max planck institut
informatik

Topics in Computational Social Choice Theory

Lecture 02: Introduction on Fair Cake Division

Nidhi Rathi

SIC Saarland Informatics Campus

Last Lecture: Discrete Fair Division

What is **fairness as a concept**?

How to **compute a fair allocation**?

Agents with **valuations** over items

Indivisible items

Goal: To **divide** the items among the agents in a **fair** manner

Fair Division

What is **fairness as a concept**?

How to **compute a fair allocation**?

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- Focus on **provable guarantees**.
- **Computational Perspective**: work towards algorithms & hardness results and approximation algorithms

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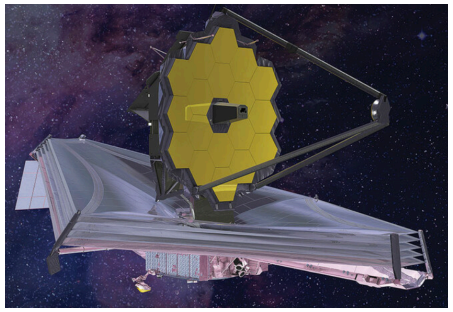

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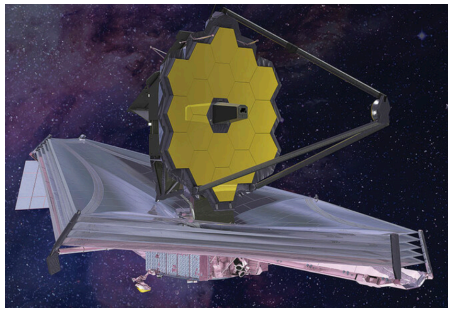

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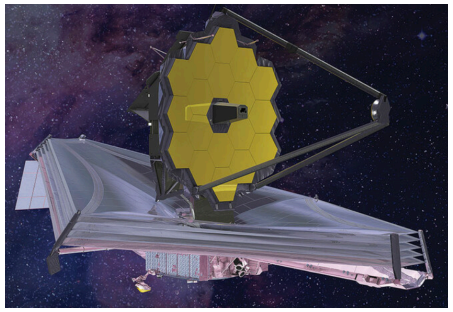

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Cake-Cutting

How to ***fairly*** cut the cake?

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How to ***fairly*** divide a cake
among agents with **differing preferences**?

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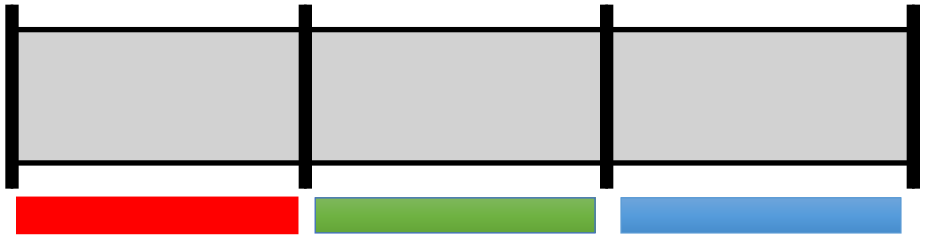

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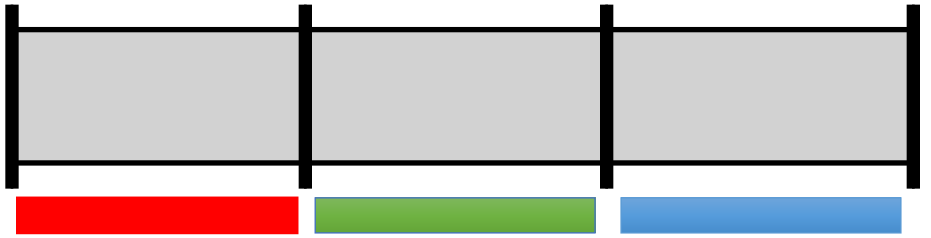

Fair

Why is this problem interesting?

I only like vanilla

I like chocolate and vanilla

I love fruits

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Is this division fair?

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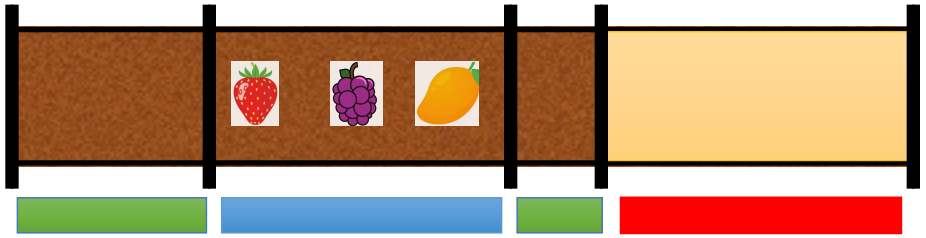

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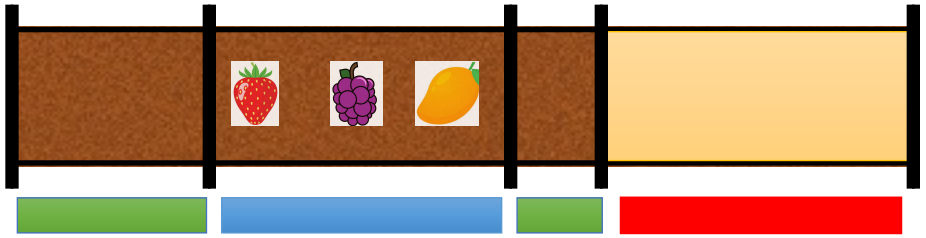

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Preferences matter!

Cut-and-choose Protocol

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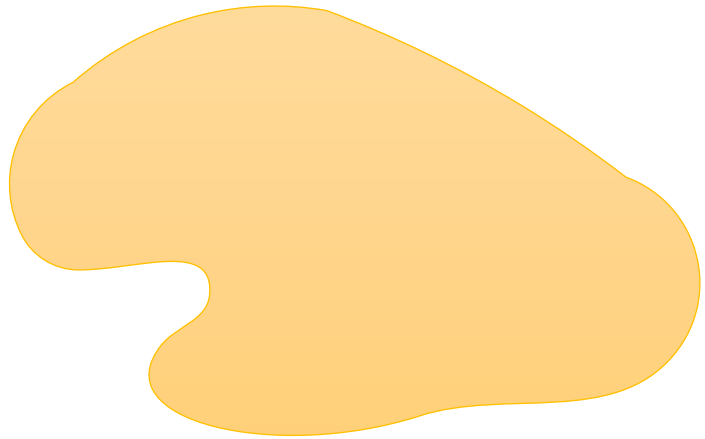

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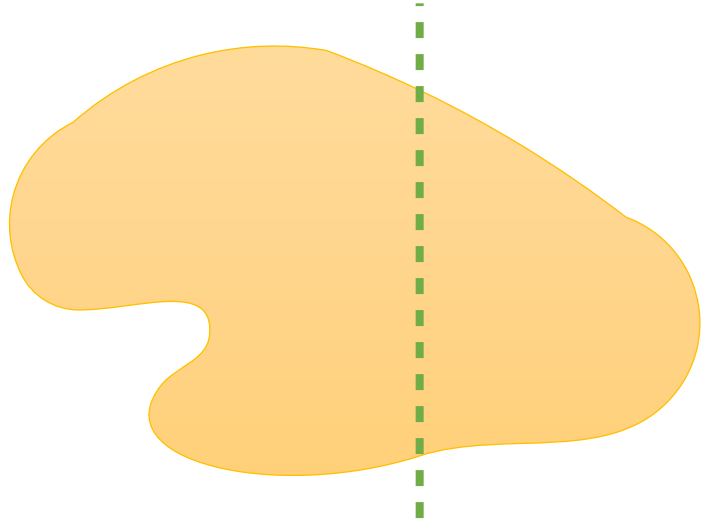

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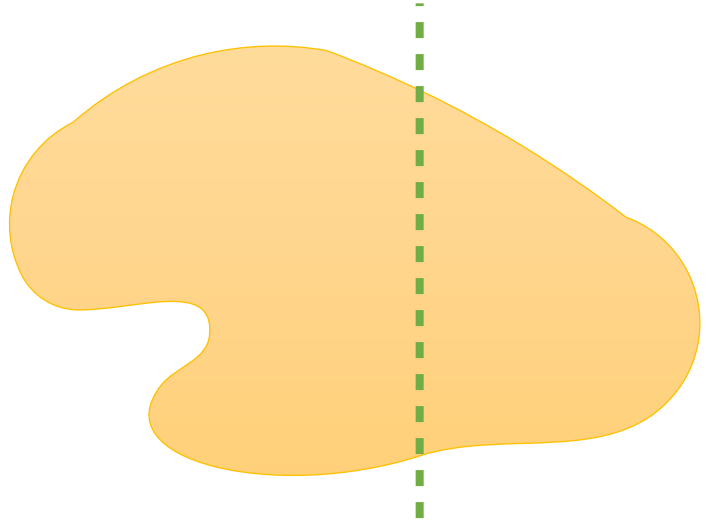

