max planck institut
informatik

Topics in Computational Social Choice Theory

Lecture 02: Introduction on Fair Cake Division

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Saarland Informatics
Campus

Last Lecture: Discrete Fair Division

What is **fairness as a concept**?

How to **compute a fair allocation**?

Agents with **valuations** over items

Indivisible items

Goal: To **divide** the items among the agents in a **fair** manner

Fair Division

What is **fairness as a concept**?

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Goal: To **divide** the resource among the agents in a **fair** manner

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- **Mathematical study** of fairly allocating resources among agents with distinct preferences, but equal entitlements.
- Focus on **provable guarantees**.
- **Computational Perspective**: work towards algorithms & hardness results and approximation algorithms

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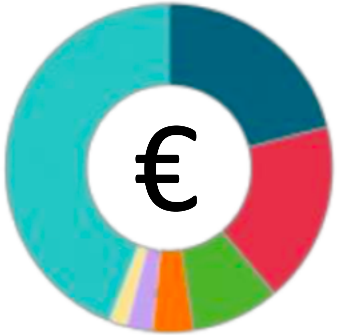

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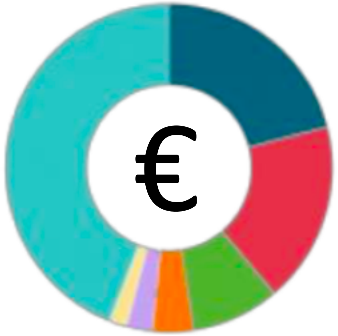

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Cake-Cutting

How to ***fairly*** cut the cake?

Cake-Cutting

How to ***fairly*** divide a cake
among agents with **differing preferences**?

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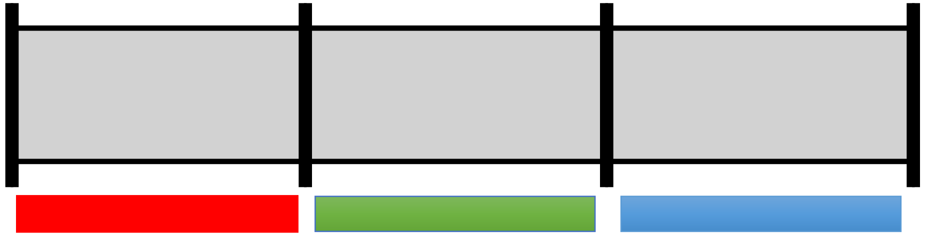

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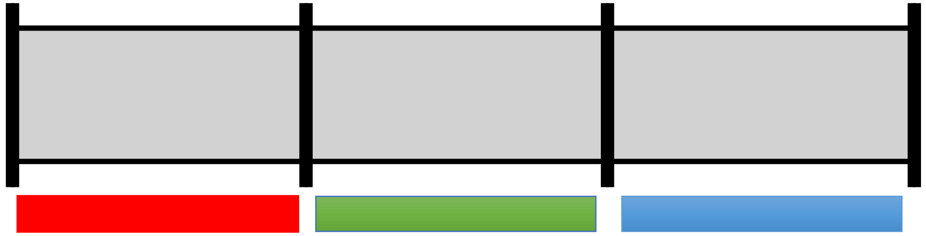


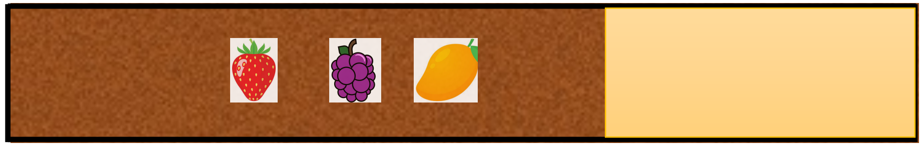
Fair

Why is this problem interesting?

I only like vanilla

I like chocolate and vanilla

I love fruits



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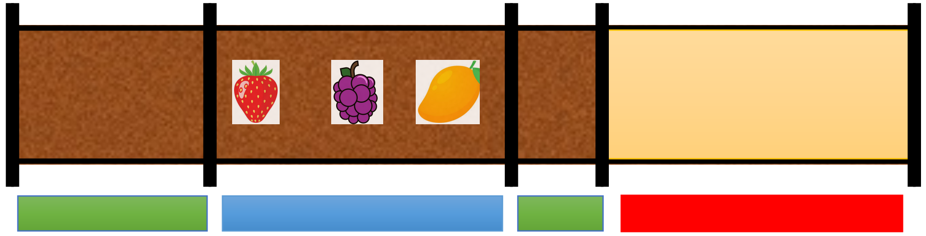

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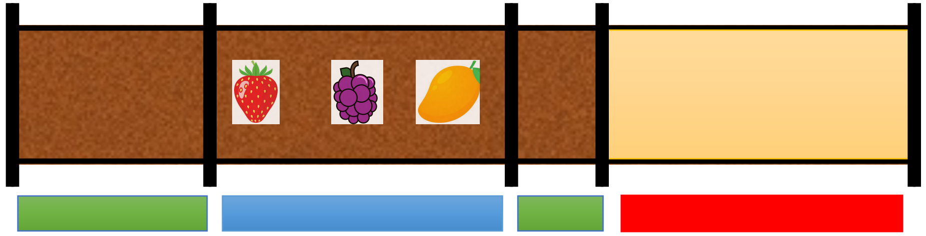

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Preferences matter!

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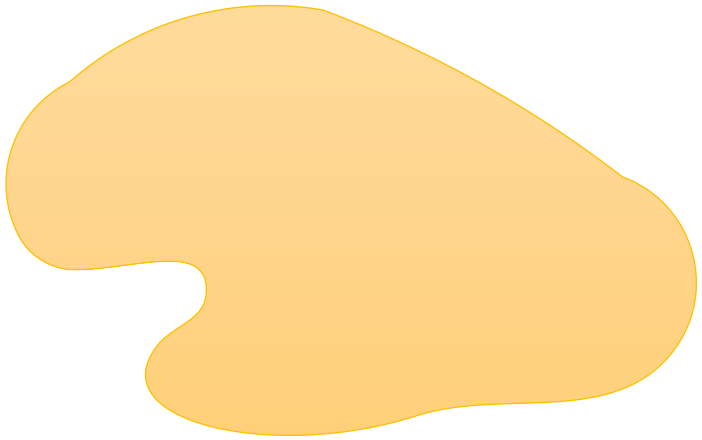

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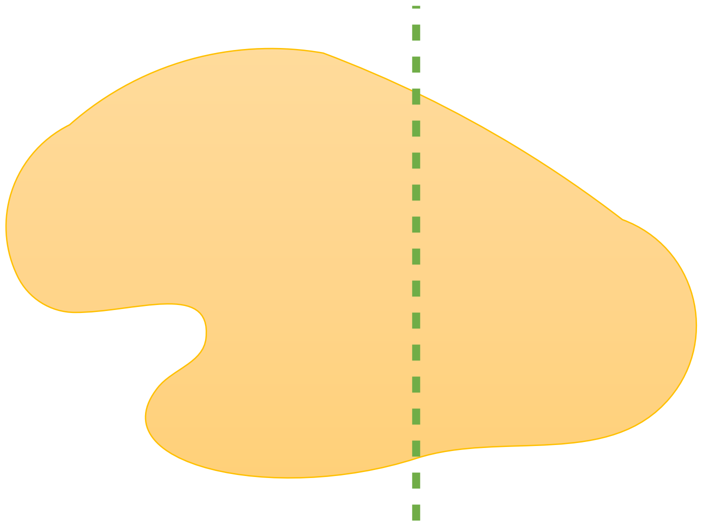

