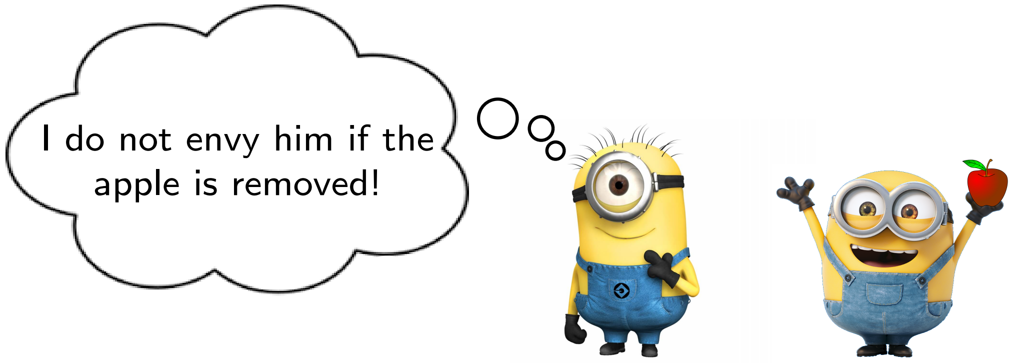

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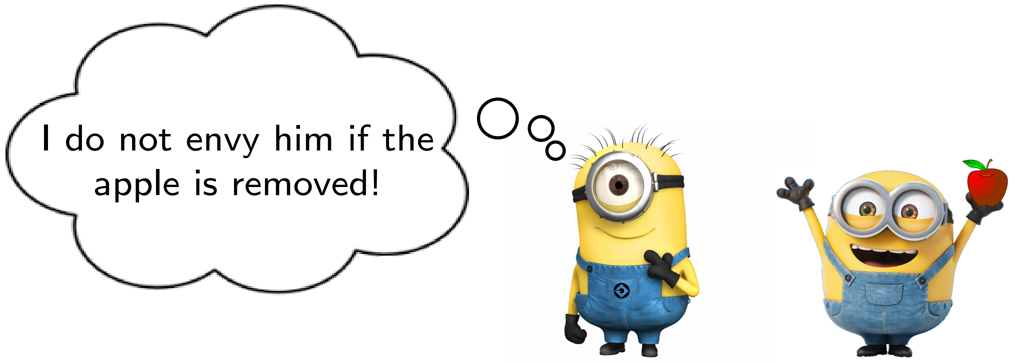

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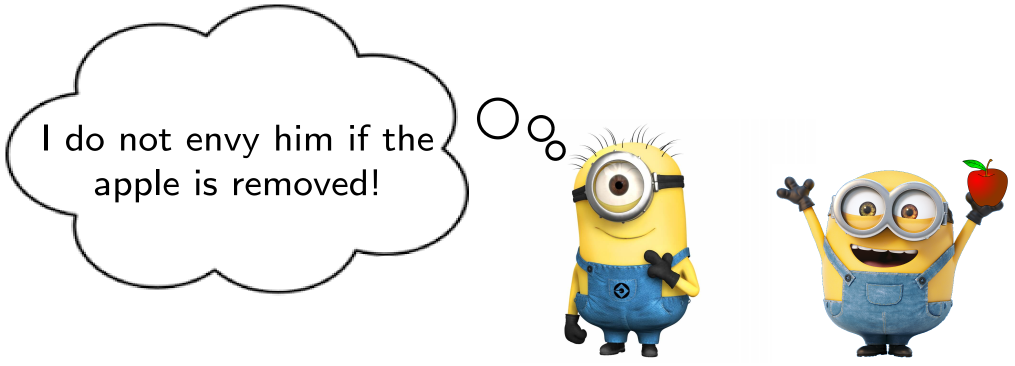
Do complete EF1 allocations always exist?

- YES for monotone valuations!



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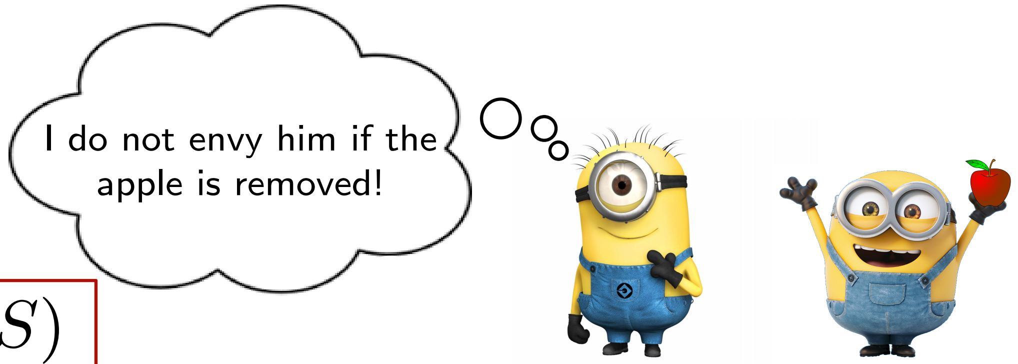

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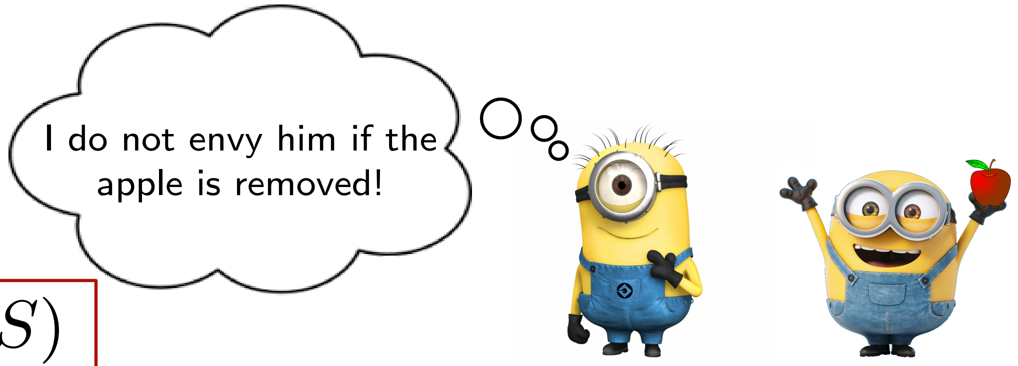

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[Lipton, Markakis, Mossel, Saberi 2004]

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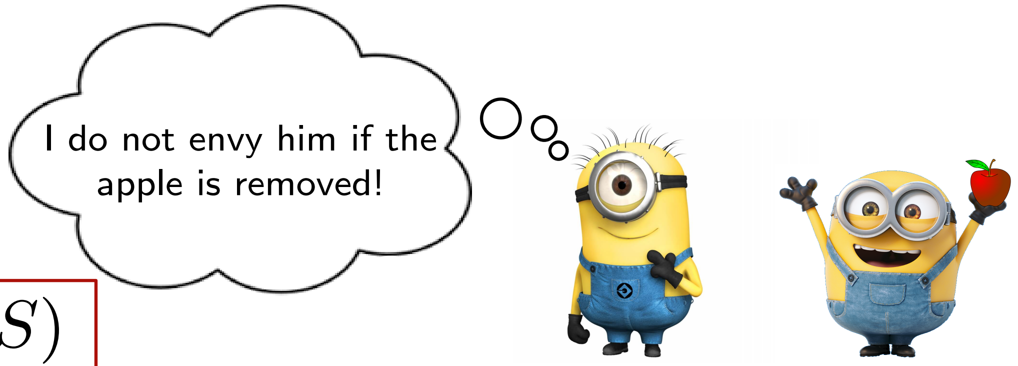

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- Today: A polynomial time algorithm to find a complete EF1 allocation for additive valuations.