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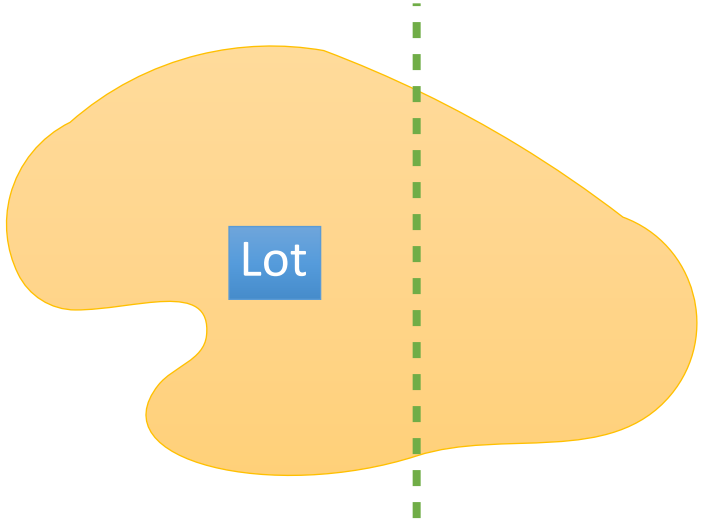

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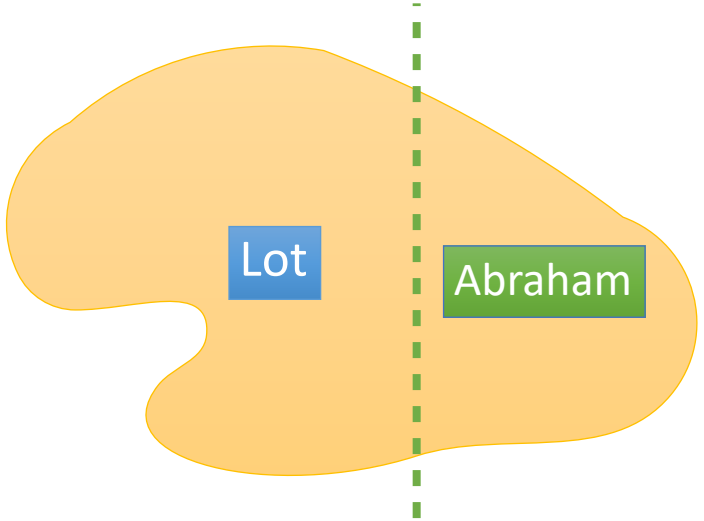

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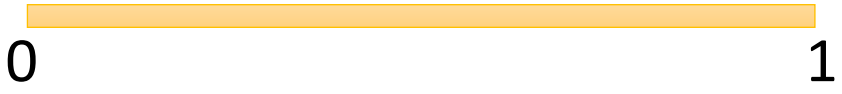

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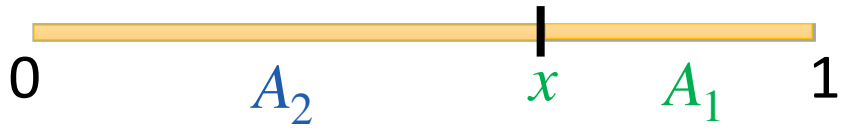

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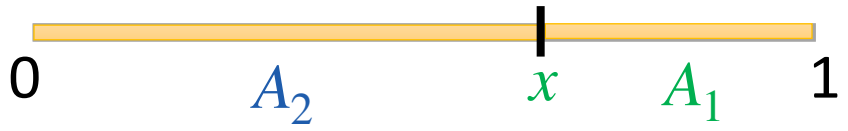

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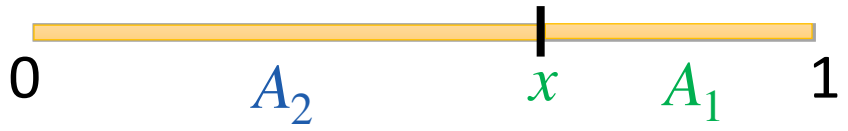

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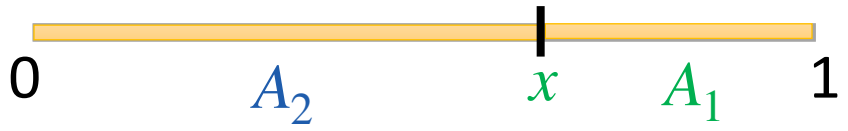

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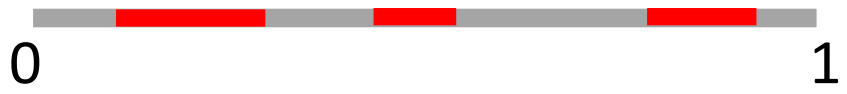
- **Piece** of a cake: finite union of subintervals of [0,1]



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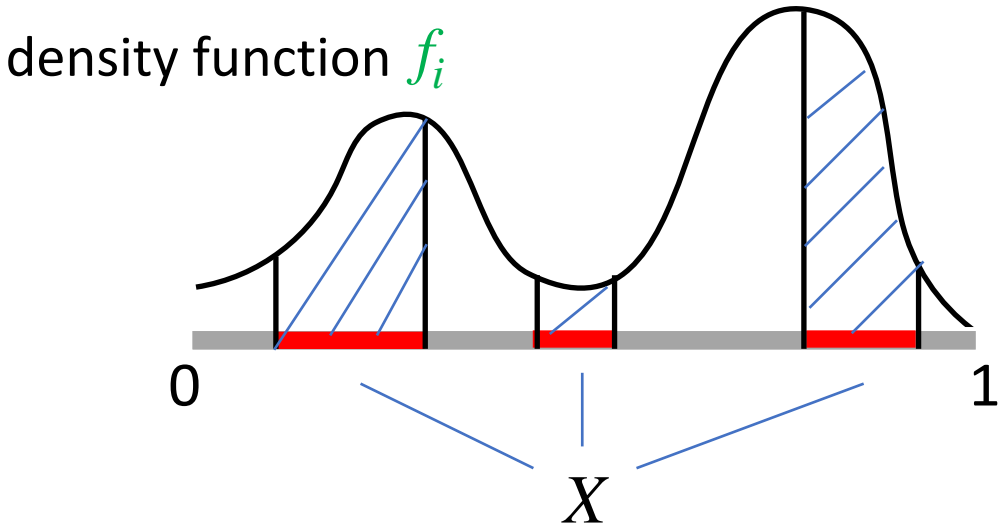

$$v_i(X) = \int_{x \in X} f_i(x)\ dx$$

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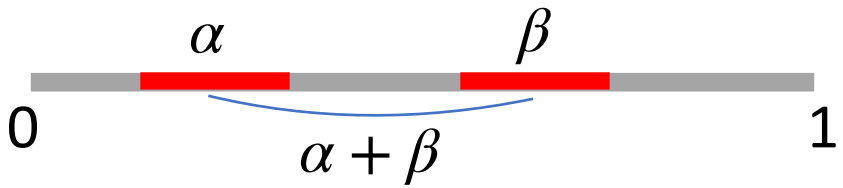

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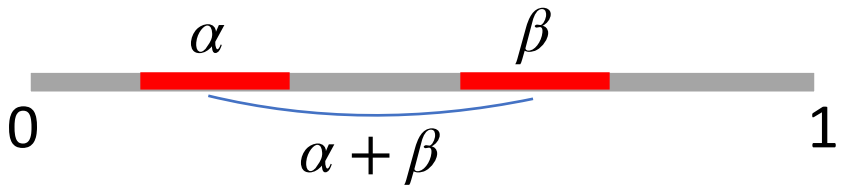
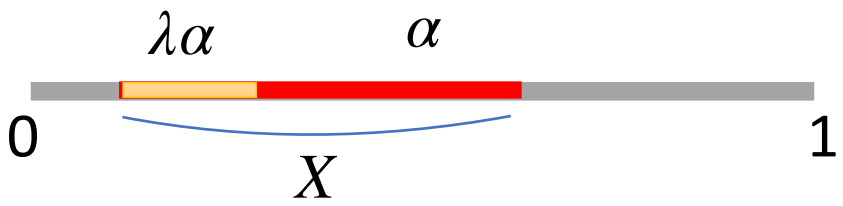


  <u>Divisible:</u>

  For any $X \subseteq [0,1]$ and $\lambda \in [0,1]$, there exists a $Y \subseteq X$ s.t. $v_i(Y) = \lambda v_i(X)$



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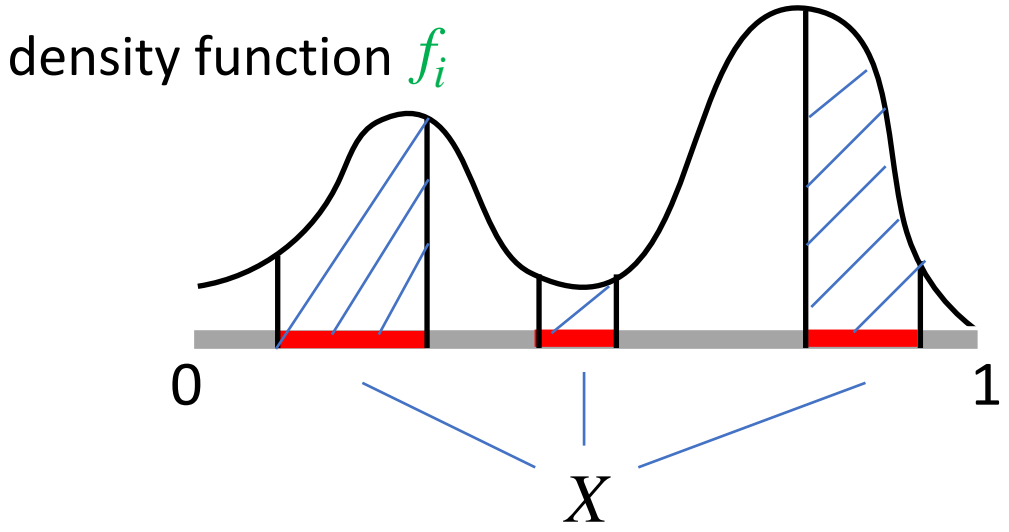

$$v_i(X) = \int_{x \in X} f_i(x)\, dx$$

v_i is a probability distribution over $[0,1]$

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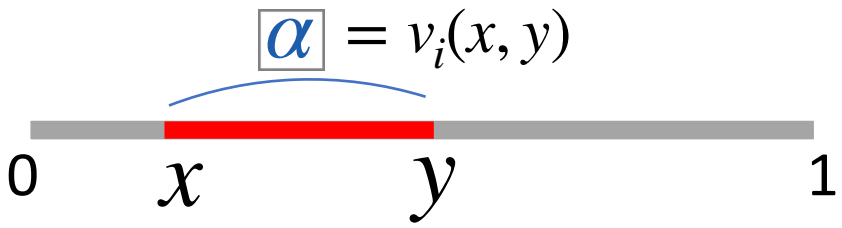