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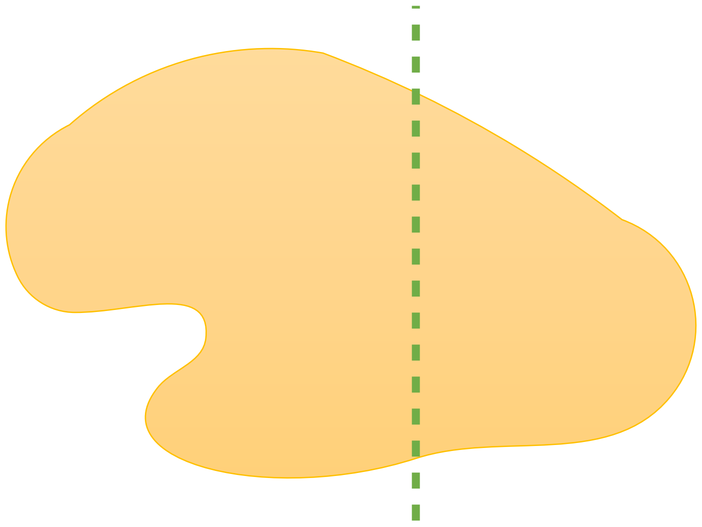

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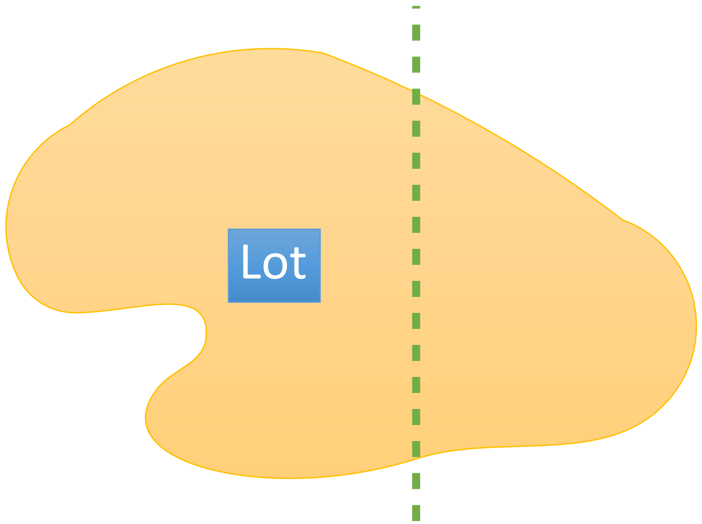

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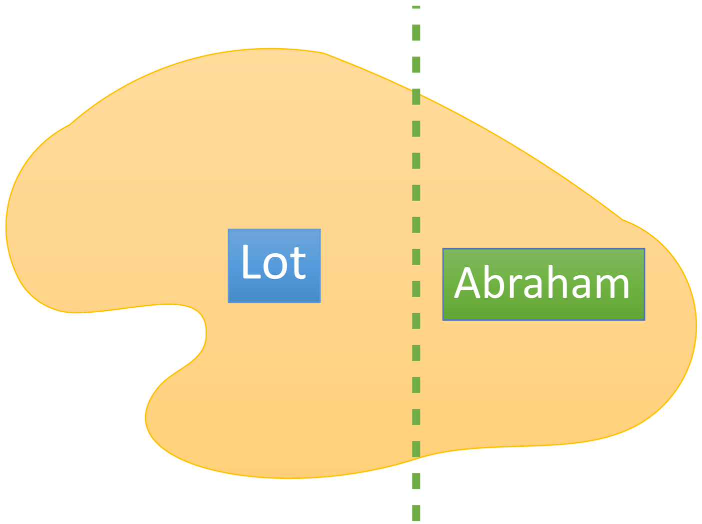

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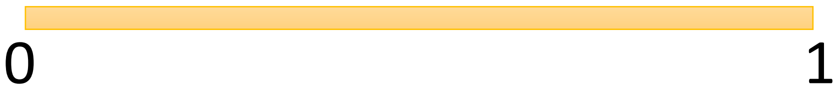

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Envy-freeness

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- Set of **agents**: {1,2, …, n}
- **Piece** of a cake: finite union of subintervals of [0,1]

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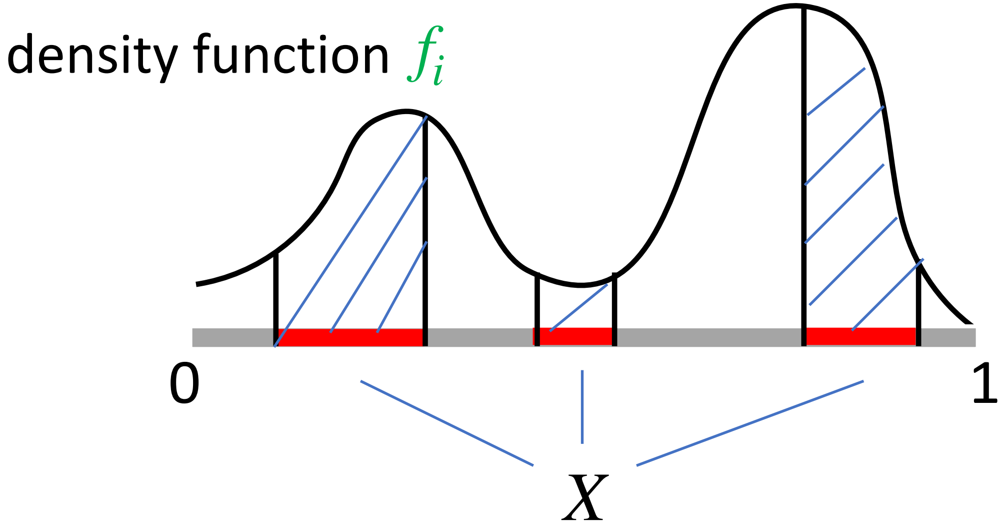

$$v_i(X) = \int_{x \in X} f_i(x)\, dx$$

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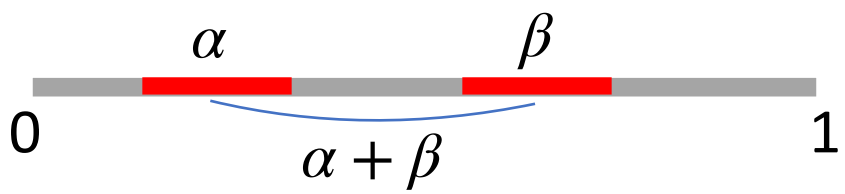
  For disjoint $X, Y \subset [0,1]$, we have $v_i(X \cup Y) = v_i(X) + v_i(Y)$



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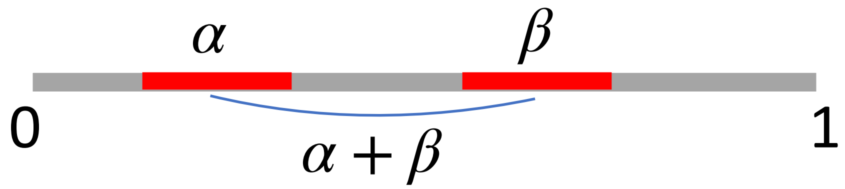
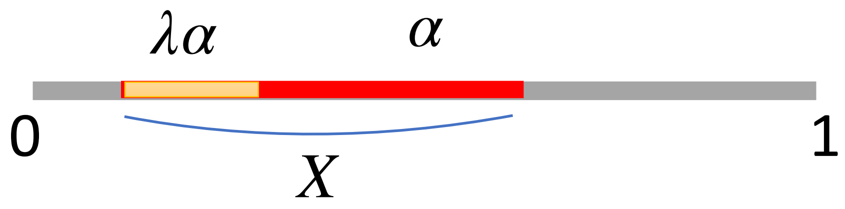