Round-Robin Algorithm

- Fix an ordering of the agents, say $a_1, a_2, \ldots, a_n$.
- Agents take turns according to the ordering $(a_1, a_2, \ldots, a_n, a_1, a_2, \ldots, a_n, \ldots)$ to pick their favorite items from the set of the remaining items.

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	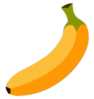	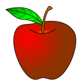			
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	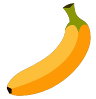	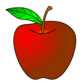			
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	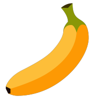	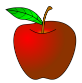			
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	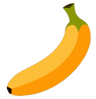	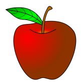			
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Theorem: For additive valuations, Round-Robin returns an EF1 allocation in polynomial time.

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First round:	●	●	●	$\cdots$	●

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First round: ● ● ● ⋯ ●

Second round:

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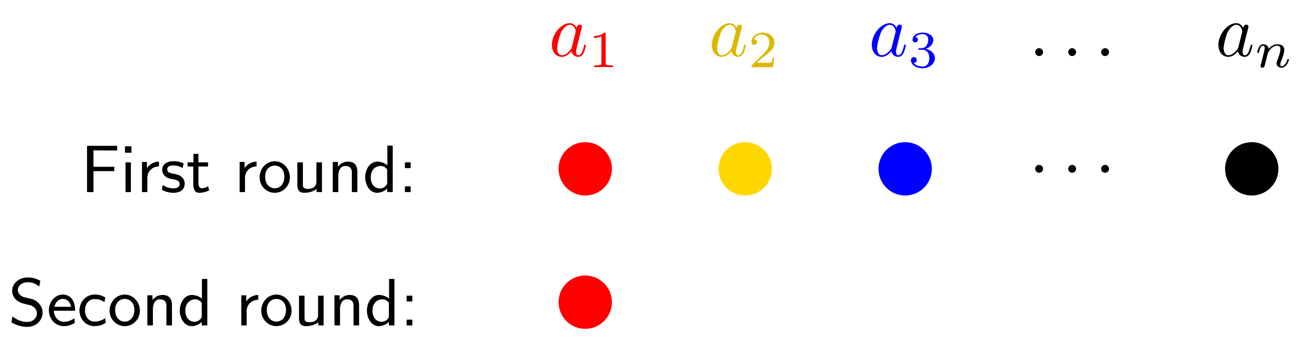

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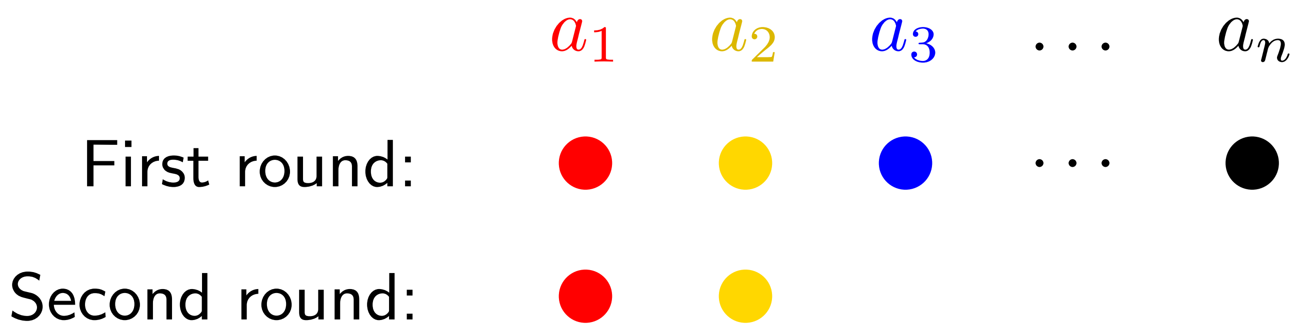

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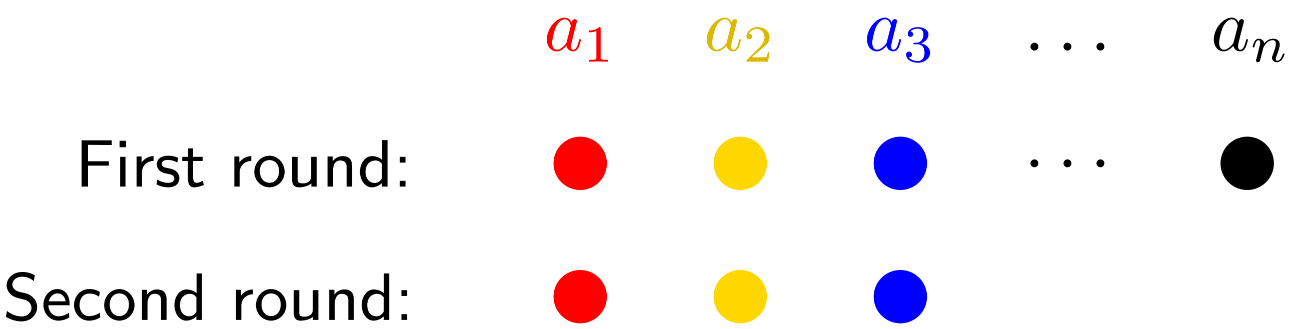

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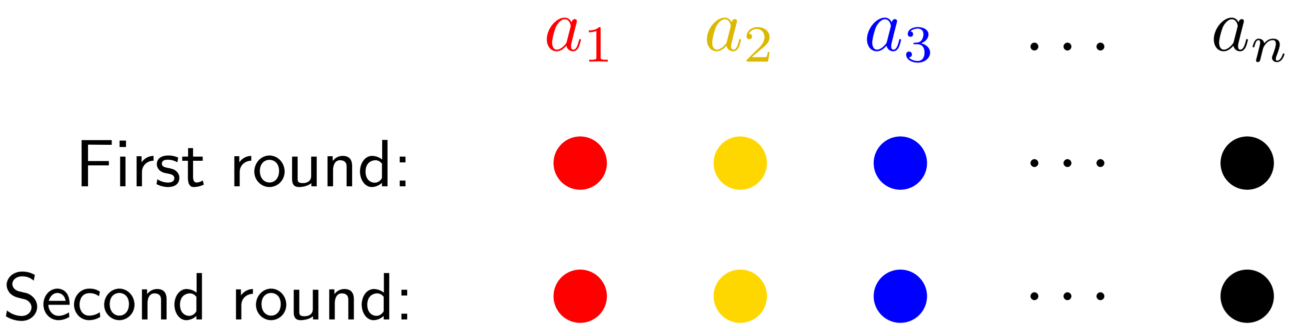

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	a_1	a_2	a_3	$\dots$	a_n
First round:	●	●	●	$\cdots$	●
Second round:	●	●	●	$\cdots$	●
$\vdots$					
Last round:					