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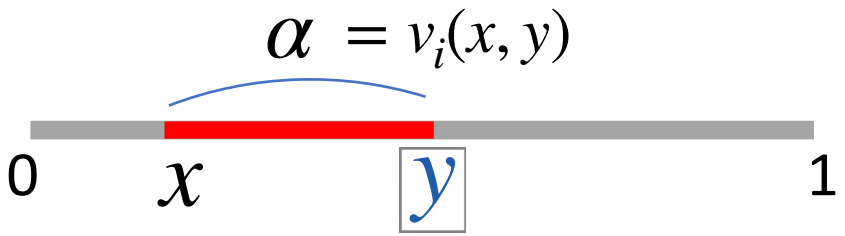

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  [Steinhaus, 1948]

- **Envy-freeness:** for every pair $i, j \in [n]$ of agents, we have $v_i(A_i) \geq v_i(A_j)$

  [Foley 1967]

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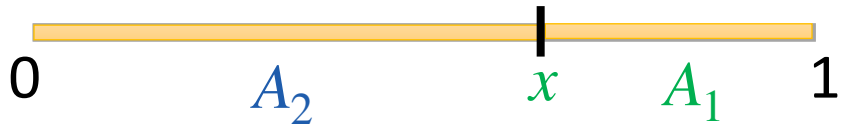

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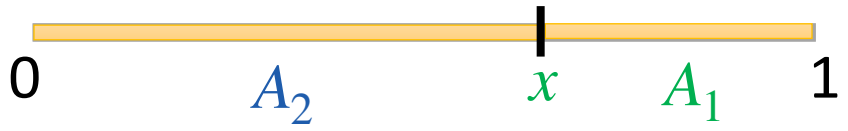

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EF and **Prop** are *equivalent for two agents*

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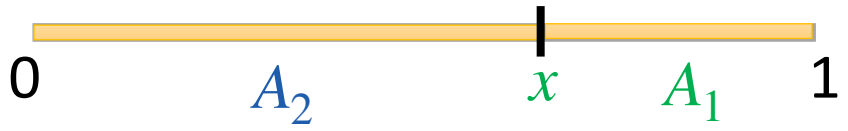

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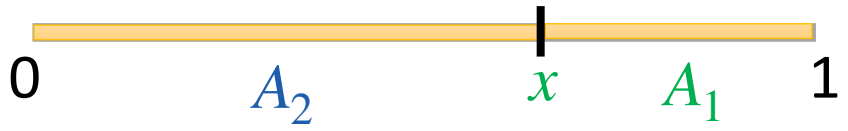

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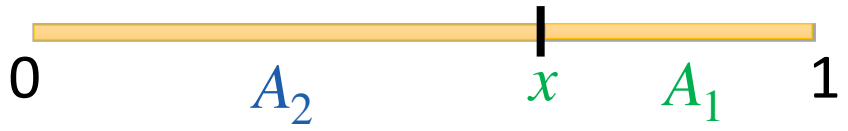

The cut-and-choose outcome is **EF** and **Prop**

Can cut-and-choose be implemented in RW model? Yes!

$$\text{cut}_1(0,1/2) = x$$

$$\text{eval}_2(0,x)$$

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1. Abraham (agent 1) cuts the cake $[0,1]$ into two pieces of equal value to him.
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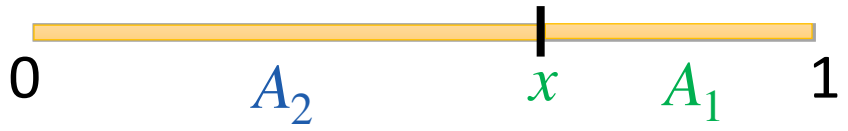
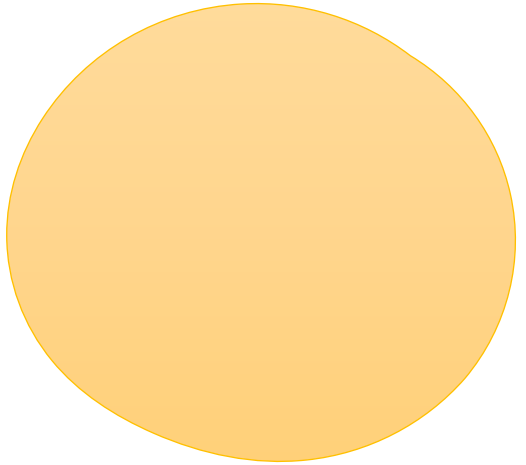


The cut-and-choose outcome is **EF** and **Prop**

For two agents, an EF/Prop cake division can be computed using two queries

Fairness Notions

- Proportionality: for each agent $i \in [n]$, we have $v_i(A_i) \geq 1/n$
- Envy-freeness: for every pair $i, j \in [n]$ of agents, we have $v_i(A_i) \geq v_i(A_j)$



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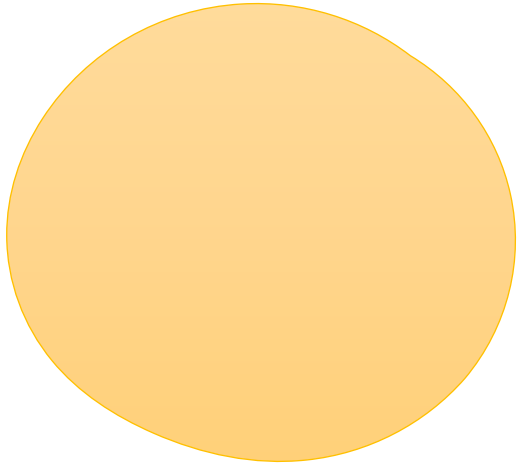
EF $\Longrightarrow$ Prop for *any* number of agents



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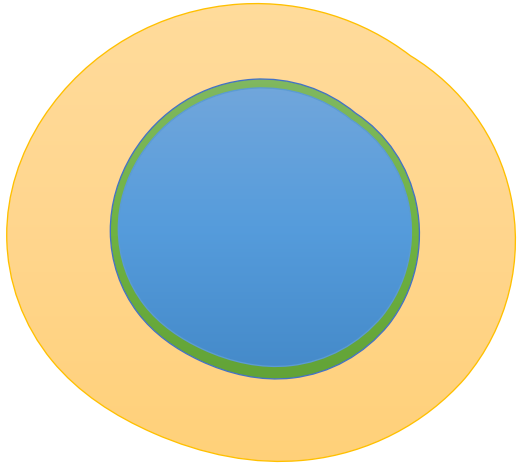

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EF $\Longrightarrow$ Prop for *any* number of agents

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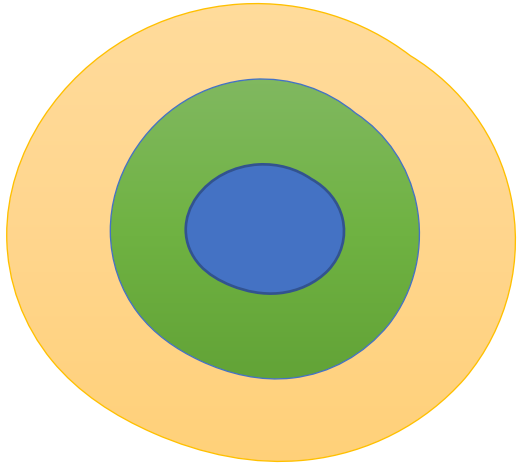

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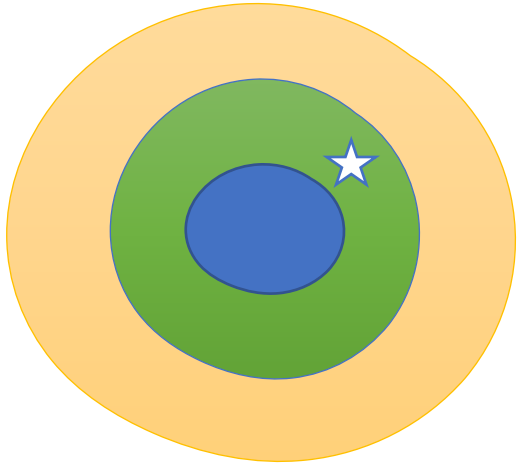


A proportional cake division always exists and can be computed efficiently

A proportional cake division always exists and can be computed efficiently

(i) Moving-knife Protocol - Dubins and Spanier [1961]
(ii) Even-Paz Protocol [1984]

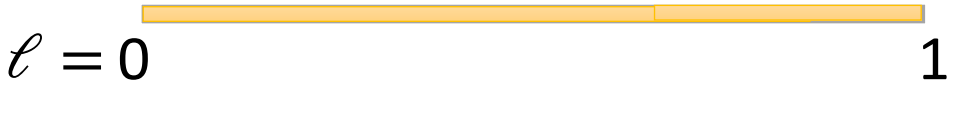
Reference: Handbook of Computational Social Choice, see Chapter 13 by Ariel Procaccia.