<u>Divisible:</u>

For any $X \subseteq [0,1]$ and $\lambda \in [0,1]$, there exists a $Y \subseteq X$ s.t. $v_i(Y) = \lambda v_i(X)$



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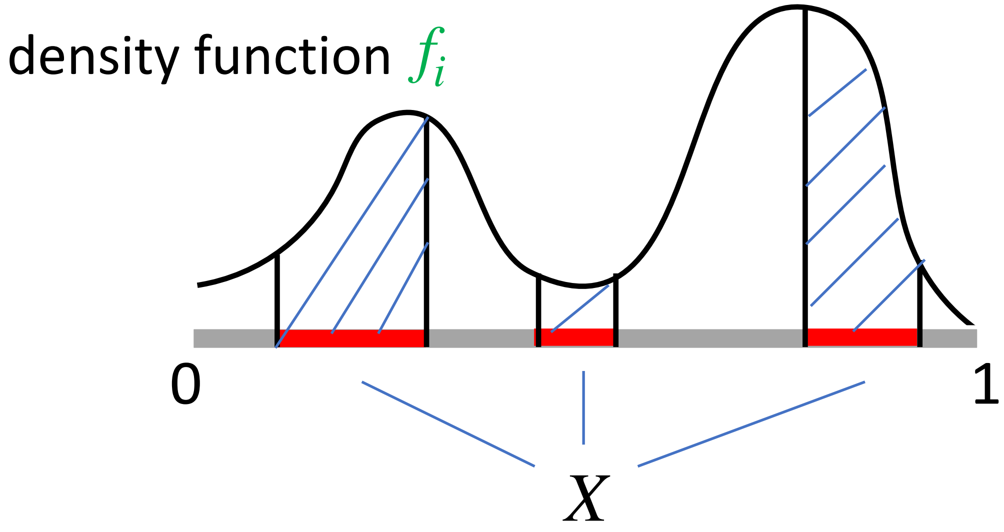

$$v_i(X) = \int_{x \in X} f_i(x)\ dx$$

v_i is a probability distribution over $[0,1]$

Robertson-Webb Query Model

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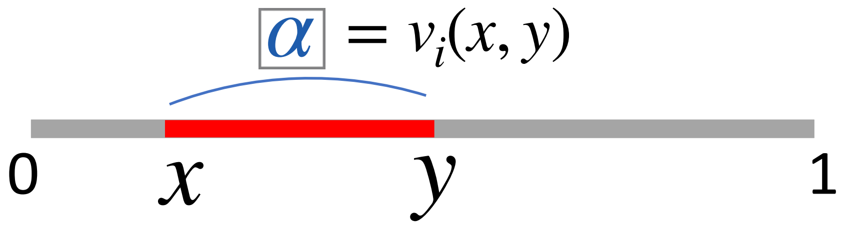

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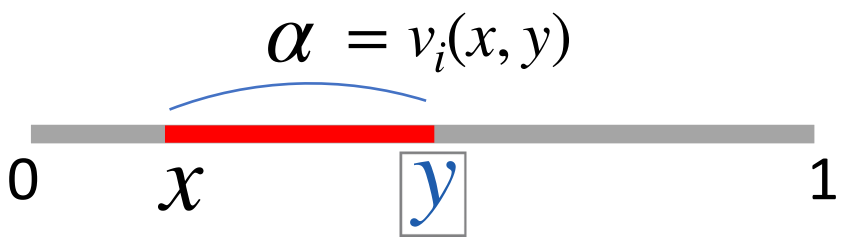

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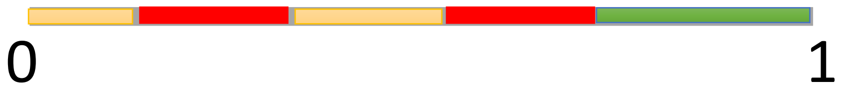
Allocation:

A partition $A = (A_1, A_2, \ldots, A_n)$ of the cake $[0,1]$ where piece A_i belongs to agent i



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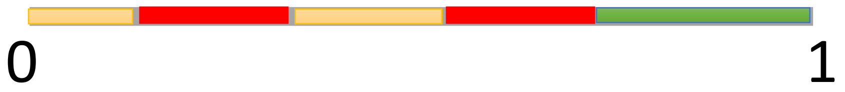

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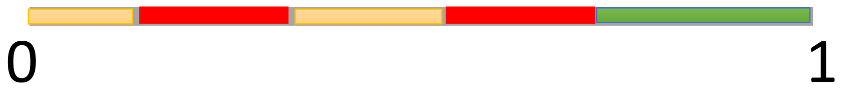

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  [Foley 1967]

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EF and **Prop** are *equivalent for two agents*

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Can cut-and-choose be implemented in RW model? Yes!

$$\text{cut}_1(0,1/2) = x$$

$$\text{eval}_2(0,x)$$

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For two agents, an EF/Prop cake division can be computed using two queries

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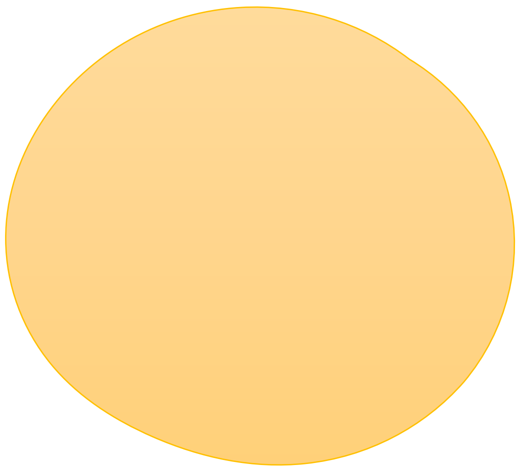

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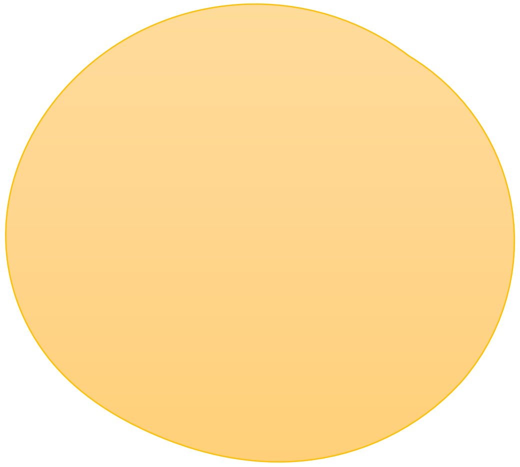
EF $\Longrightarrow$ Prop for *any* number of agents



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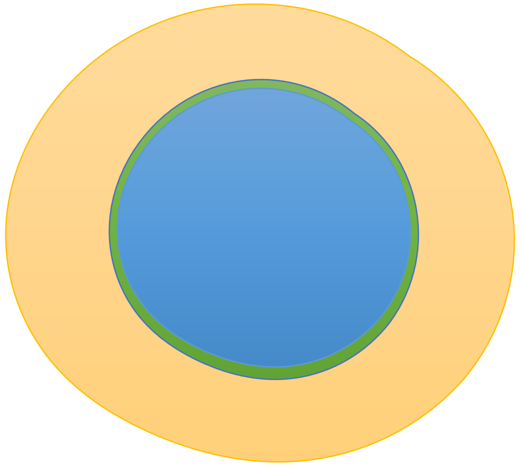
Prop $\Longrightarrow$ EF for *two* agents