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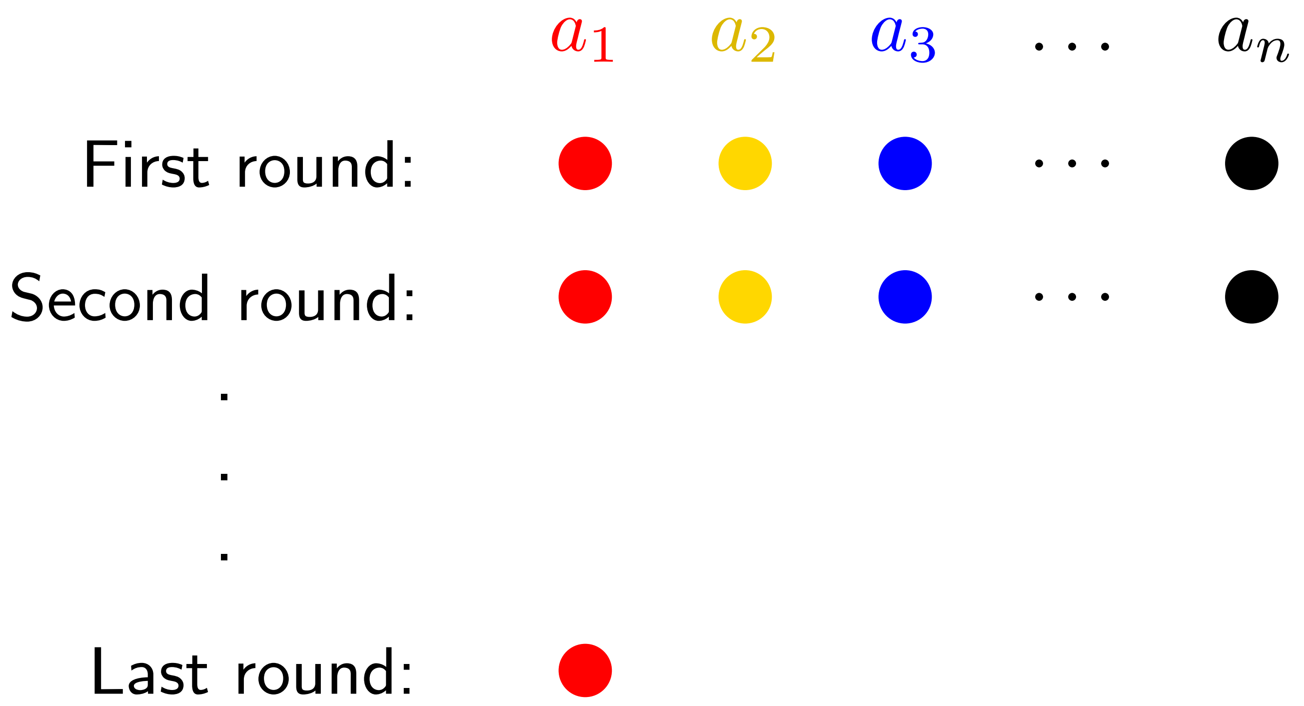

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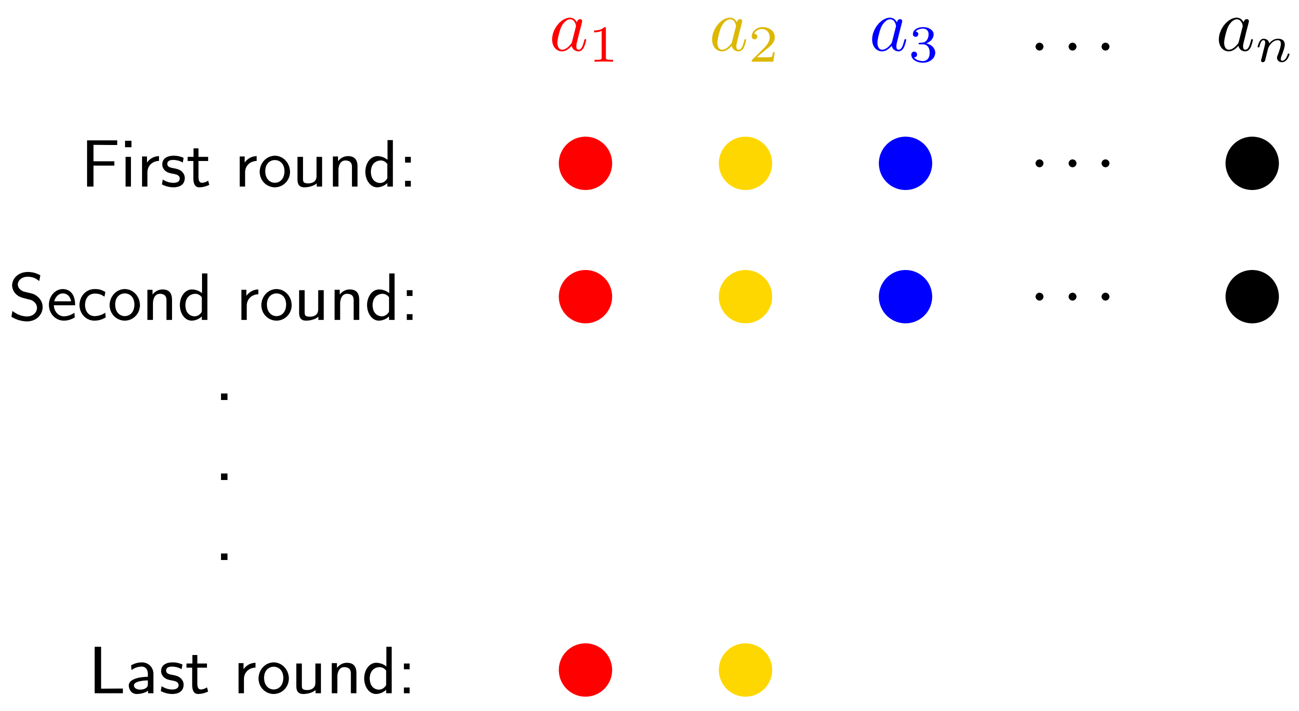

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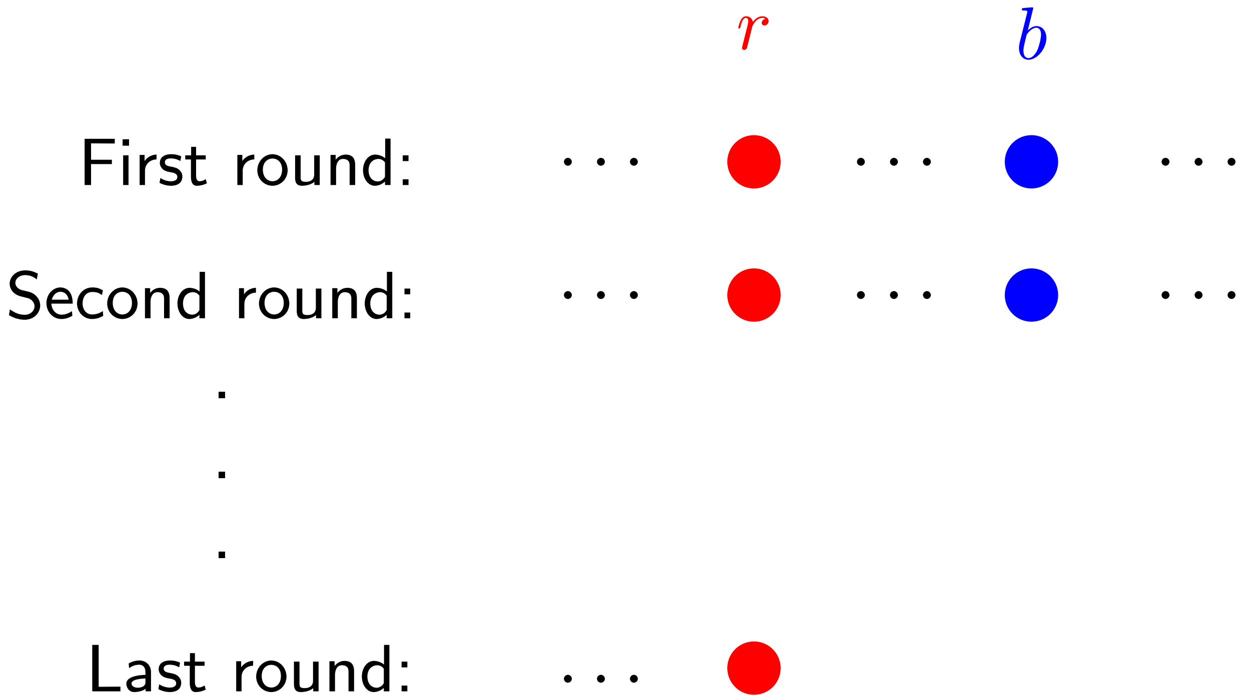