Moving-Knife Protocol (Dubins-Spanier)

An efficient proportional cake division protocol for any number of agents

Moving-Knife Protocol (Dubins-Spanier)

1. Initialize $\ell = 0$ and $W = [n]$



$W = \{1,2,3,4\}$

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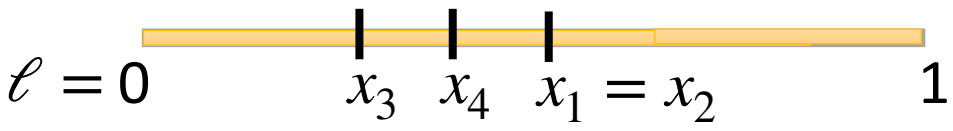

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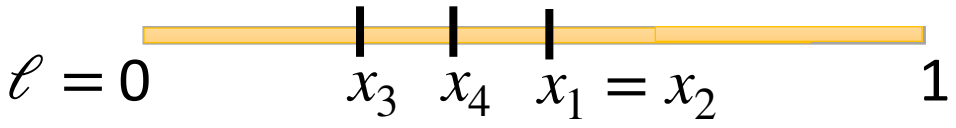

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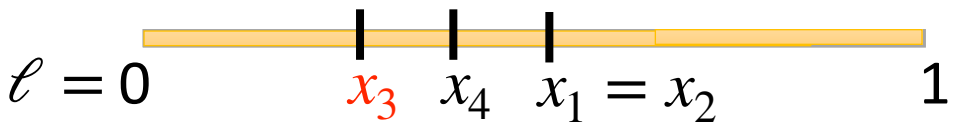

$i^* = 3$

$W = \{1,2,3,4\}$

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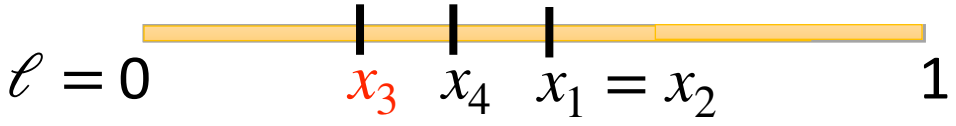

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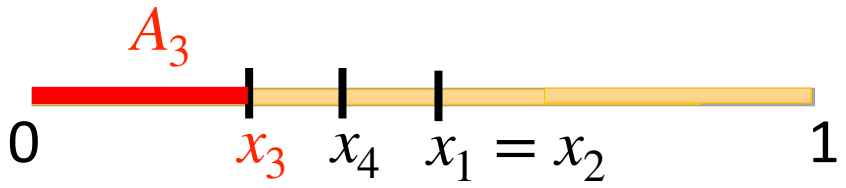

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$v_3(A_3) = 1/4$

$W = \{1,2,4\}$

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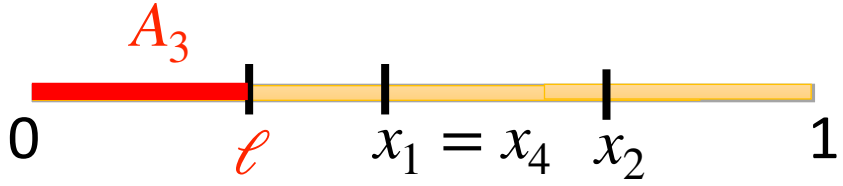

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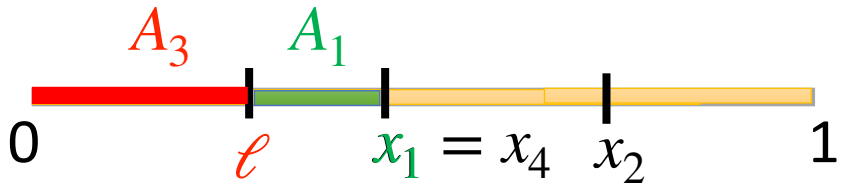

$i^* = 1$

$W = \{1,2,4\}$

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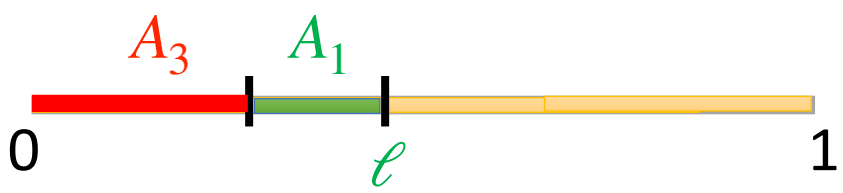

$v_3(A_3) = 1/4$

$v_1(A_1) = 1/4$

$W = \{2,4\}$

Moving-Knife Protocol (Dubins-Spanier)

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$v_3(A_3) = 1/4$

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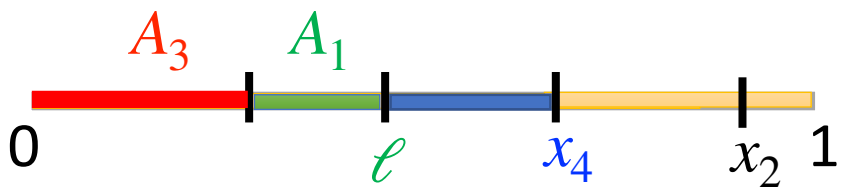

$i^* = 4$

$W = \{2,4\}$

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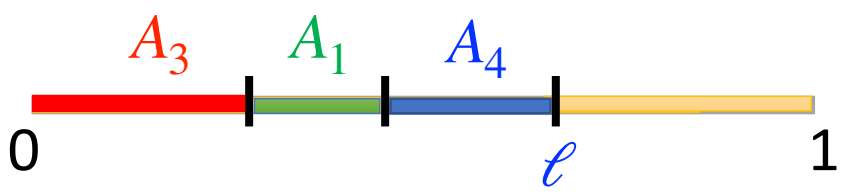

$v_3(A_3) = 1/4$
$v_1(A_1) = 1/4$
$v_4(A_4) = 1/4$

$W = \{2\}$

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2. While $|W| > 1$,
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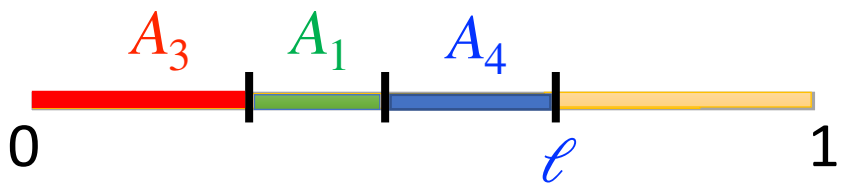
3. Give the remaining piece to the agent left in W



$v_3(A_3) = 1/4$
$v_1(A_1) = 1/4$
$v_4(A_4) = 1/4$

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Moving-Knife Protocol (Dubins-Spanier)

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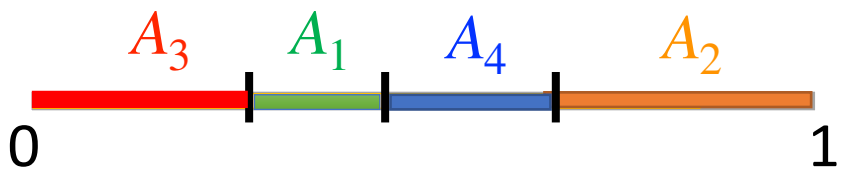

$v_3(A_3) = 1/4$

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$v_4(A_4) = 1/4$

$W = \varnothing$

Moving-Knife Protocol (Dubins-Spanier)

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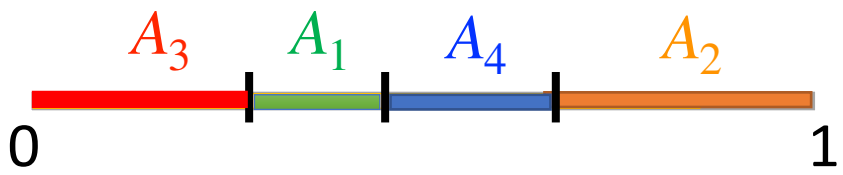

$W = \varnothing$

$v_3(A_3) = 1/4$

$v_1(A_1) = 1/4$

$v_4(A_4) = 1/4$

$v_2(A_2) \geq 1/4$

Moving-Knife Protocol (Dubins-Spanier)

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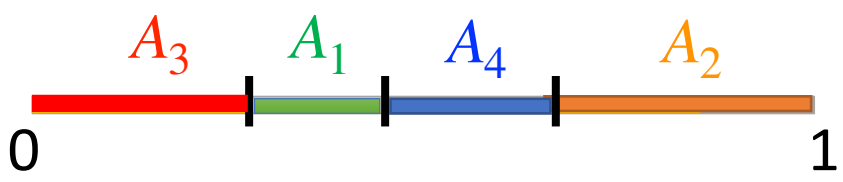

In general,

$v_i(A_i) = 1/n$ for all agents

$v_i(A_i) \geq 1/n$ for the last agent

Moving-Knife Protocol (Dubins-Spanier)

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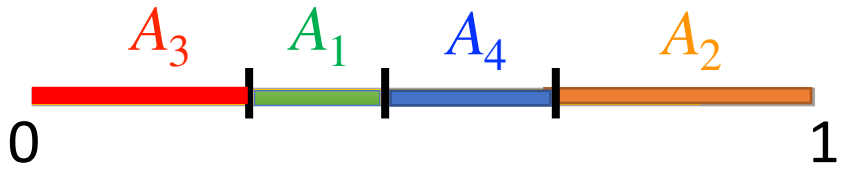