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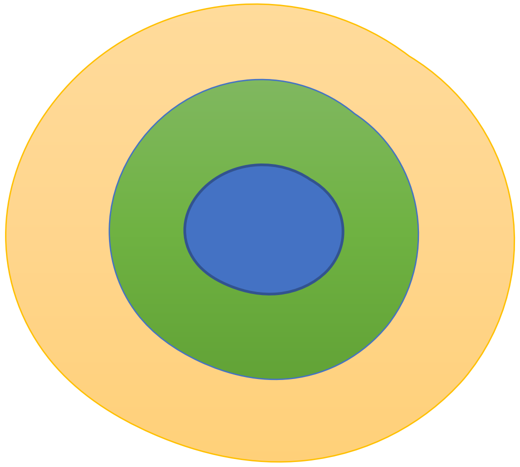
Prop $\Longrightarrow$ EF for *two* agents *(but no more)*



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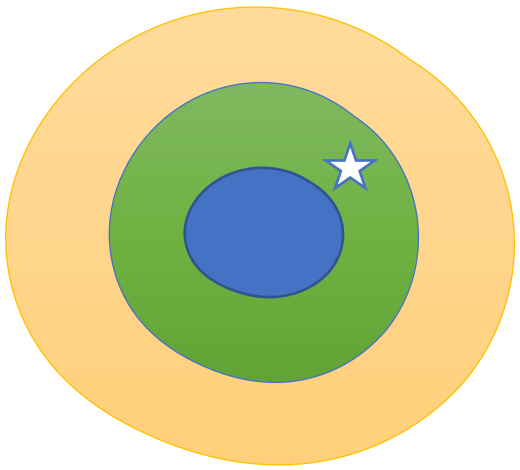

All allocations

A proportional cake division always exists and can be computed efficiently

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(i) Moving-knife Protocol - Dubins and Spanier [1961]
(ii) Even-Paz Protocol [1984]

Reference: Handbook of Computational Social Choice, see Chapter 13 by Ariel Procaccia.

Moving-Knife Protocol (Dubins-Spanier)

An efficient proportional cake division protocol for any number of agents

Moving-Knife Protocol (Dubins-Spanier)

1. Initialize $\ell = 0$ and $W = [n]$

$\ell = 0$ 1

$W = \{1,2,3,4\}$

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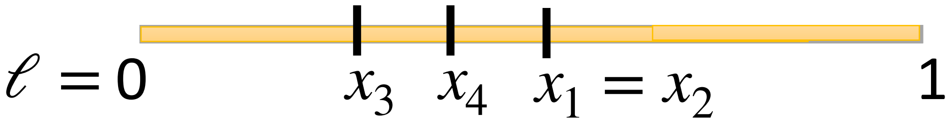

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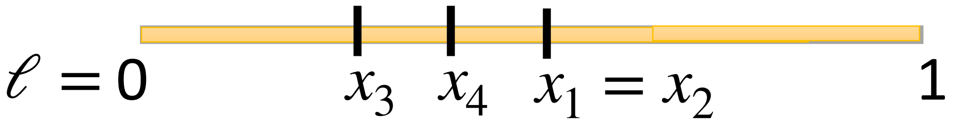
   - Each agent $i \in W$ marks $x_i \in [\ell, 1]$ such that $v_i([\ell, x_i]) = 1/n$
   - Set $i^* = \operatorname{argmin}_{i \in W} x_i$



$W = \{1,2,3,4\}$

Moving-Knife Protocol (Dubins-Spanier)

1. Initialize $\ell = 0$ and $W = [n]$
2. While $|W| > 1$,
   - Each agent $i \in W$ marks $x_i \in [\ell, 1]$ such that $v_i([\ell, x_i]) = 1/n$
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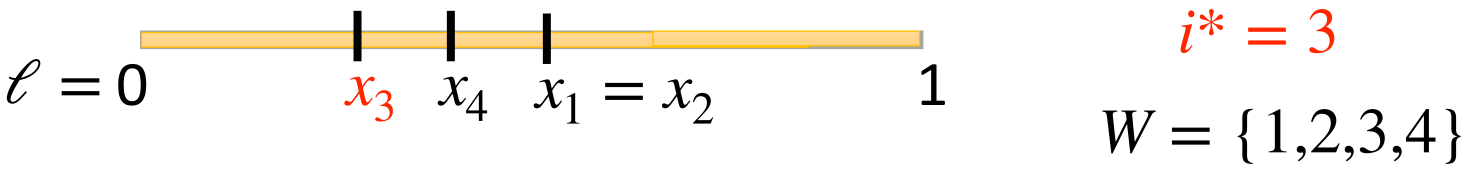

Moving-Knife Protocol (Dubins-Spanier)

1. Initialize $\ell = 0$ and $W = [n]$
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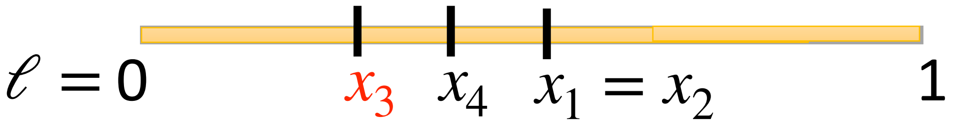
   - Set $i^* = \underset{i \in W}{\operatorname{argmin}} \; x_i$
   - Set $A_{i^*} = [\ell, x_{i^*}]$



$i^* = 3$

$W = \{1,2,3,4\}$

Moving-Knife Protocol (Dubins-Spanier)

1. Initialize $\ell = 0$ and $W = [n]$
2. While $|W| > 1$,
   - Each agent $i \in W$ marks $x_i \in [\ell, 1]$ such that $v_i([\ell, x_i]) = 1/n$
   - Set $i^* = \operatorname{argmin}_{i \in W} x_i$
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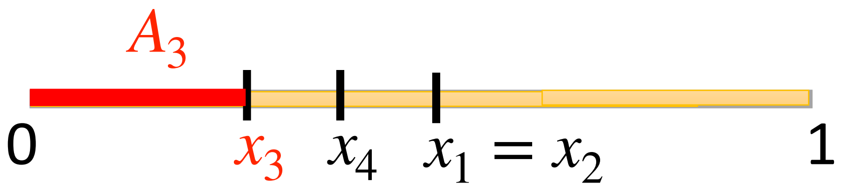

$i^* = 3$

$W = \{1,2,3,4\}$

Moving-Knife Protocol (Dubins-Spanier)

1. Initialize $\ell = 0$ and $W = [n]$
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   - Set $i^* = \operatorname{argmin}_{i \in W} \; x_i$
   - Set $A_{i^*} = [\ell, x_{i^*}]$
   - Update $\ell = x_i$ and $W = W \backslash \{i^*\}$

$v_3(A_3) = 1/4$

$W = \{1,2,4\}$

Moving-Knife Protocol (Dubins-Spanier)

1. Initialize $\ell = 0$ and $W = [n]$
2. While $|W| > 1$,
   - Each agent $i \in W$ marks $x_i \in [\ell, 1]$ such that $v_i([\ell, x_i]) = 1/n$
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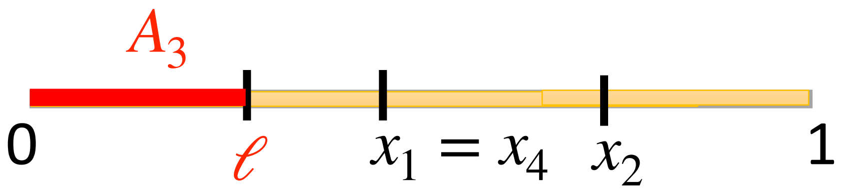
$v_3(A_3) = 1/4$