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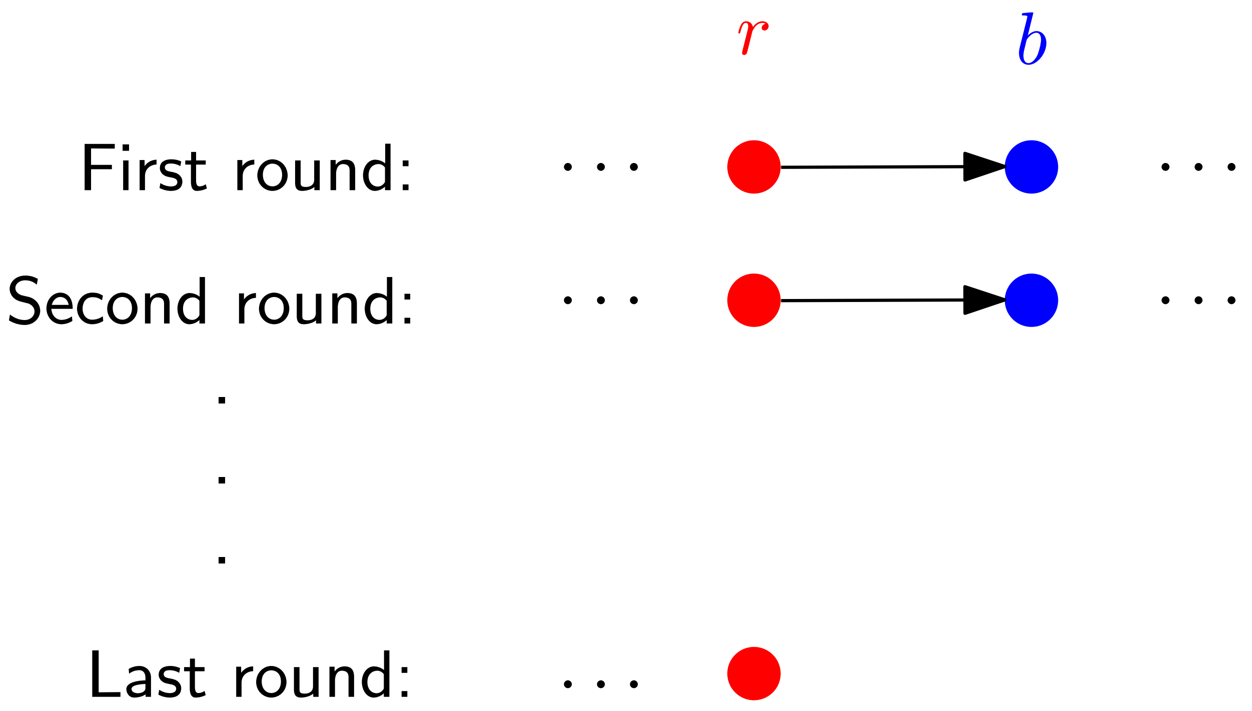

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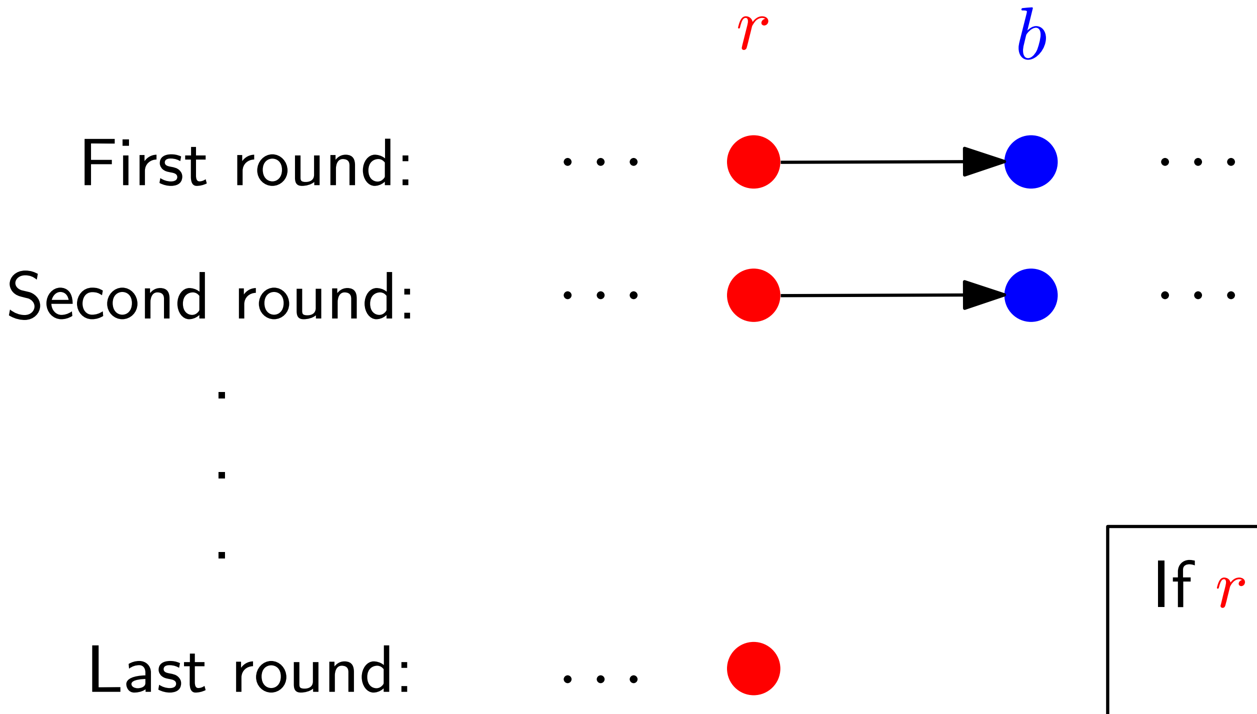