Prop

Moving-Knife Protocol (Dubins-Spanier)

1. Initialize $\ell = 0$ and $W = [n]$
2. While $|W| > 1$,
   - $\text{cut}_i([\ell, 1],\ 1/n)$ to each agent $i \in W$
   - Set $i^* = \underset{i \in W}{\text{argmin}}\ x_i$
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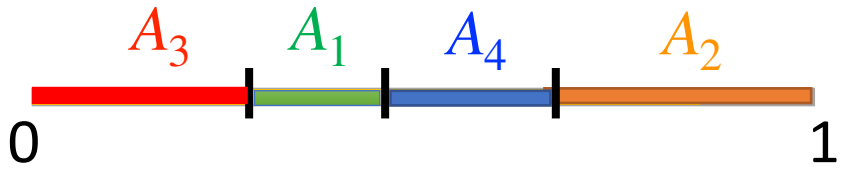

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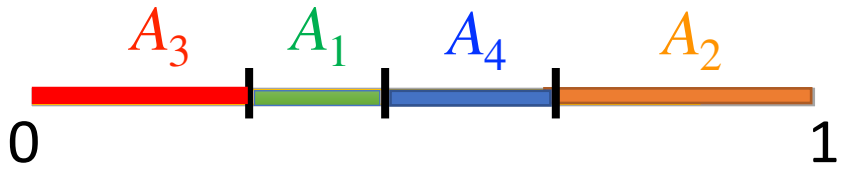
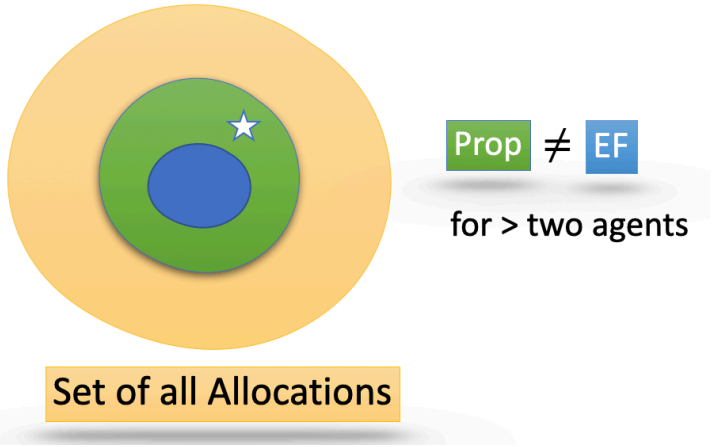


Prop A total of $\mathcal{O}(n^2)$ queries

Query Complexity of Proportionality

Source: ‘Computational Social Choice’ Lecture Slides by Rohit Vaish, IITD

Query Complexity of Proportionality

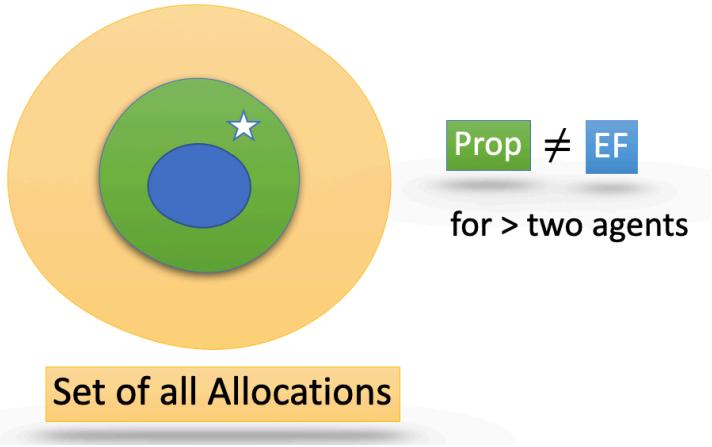


Prop ≠ EF

for > two agents

Source: ‘Computational Social Choice’ Lecture Slides by Rohit Vaish, IITD

Query Complexity of Proportionality



Prop ≠ EF

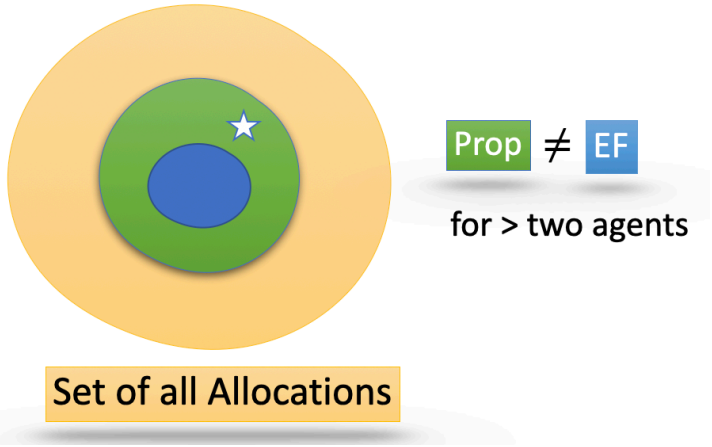
for > two agents

$\mathcal{O}(n^2)$ — Dubins-Spanier, *Amer. Math. Mon. 1961*

2 queries for $n = 2$ — Cut-and-choose

Source: 'Computational Social Choice' Lecture Slides by Rohit Vaish, IITD

Query Complexity of Proportionality



Prop $\neq$ EF

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$\mathcal{O}(n^2)$ Dubins-Spanier, *Amer. Math. Mon. 1961*

$\mathcal{O}(n \log n)$ Even-Paz [1984] (via recursion)

2 queries for $n = 2$ Cut-and-choose

Source: 'Computational Social Choice' Lecture Slides by Rohit Vaish, IITD

Query Complexity of Proportionality

Set of all Allocations

Prop ≠ EF

for > two agents

$\mathcal{O}(n^2)$ Dubins-Spanier, *Amer. Math. Mon. 1961*

$\mathcal{O}(n \log n)$ Even-Paz [1984] (via recursion)

$\Omega(n \log n)$ Edmonds & Pruhs, *TALG 2011]*

2 queries for $n = 2$ Cut-and-choose

Source: 'Computational Social Choice' Lecture Slides by Rohit Vaish, IITD

Envy-free Protocol

Envy-free Protocol for 3 agents

Envy-free Protocol for 3 agents

make three
equal pieces

Envy-free Protocol for 3 agents

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make my
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Envy-free Protocol for 3 agents

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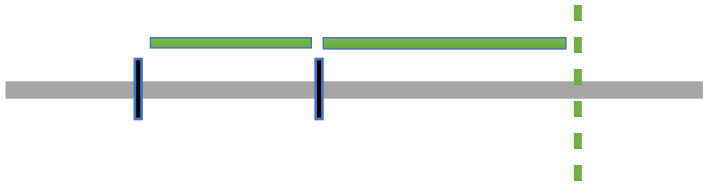

Envy-free Protocol for 3 agents

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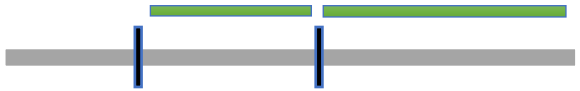
equal pieces

make my
top two
pieces equal



Trimmings

Envy-free Protocol for 3 agents

make three
equal pieces

make my
top two
pieces equal

I pick first

Trimmings

Envy-free Protocol for 3 agents

make three
equal pieces

make my
top two
pieces equal

I pick second
(one of the trimmed pieces)

I pick first

Trimmings

Envy-free Protocol for 3 agents

make three
equal pieces

I pick last

make my
top two
pieces equal

I pick second
(one of the trimmed pieces)

I pick first

Trimmings

Envy-free Protocol for 3 agents

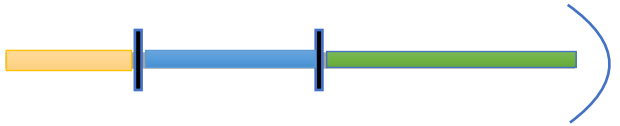
make three
equal pieces

I pick last
(untrimmed piece)

make my
top two
pieces equal

I pick second
(one of the trimmed pieces)

I pick first



envy-free

Trimmings

Envy-free Protocol for 3 agents

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I pick last
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make my
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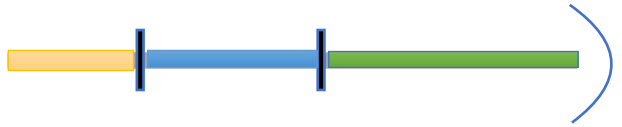
I pick second
(one of the trimmed pieces)

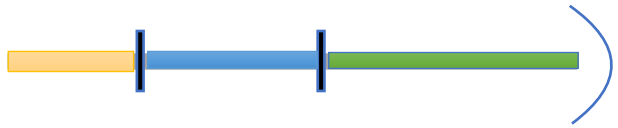
I pick first
hence EF



envy-free

Trimmings

Envy-free Protocol for 3 agents

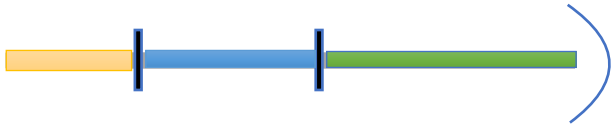


envy-free

Trimmings (T)