$W = \{1,2,4\}$

Moving-Knife Protocol (Dubins-Spanier)

1. Initialize $\ell = 0$ and $W = [n]$
2. While $|W| > 1$,
   - Each agent $i \in W$ marks $x_i \in [\ell, 1]$ such that $v_i([\ell, x_i]) = 1/n$
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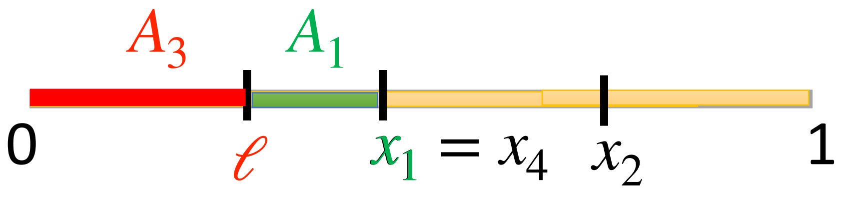
   - Set $A_{i^*} = [\ell, x_{i^*}]$
   - Update $\ell = x_i$ and $W = W \backslash \{i^*\}$

$v_3(A_3) = 1/4$

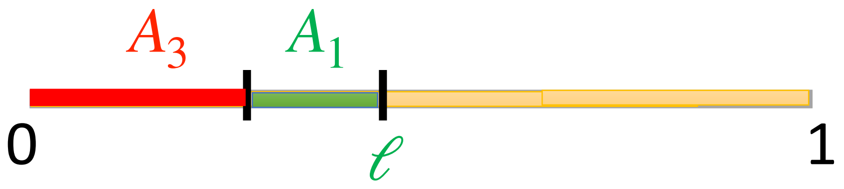


$i^* = 1$

$W = \{1,2,4\}$

Moving-Knife Protocol (Dubins-Spanier)

1. Initialize $\ell = 0$ and $W = [n]$
2. While $|W| > 1$,
   - Each agent $i \in W$ marks $x_i \in [\ell, 1]$ such that $v_i([\ell, x_i]) = 1/n$
   - Set $i^* = \operatorname{argmin}_{i \in W} x_i$
   - Set $A_{i^*} = [\ell, x_{i^*}]$
   - Update $\ell = x_i$ and $W = W \setminus \{i^*\}$

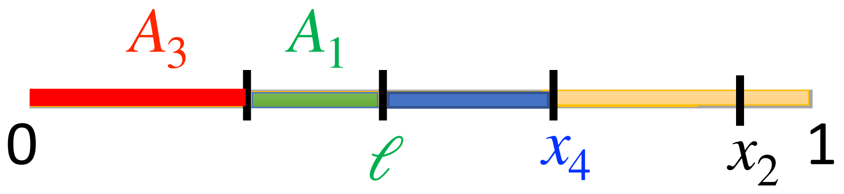


$v_3(A_3) = 1/4$
$v_1(A_1) = 1/4$

$W = \{2,4\}$

Moving-Knife Protocol (Dubins-Spanier)

1. Initialize $\ell = 0$ and $W = [n]$
2. While $|W| > 1$,
   - Each agent $i \in W$ marks $x_i \in [\ell, 1]$ such that $v_i([\ell, x_i]) = 1/n$
   - Set $i^* = \operatorname*{argmin}_{i \in W} \; x_i$
   - Set $A_{i^*} = [\ell, x_{i^*}]$
   - Update $\ell = x_i$ and $W = W \backslash \{i^*\}$



$v_3(A_3) = 1/4$

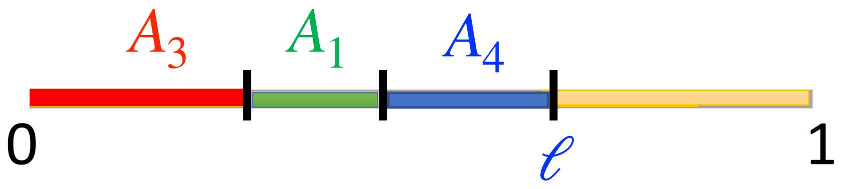
$v_1(A_1) = 1/4$

$i^* = 4$

$W = \{2,4\}$

Moving-Knife Protocol (Dubins-Spanier)

1. Initialize $\ell = 0$ and $W = [n]$
2. While $|W| > 1$,
   - Each agent $i \in W$ marks $x_i \in [\ell, 1]$ such that $v_i([\ell, x_i]) = 1/n$
   - Set $i^* = \underset{i \in W}{\text{argmin}} \; x_i$
   - Set $A_{i^*} = [\ell, x_{i^*}]$
   - Update $\ell = x_i$ and $W = W \setminus \{i^*\}$



$v_3(A_3) = 1/4$

$v_1(A_1) = 1/4$

$v_4(A_4) = 1/4$

$W = \{2\}$

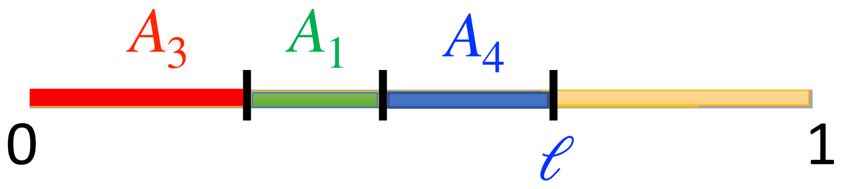
Moving-Knife Protocol (Dubins-Spanier)

1. Initialize $\ell = 0$ and $W = [n]$
2. While $|W| > 1$,
   - Each agent $i \in W$ marks $x_i \in [\ell, 1]$ such that $v_i([\ell, x_i]) = 1/n$
   - Set $i^* = \underset{i \in W}{\operatorname{argmin}} \; x_i$
   - Set $A_{i^*} = [\ell, x_{i^*}]$
   - Update $\ell = x_i$ and $W = W \backslash \{i^*\}$
3. Give the remaining piece to the agent left in W

$v_3(A_3) = 1/4$

$v_1(A_1) = 1/4$

$v_4(A_4) = 1/4$



$W = \{2\}$

Moving-Knife Protocol (Dubins-Spanier)

1. Initialize $\ell = 0$ and $W = [n]$
2. While $|W| > 1$,
   - Each agent $i \in W$ marks $x_i \in [\ell, 1]$ such that $v_i([\ell, x_i]) = 1/n$
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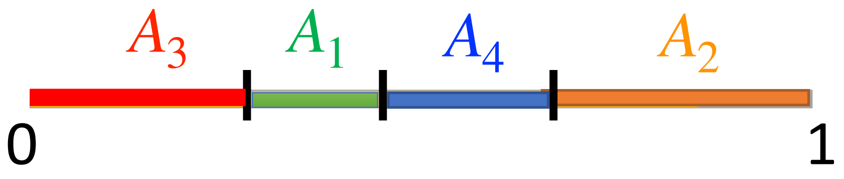
   - Set $A_{i^*} = [\ell, x_{i^*}]$
   - Update $\ell = x_i$ and $W = W \backslash \{i^*\}$
3. Give the remaining piece to the agent left in W



$W = \varnothing$

$v_3(A_3) = 1/4$

$v_1(A_1) = 1/4$

$v_4(A_4) = 1/4$

Moving-Knife Protocol (Dubins-Spanier)