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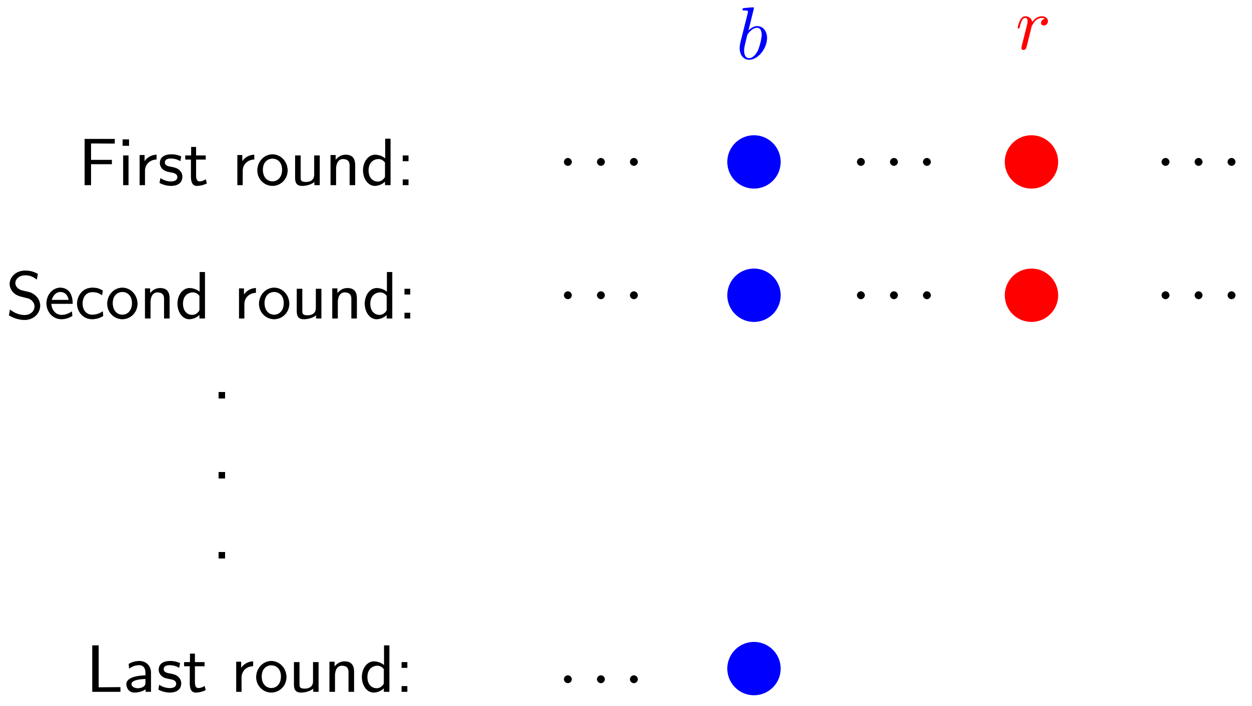

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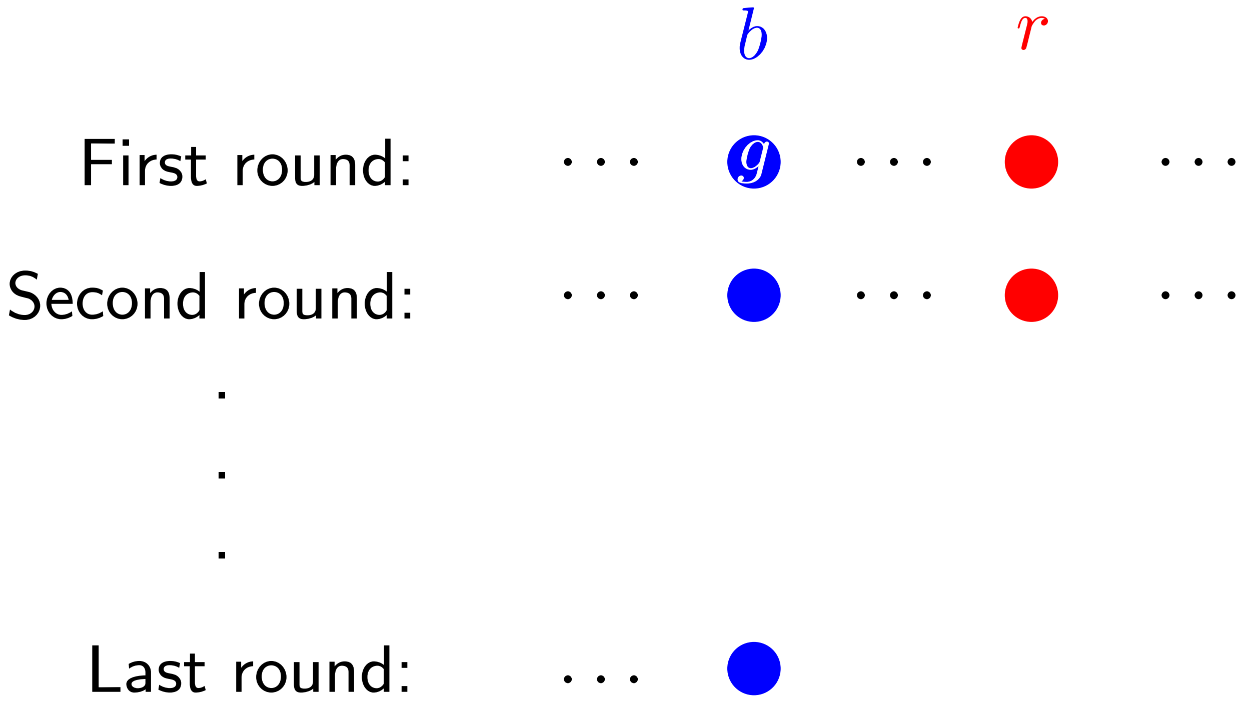

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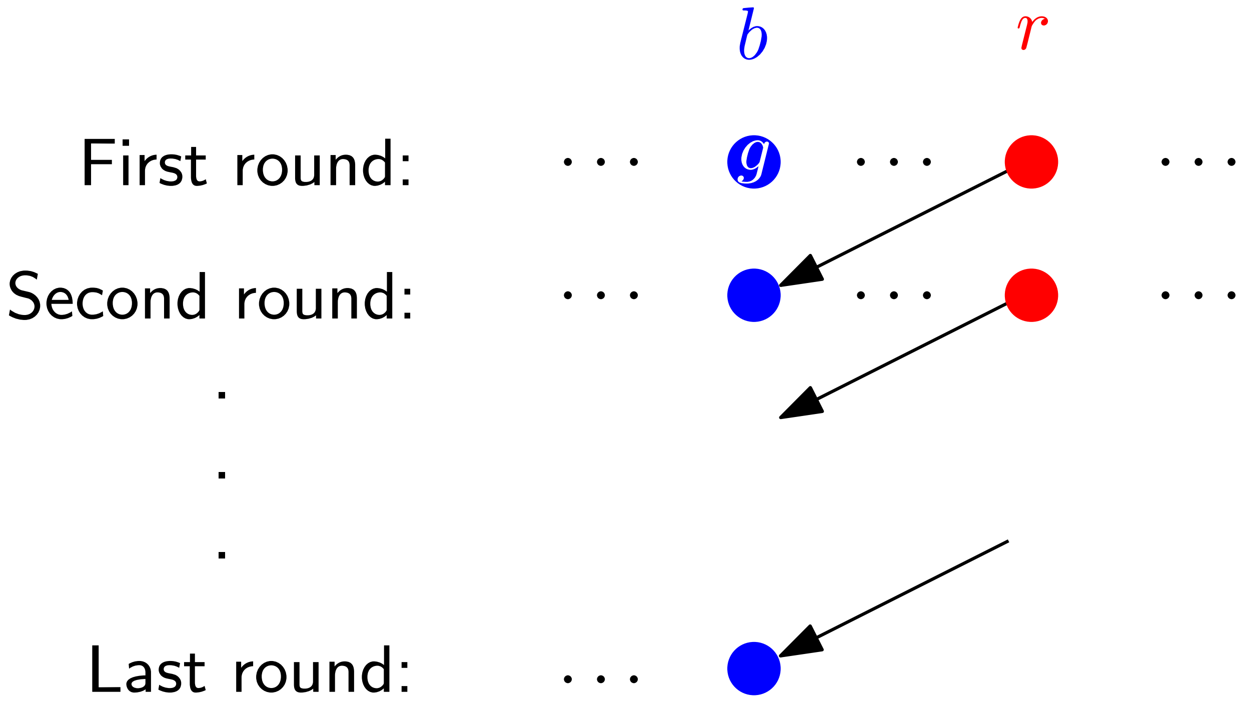