Envy-free Protocol for 3 agents

equi-divide T
& pick last



envy-free

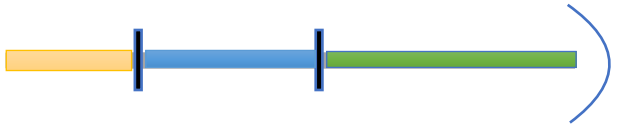
Trimmings (T)

Envy-free Protocol for 3 agents

I pick second

I pick first

equi-divide T
& pick last



envy-free

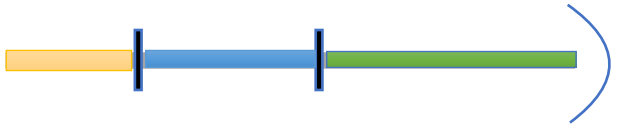
Trimmings (T)

Envy-free Protocol for 3 agents

I pick second

I pick first,
hence EF

equi-divide T
& pick last



envy-free

Trimmings (T)

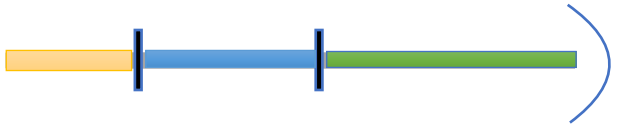
Envy-free Protocol for 3 agents

I pick second

Advantage from first round, hence EF

I pick first, hence EF

equi-divide T & pick last



envy-free

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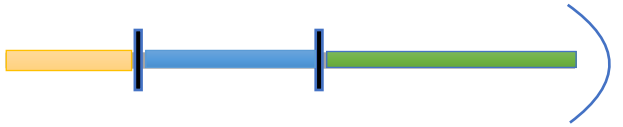
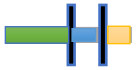


envy-free



Trimmings (T)

Envy-free Protocol for 3 agents

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Advantage from first round, hence EF

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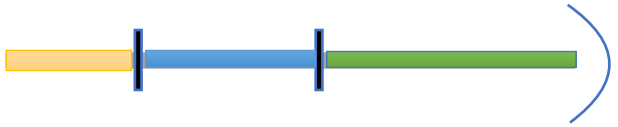
equi-divide T & pick last

I equi-divided hence EF



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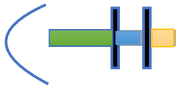
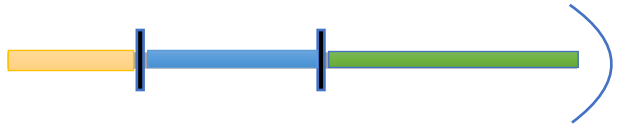


envy-free

Trimmings (T)

Envy-free Protocol for 3 agents



envy-free

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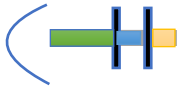
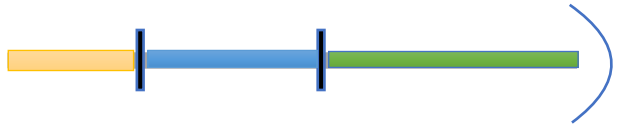


Trimmings (T)

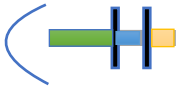
Hence, we find an envy-free cake division

Envy-free Protocol for 3 agents



envy-free

envy-free

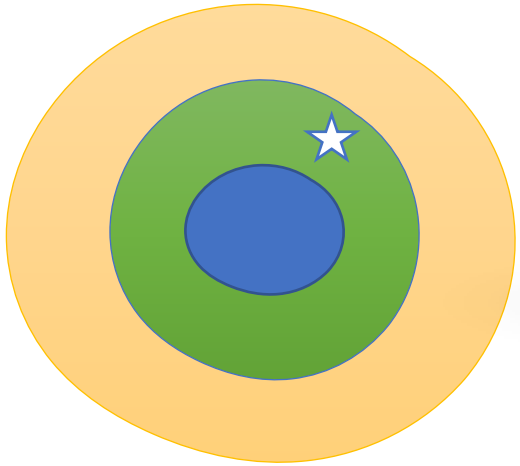


Trimmings (T)

Selfridge-Conway protocol finds an EF cake division among *three* agents using $\mathcal{O}(1)$ queries

Existence of Envy-free Cake Divisions

Existence of Envy-free Cake Divisions



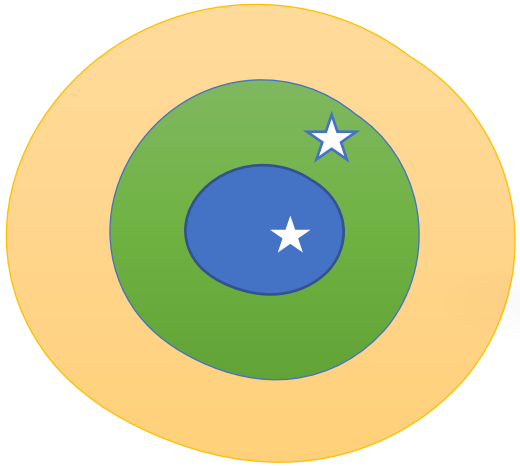
All allocations

EF ⇌ Prop

Stromquist [1980], Su [1999]

Envy-free cake division exist for any number of agents (Lecture 04)

Existence of Envy-free Cake Divisions



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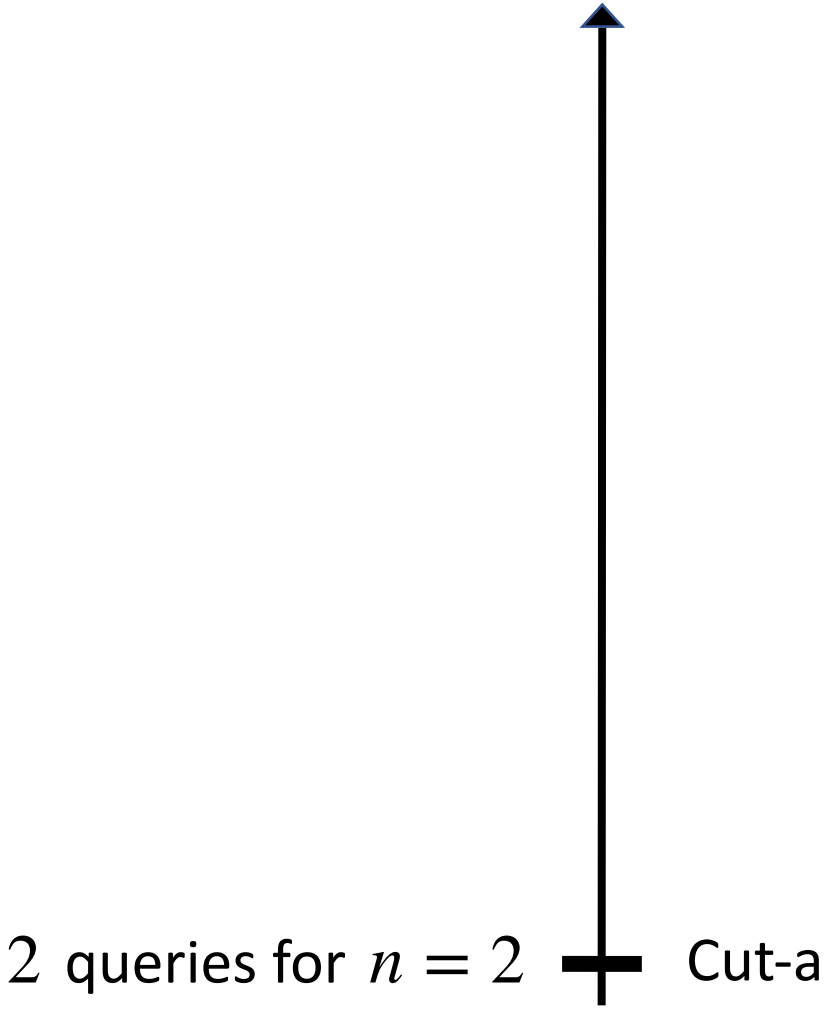
Query Complexity of Envy-freeness