1. Initialize $\ell = 0$ and $W = [n]$
2. While $|W| > 1$,
   - Each agent $i \in W$ marks $x_i \in [\ell, 1]$ such that $v_i([\ell, x_i]) = 1/n$
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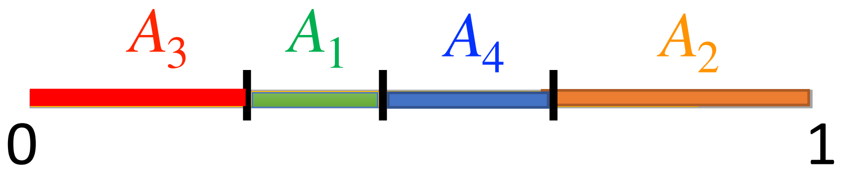
   - Set $A_{i^*} = [\ell, x_{i^*}]$
   - Update $\ell = x_i$ and $W = W \backslash \{i^*\}$
3. Give the remaining piece to the agent left in W



$W = \varnothing$

$v_3(A_3) = 1/4$

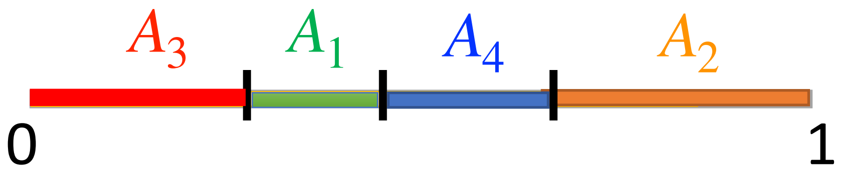
$v_1(A_1) = 1/4$

$v_4(A_4) = 1/4$

$v_2(A_2) \geq 1/4$

Moving-Knife Protocol (Dubins-Spanier)

1. Initialize $\ell = 0$ and $W = [n]$
2. While $|W| > 1$,
   - Each agent $i \in W$ marks $x_i \in [\ell, 1]$ such that $v_i([\ell, x_i]) = 1/n$
   - Set $i^* = \underset{i \in W}{\operatorname{argmin}} \; x_i$
   - Set $A_{i^*} = [\ell, x_{i^*}]$
   - Update $\ell = x_i$ and $W = W \setminus \{i^*\}$
3. Give the remaining piece to the agent left in W



In general,

$v_i(A_i) = 1/n$ for all agents

$v_i(A_i) \geq 1/n$ for the last agent

Moving-Knife Protocol (Dubins-Spanier)

1. Initialize $\ell = 0$ and $W = [n]$
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   - Update $\ell = x_i$ and $W = W \setminus \{i^*\}$
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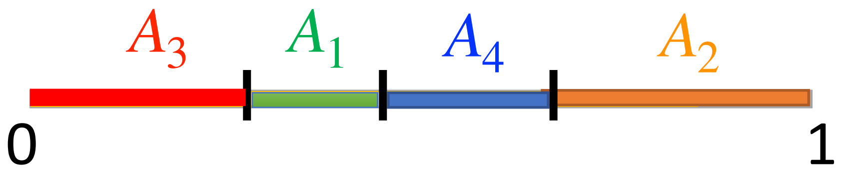
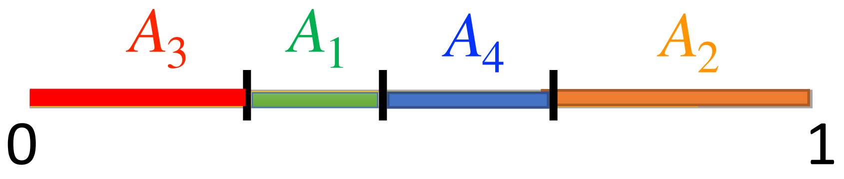


Prop

Moving-Knife Protocol (Dubins-Spanier)

1. Initialize $\ell = 0$ and $W = [n]$
2. While $|W| > 1$,
   - $\text{cut}_i([\ell, 1],\ 1/n)$ to each agent $i \in W$
   - Set $i^* = \text{argmin}_{i \in W}\ x_i$
   - Set $A_{i^*} = [\ell, x_{i^*}]$
   - Update $\ell = x_i$ and $W = W \setminus \{i^*\}$
3. Give the remaining piece to the agent left in W



Prop

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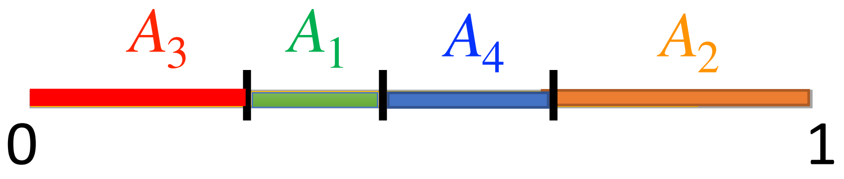