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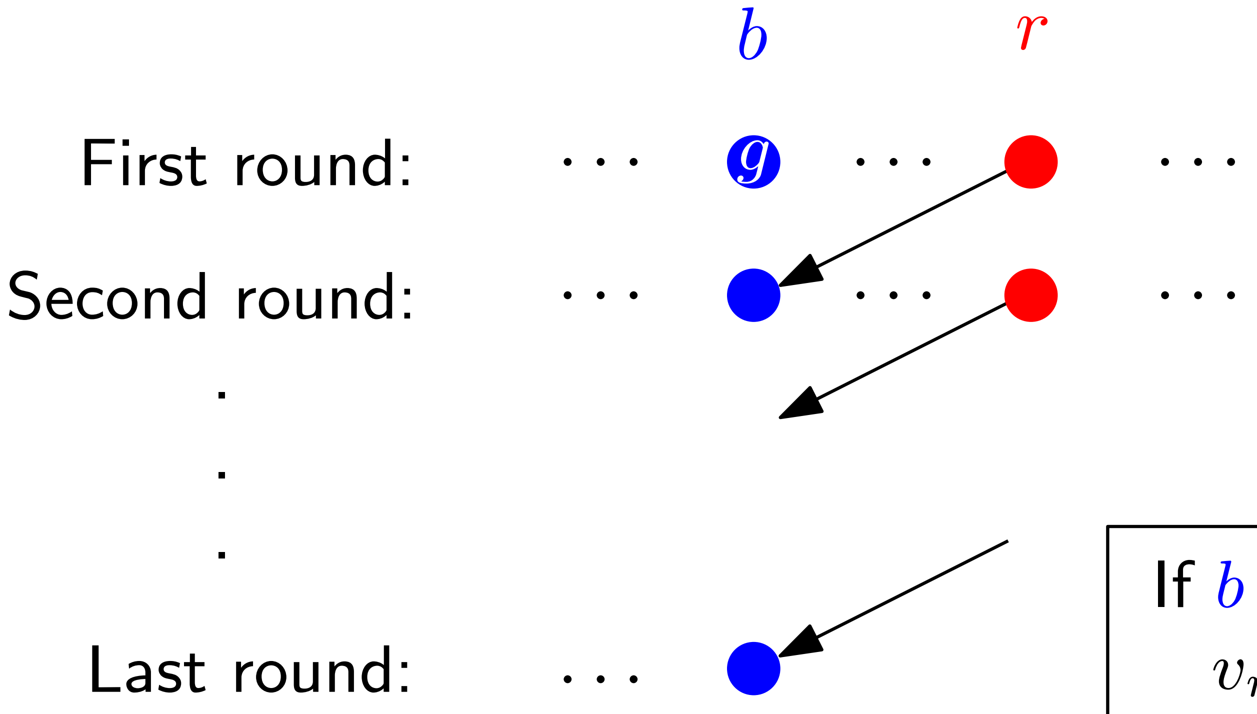

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Definition: An allocation X is **envy free up to any item** or **EFX**, if and only if for all agents a_i, a_j, and for all goods $g \in X_j$: $v_i(X_i) \geq v_i(X_j \setminus \{g\})$.

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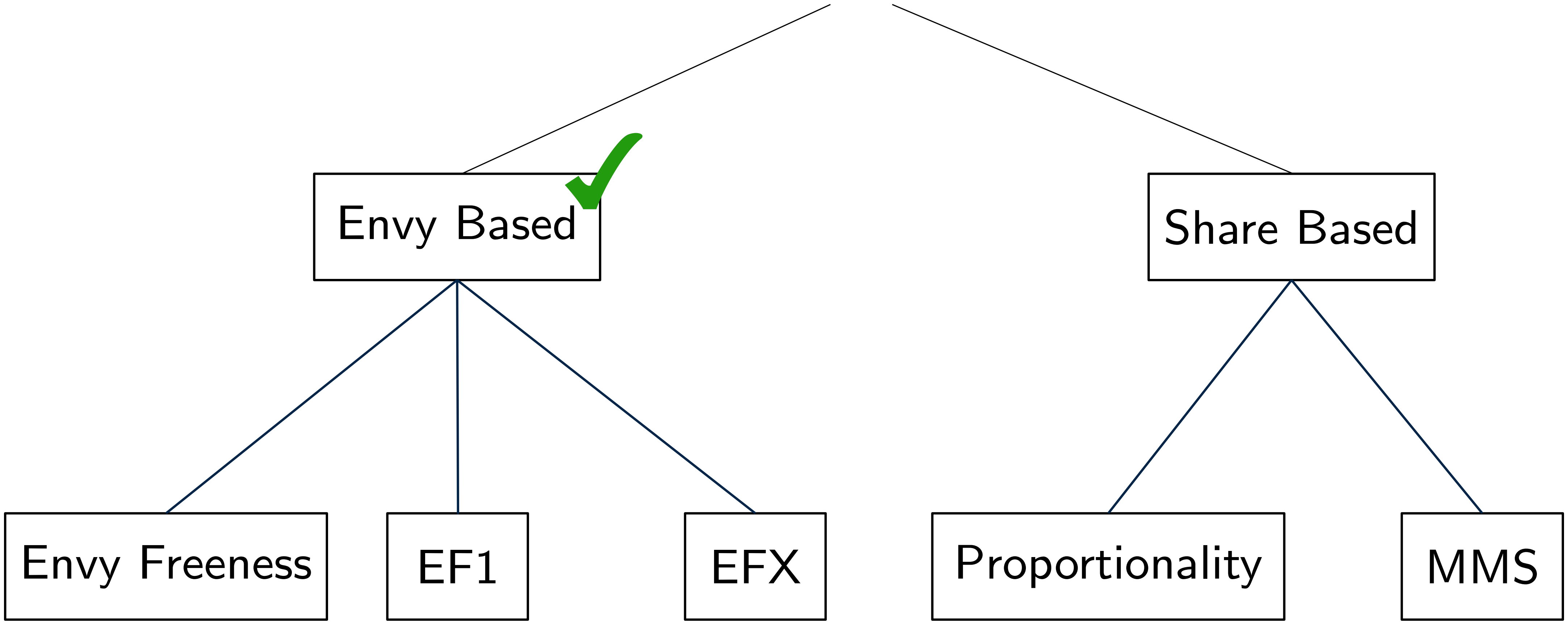
In this seminar we will see:

- Complete EFX allocations exist for 3 agents if at least one has an additive valuation. [Akrami, Alon, Chaudhury, Garg, Mehlhorn, Mehta 2023]
- "Good" partial EFX allocations exists. [Chaudhury, Kavitha, Mehlhorn, Sgouritsa 2020]

Fairness


Envy Based
Share Based
Envy Freeness
EF1
EFX
Proportionality
MMS

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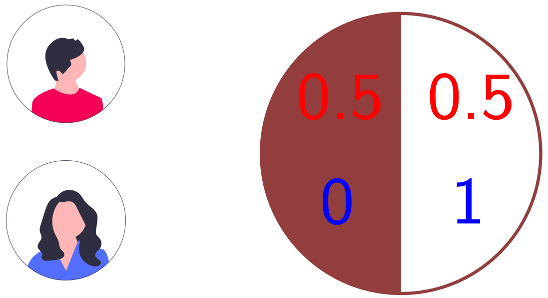
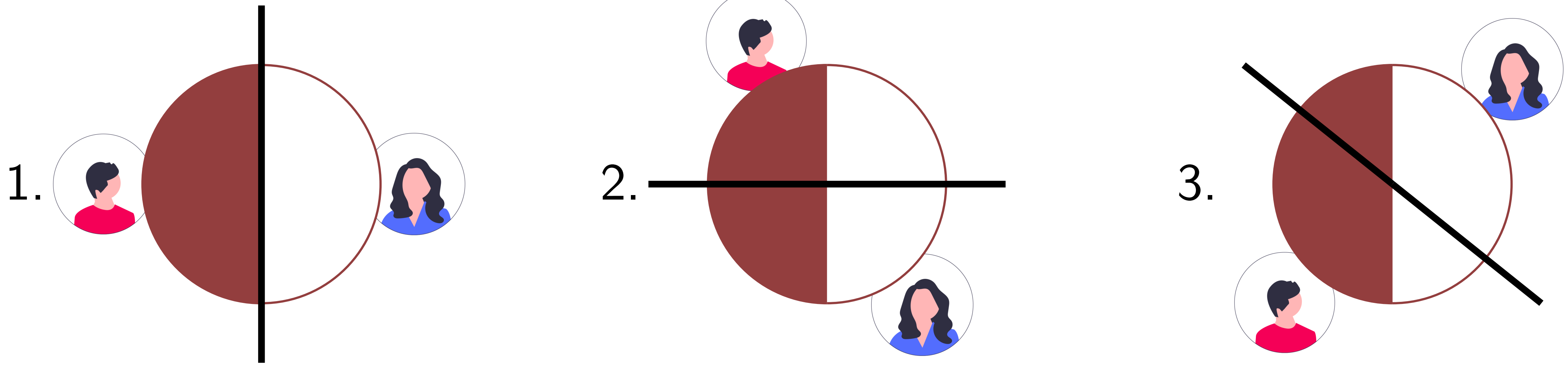


Which allocation is proportional?



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	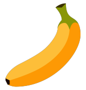	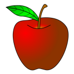			
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Definition: For all $\alpha \in [0, 1]$, an allocation X is α-MMS, if for all agents a_i, $v_i(X_i) \geq \alpha \cdot \mathsf{MMS}_i$.

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Do MMS allocations always exist? NO! [Procaccia, Wang 2014]

Definition: For all $\alpha \in [0,1]$, an allocation X is α-MMS, if for all agents a_i, $v_i(X_i) \geq \alpha \cdot \text{MMS}_i$.

- The best known α: $3/4 + 3/3836$ [Akrami, Garg 2024]

Maximin Share

Definition: For all agents a_i, maximin share of agent i is

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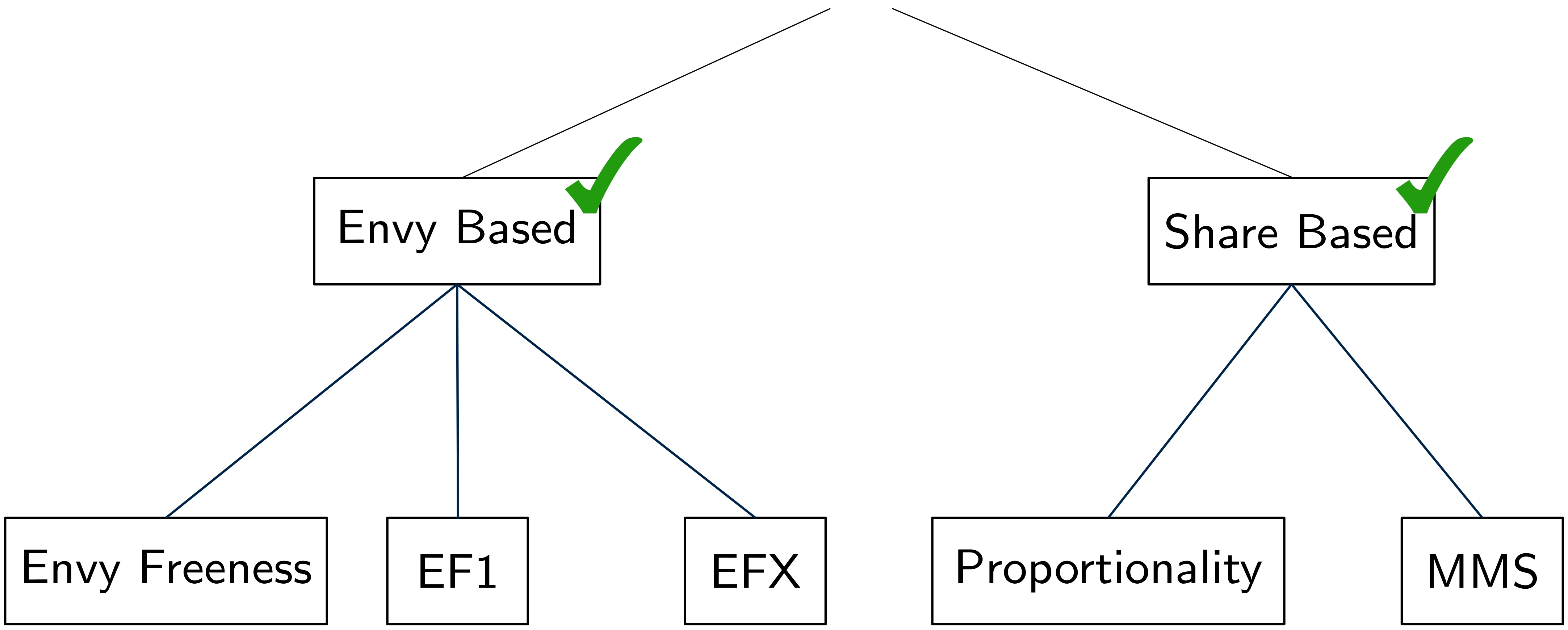
- The best known α: $3/4 + 3/3836$ [Akrami, Garg 2024]

In this seminar we will see:

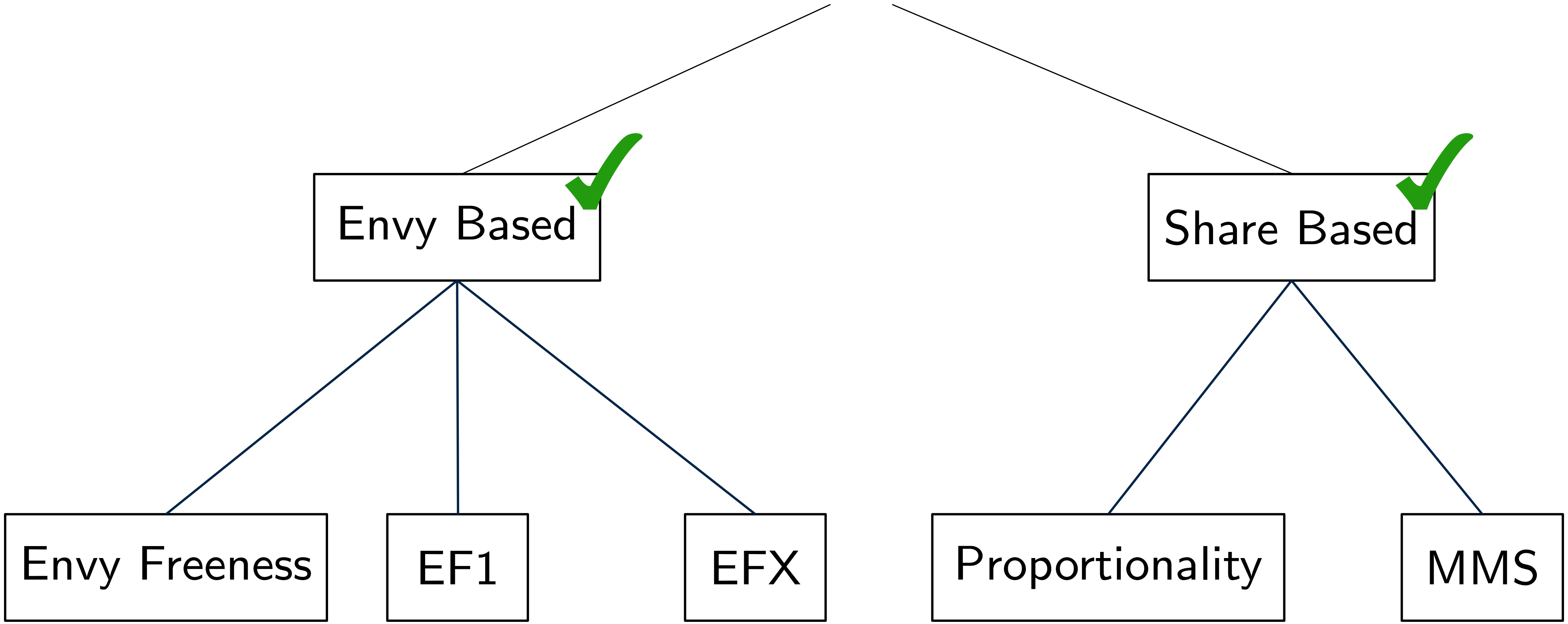
- $3/4$-MMS allocations exist. [Ghodsi, Hajiaghayi, Seddighin, Seddighin, Yami 2018] [Garg, Taki 2020] [Akrami, Garg, Taki 2023]

Fairness



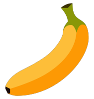
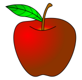
Envy Based
Share Based
Envy Freeness
EF1
EFX
Proportionality
MMS

Fairness



Are we done?

Are we done?

		
	100	1
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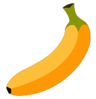
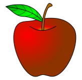
Are we done?

		
	100	1
	1	100

Is the allocation "fair"?

Are we done?

	100	1
	1	100

Is the allocation "fair"?

- EF1?

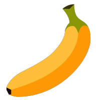
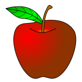
Are we done?

	100	1
	1	100

Is the allocation "fair"?

- EF1?
- EFX?

Are we done?

		
	100	1
	1	100

Is the allocation "fair"?

- EF1?
- EFX?
- MMS?

Efficiency

Divide **indivisible items** among **agents** in a **fair** and **efficient** manner.

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	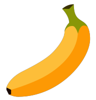	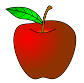
	100	1
	1	100

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	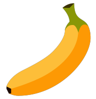	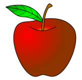
	100	1
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	100	1
	1	100

Is the allocation pareto optimal?

Fairness and Efficiency

	100	1
	1	100

✓ Fair

✗ Efficient

	100	1
	1	100

✗ Fair

✓ Efficient

	100	1
	1	100

✓ Fair

✓ Efficient

Fairness and Efficiency

	100	**1**
	1	100

✓ Fair

✗ Efficient

	100	**1**
	1	100

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✓ Efficient

	100	1
	1	**100**

✓ Fair

✓ Efficient

In this seminar we will see:

- EF1+PO allocations exist an can be computed in pseudopolynomial time.

[Barman, Krishnamurthy, Vaish 2018]

Nash Soical Welfare

Definition: Nash social welfare of an allocation X is

$$\mathsf{NSW}(X) = \left(\prod_{a_i \in N} v_i(X_i) \right)^{1/n} .$$

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In this seminar we will see:

- MNW $\implies$ EF1 + PO [Caragiannis, Kurokawa, Moulin, Procaccia, Shah, Wang 2016]

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In this seminar we will see:

- MNW $\implies$ EF1 + PO [Caragiannis, Kurokawa, Moulin, Procaccia, Shah, Wang 2016]
- 1.45^{-1}-MNW allocations can be computed in polynomial time. [Barman, Krishnamurthy, Vaish 2018]

Recap

Divide **items** among **agents** in a **fair** and **efficient** manner.

Notions of fairness: envy freeness, EF1, EFX, proportionality, MMS, . . .

Notions of efficiency: pareto optimality, MNW . . .

Seminar Overview

23.04: Introduction on Discrete Fair Division (HA)

30.04: Introduction on Cake Cutting (NR)

07.05: EFX: A Simpler Approach and an (Almost) Optimal Guarantee via Rainbow Cycle Number [Akrami, Alon, Chaudhury, Garg, Mehlhorn, Mehta 2023] (HA)
- EFX for 3 agents

14.05: Rental Harmony: Sperner's Lemma in Fair Division [Su 1999] (NR)

21.05: no lecture

28.05: Fair and Efficient Cake Division with Connected Pieces [Arunachaleswaran, Barman, Kumar, Rathi 2019] (student talk)
-

Seminar Overview

04.06: The Unreasonable Fairness of Maximum Nash Welfare [Caragiannis, Kurokawa, Moulin, Procaccia, Shah, Wang 2016] (student talk)
- MNW $\implies$ EF1+PO

11.06: A Little Charity Guarantees Almost Envy-Freeness [Chaudhury, Kavitha, Mehlhorn, Sgouritsa 2020] (student talk)
- “good” partial EFX allocation

18.06: no lecture

25.06: Existence and Computation of Epistemic EFX Allocations [Caragiannis, Sharma, Garg, Rathi, Varricchio 2023] (student talk)
- a relaxation of EFX

Seminar Overview

02.07: Simplification and Improvement of MMS Approximation [Akrami, Garg, Sharma, Taki 2023] (student talk)
- $3/4$-MMS

09.07: Finding Fair and Efficient Allocations [Barman, Krishnamurthy, Vaish 2018] (student talk)
- 1.45^{-1}-MNW $+$ EF1 $+$ PO

16.07: On Approximate Envy-Freeness for Indivisible Chores and Mixed Resources [Bhaskar, Sricharan, Vaish 2021] (student talk)
-

23.07: Best of Both Worlds: Ex-Ante and Ex-Post Fairness in Resource Allocation [Freeman, Shah, Vaish 2020] (student talk)
- randomized allocations

Don't forget!

Send us your preferred list of the student papers by April 30th.