Source: ‘Computational Social Choice’ Lecture Slides by Rohit Vaish, IITD

Query Complexity of Envy-freeness



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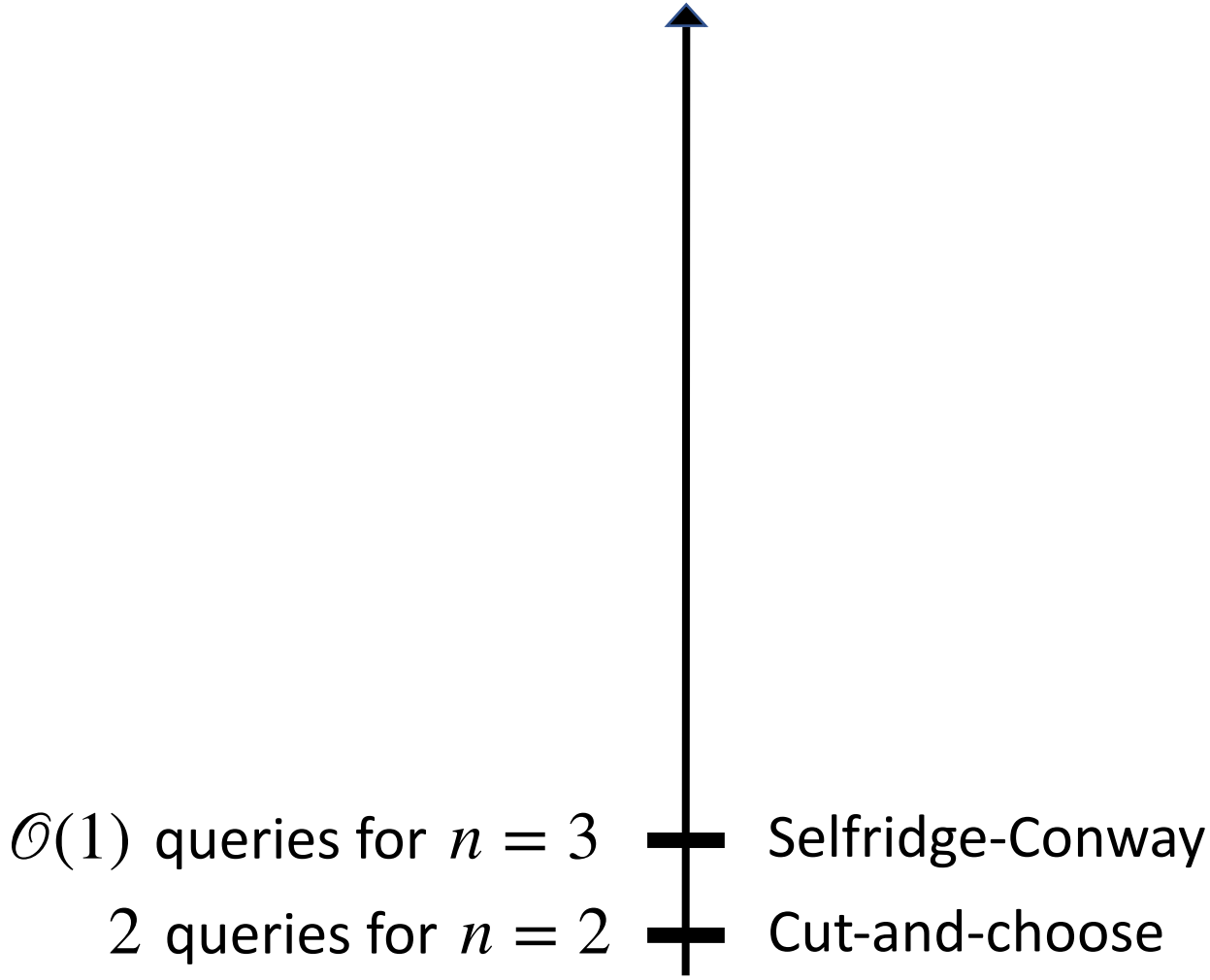

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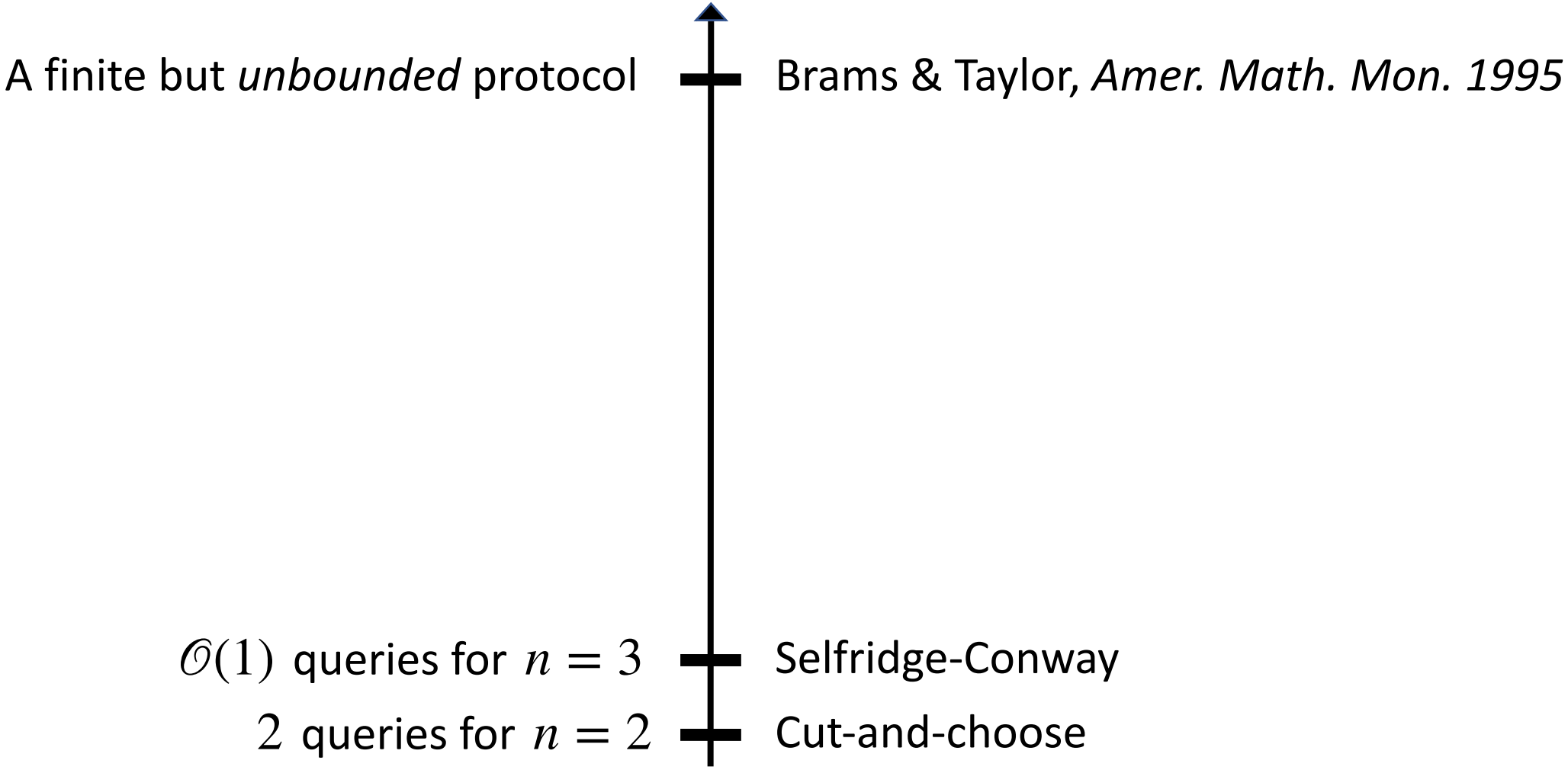

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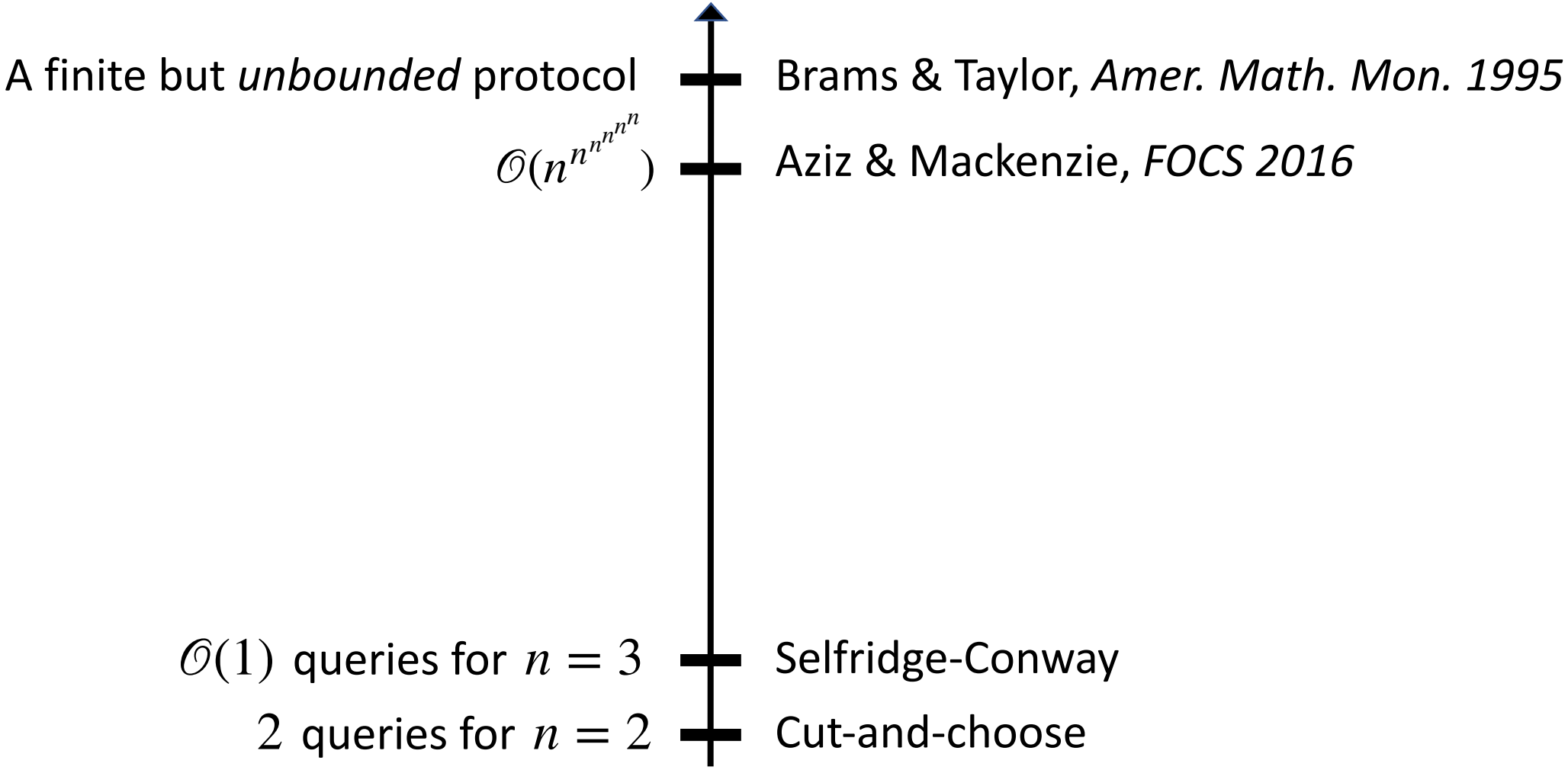