Prop

A total of $\mathcal{O}(n^2)$ queries

Query Complexity of Proportionality

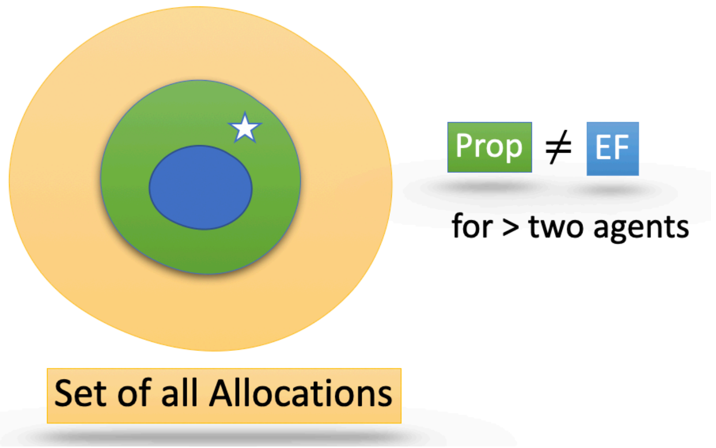
Query Complexity of Proportionality

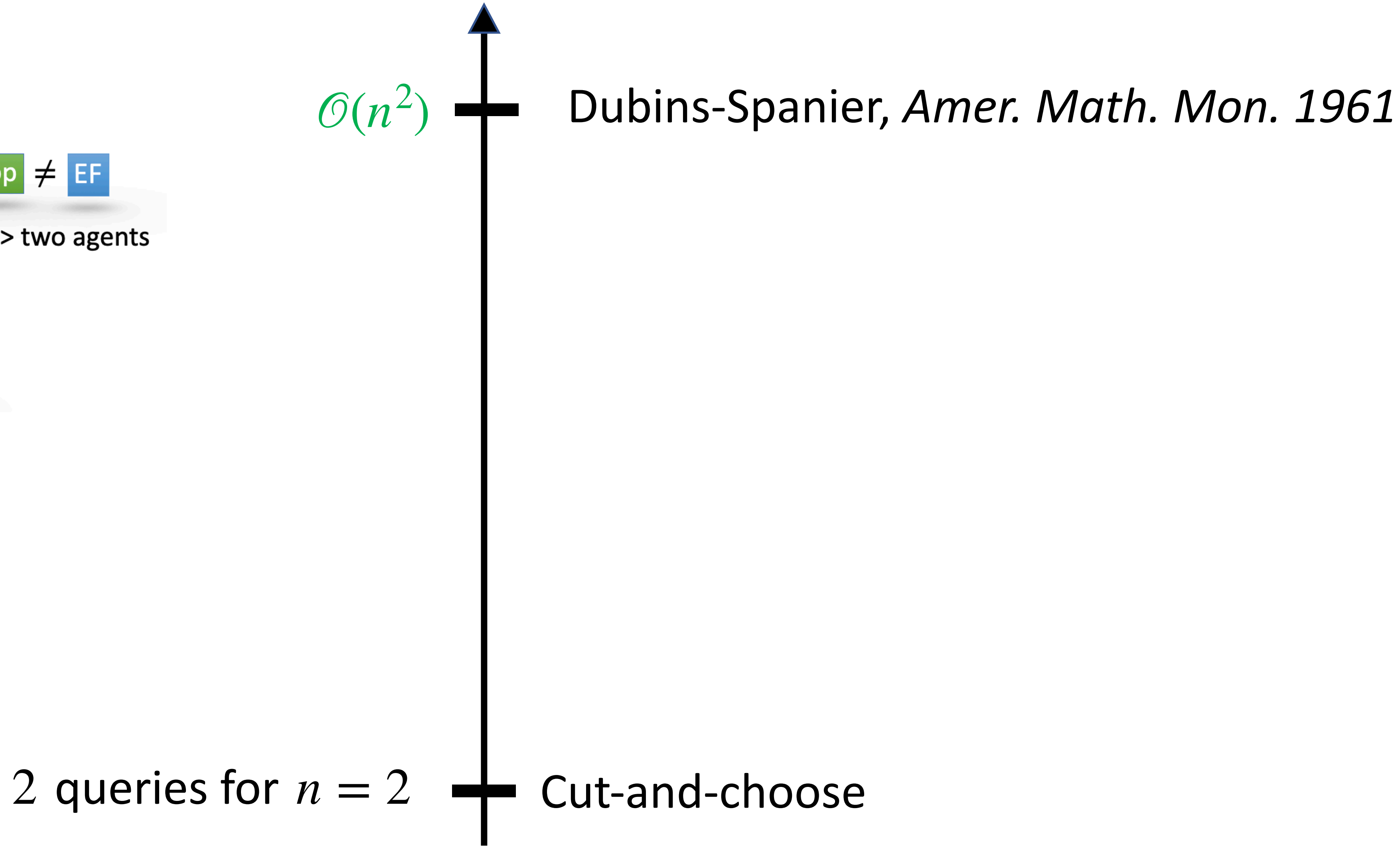
Set of all Allocations

Prop ≠ EF

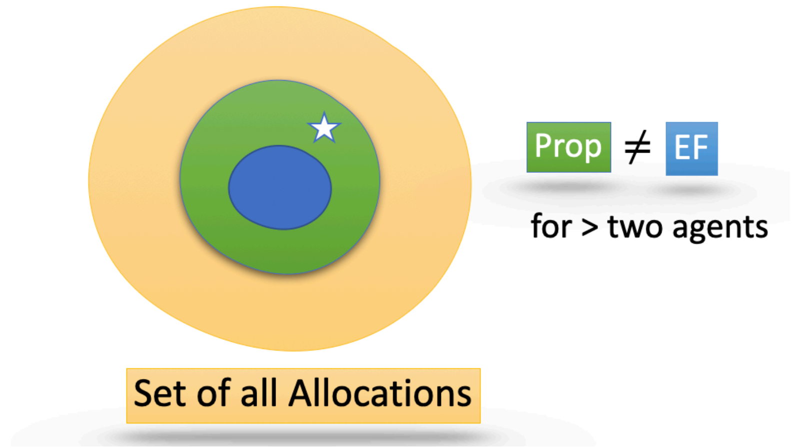
for > two agents

Query Complexity of Proportionality





Query Complexity of Proportionality

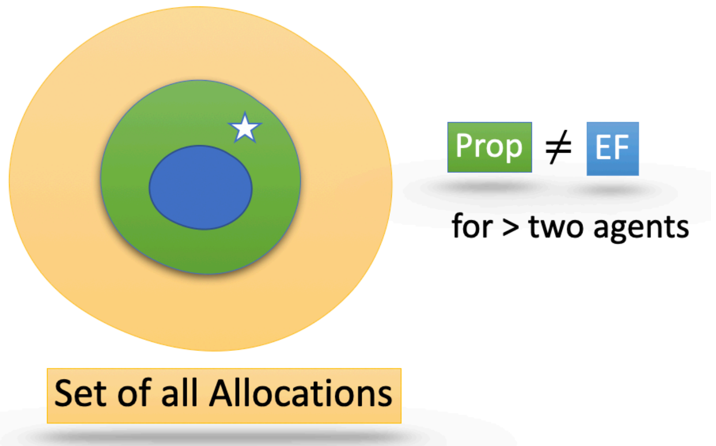


$\mathcal{O}(n^2)$ — Dubins-Spanier, *Amer. Math. Mon. 1961*

$\mathcal{O}(n \log n)$ — Even-Paz [1984] (via recursion)

2 queries for $n = 2$ — Cut-and-choose

Query Complexity of Proportionality



$\mathcal{O}(n^2)$ — Dubins-Spanier, *Amer. Math. Mon. 1961*

$\mathcal{O}(n \log n)$ — Even-Paz [1984] (via recursion)

$\Omega(n \log n)$ — Edmonds & Pruhs, *TALG 2011]*

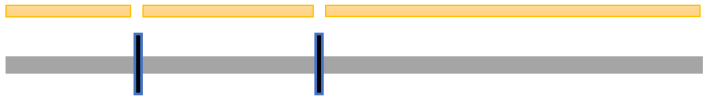
2 queries for $n = 2$ — Cut-and-choose

Envy-free Protocol

Envy-free Protocol for 3 agents

Envy-free Protocol for 3 agents

make three
equal pieces



Envy-free Protocol for 3 agents

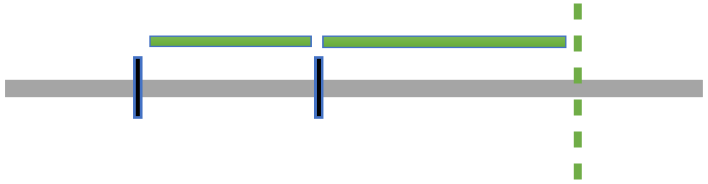
make three
equal pieces

make my
top two
pieces equal

Envy-free Protocol for 3 agents

make three
equal pieces

make my
top two
pieces equal



Envy-free Protocol for 3 agents

make three
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make my
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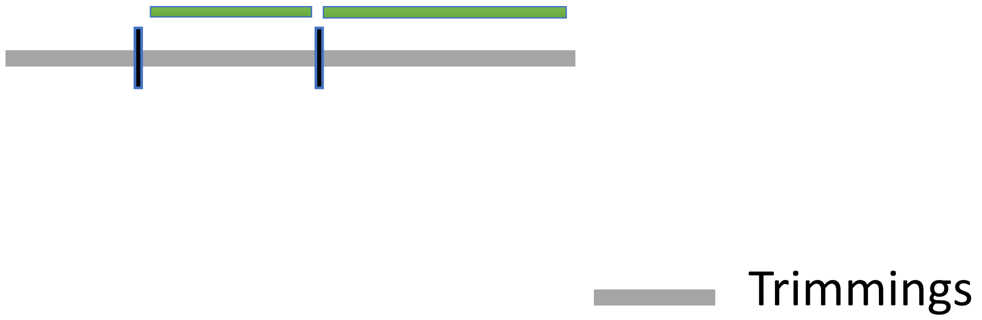


Envy-free Protocol for 3 agents

make three
equal pieces

make my
top two
pieces equal

I pick first

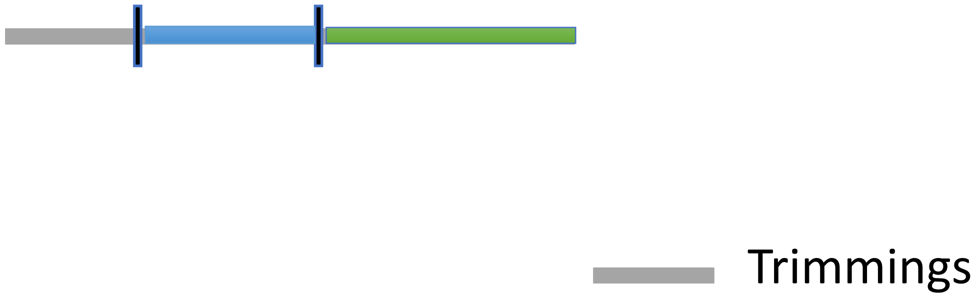
Envy-free Protocol for 3 agents

make three
equal pieces

make my
top two
pieces equal

I pick second
(one of the trimmed pieces)