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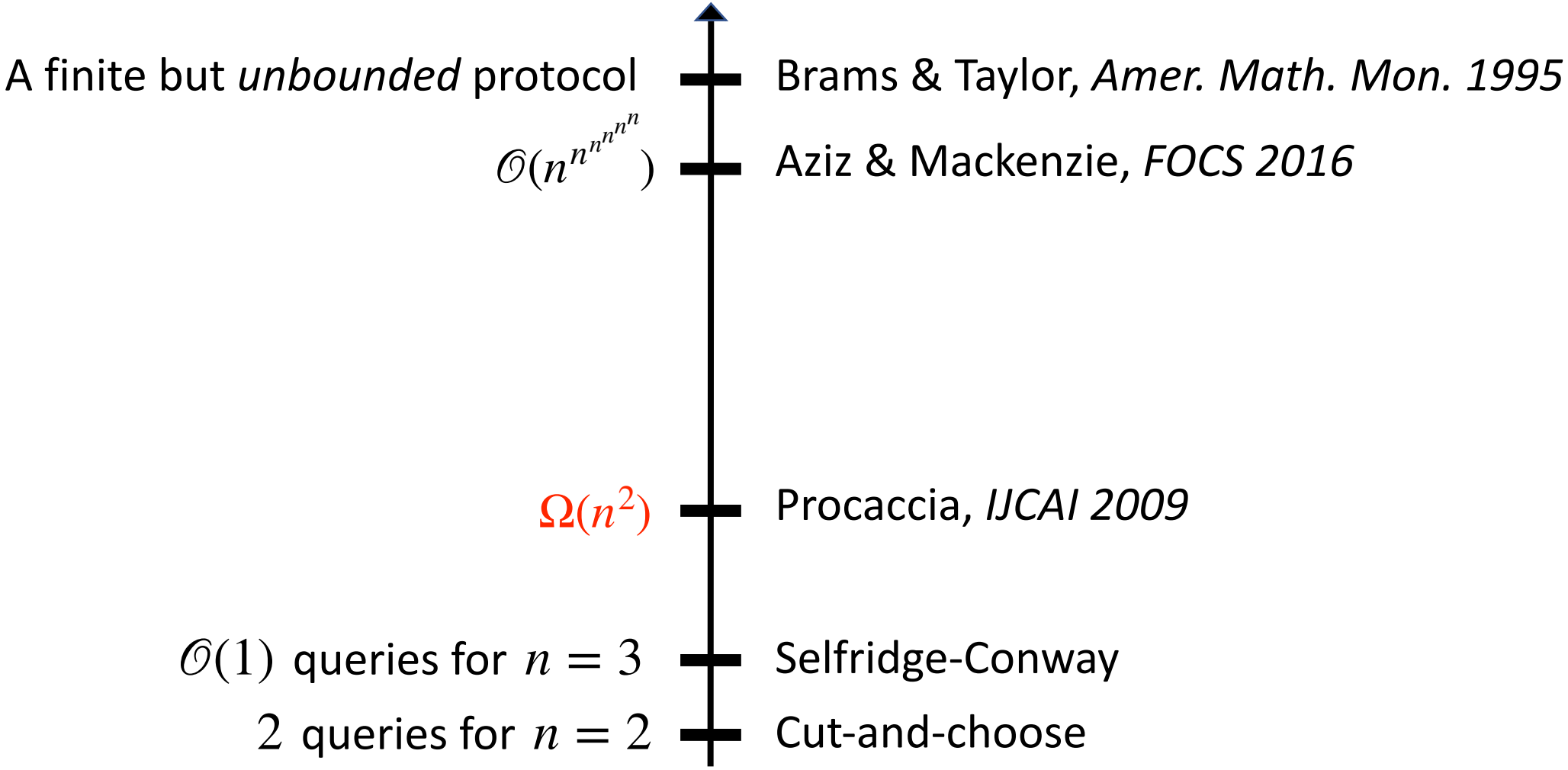

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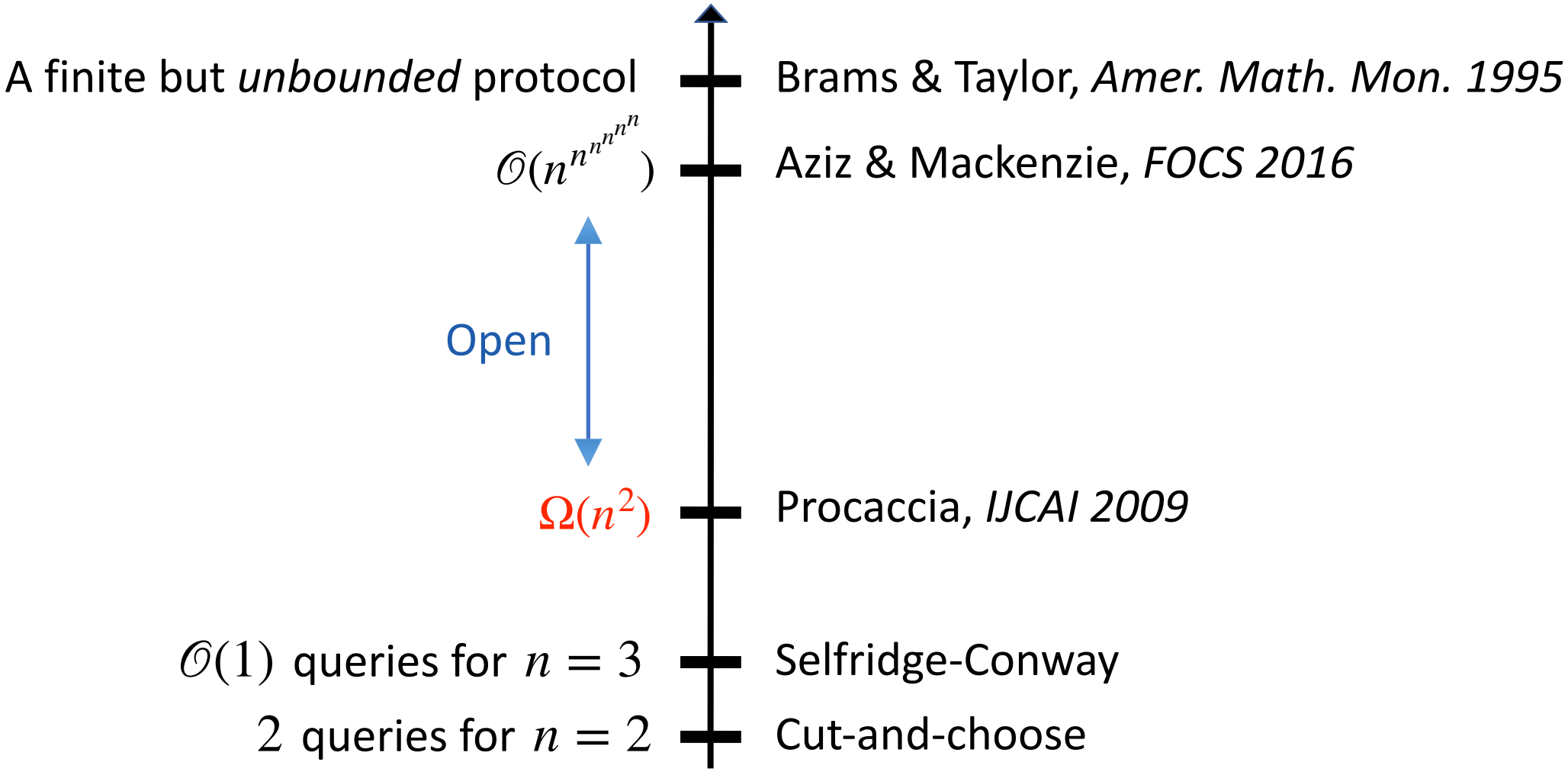

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Query Complexity of Envy-freeness

What happens when every agent wishes to have a ***contiguous*** piece of the cake?

Query Complexity of Envy-freeness

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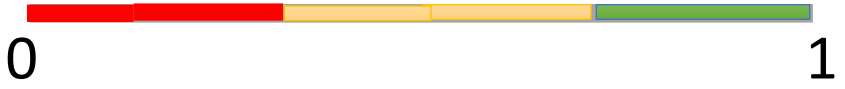

Stromquist [1980], Su [1999]

Envy-free cake division exists for any number of agents

(4th Lecture)

Query Complexity of Envy-freeness

What happens when every agent wishes to have a ***contiguous*** piece of the cake?

Stromquist [1980], Su [1999] **connected pieces**

Envy-free cake division exists for any number of agents (4th Lecture)

Query Complexity of Envy-freeness

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No finite-query protocol exists for connected EF cake division

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Stromquist [1980], Su [1999] **connected pieces**

Envy-free cake division exists for any number of agents (30 April)

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No finite-query protocol exists for connected EF cake division

[ABKR] *WINE 2019* (Fair and Efficient Cake Division with Connected Pieces)

An efficient algorithm: **1/2-EF + 1/3-NSW** allocation for *connected* EF cake division

(28 May)