I pick first



Envy-free Protocol for 3 agents

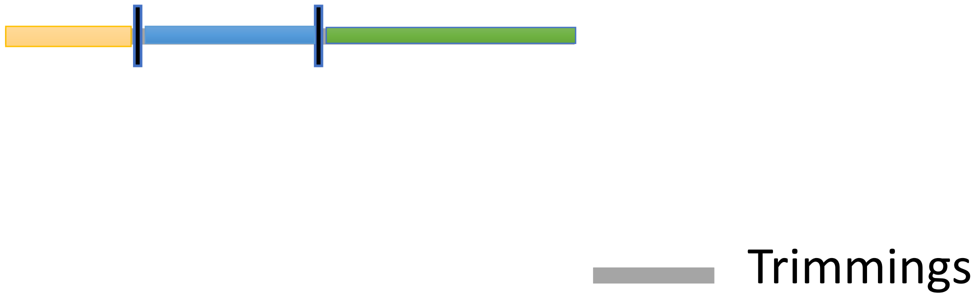
make three equal pieces

I pick last

make my top two pieces equal

I pick second *(one of the trimmed pieces)*

I pick first



Envy-free Protocol for 3 agents

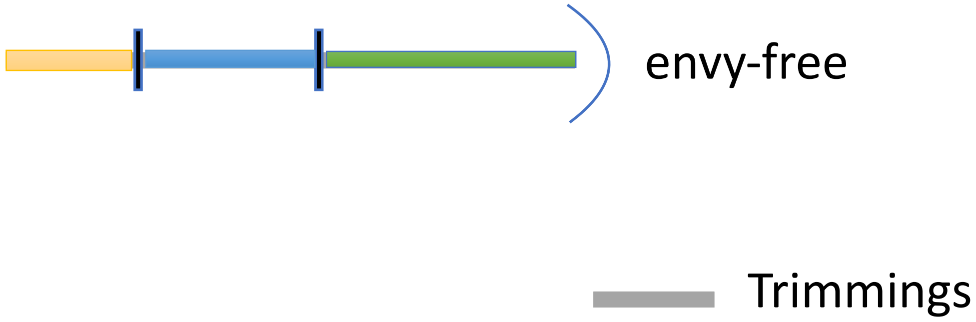
make three equal pieces

I pick last
(untrimmed piece)

make my top two pieces equal

I pick second
(one of the trimmed pieces)

I pick first



Envy-free Protocol for 3 agents

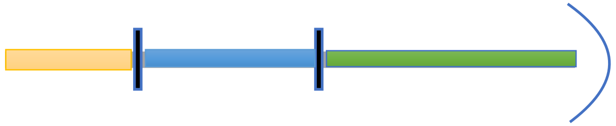
make three equal pieces

I pick last
(untrimmed piece)

make my top two pieces equal

I pick second
(one of the trimmed pieces)

I pick first hence EF



envy-free

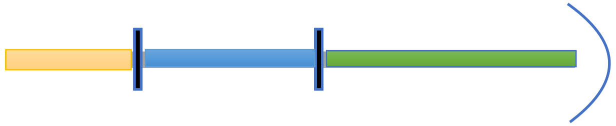
Trimmings

Envy-free Protocol for 3 agents

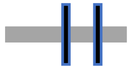
Trimmings (T)

Envy-free Protocol for 3 agents

equi-divide T
& pick last



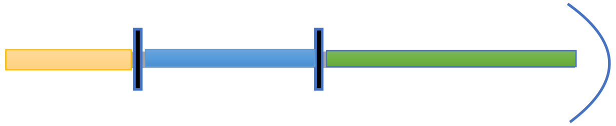
envy-free



Trimmings (T)

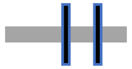
Envy-free Protocol for 3 agents

I pick second



envy-free

I pick first

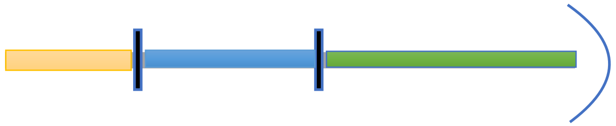


Trimmings (T)

equi-divide T
& pick last

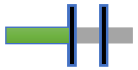
Envy-free Protocol for 3 agents

I pick second



envy-free

I pick first,
hence EF



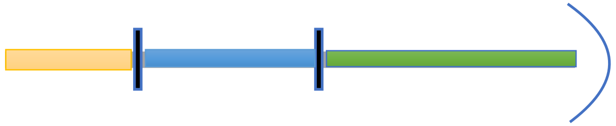
Trimmings (T)

equi-divide T
& pick last

Envy-free Protocol for 3 agents

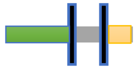
I pick second

Advantage from first round, hence EF



envy-free

I pick first,
hence EF



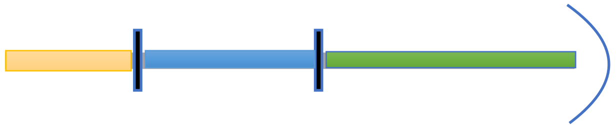
Trimmings (T)

equi-divide T
& pick last

Envy-free Protocol for 3 agents

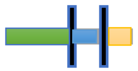
I pick second

Advantage from first round, hence EF



envy-free

I pick first, hence EF



Trimmings (T)

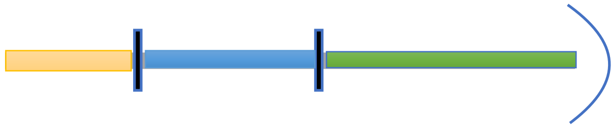
equi-divide T & pick last

I equi-divided hence EF

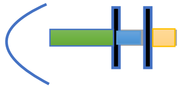
Envy-free Protocol for 3 agents

I pick second

Advantage from first round, hence EF



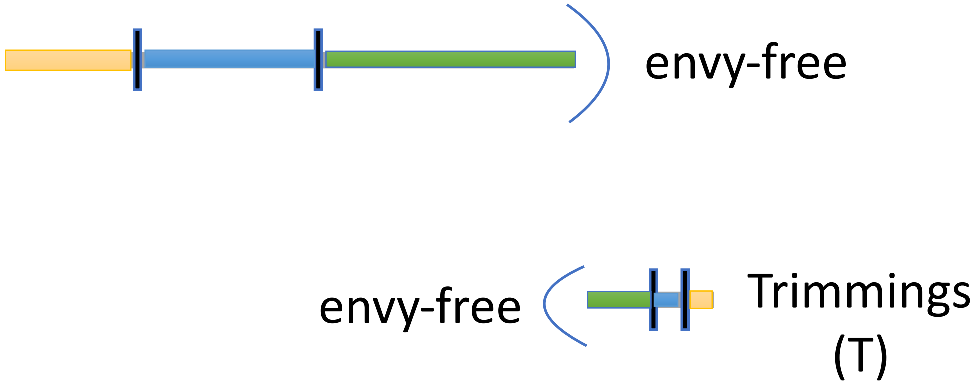
I pick first, hence EF



equi-divide T & pick last

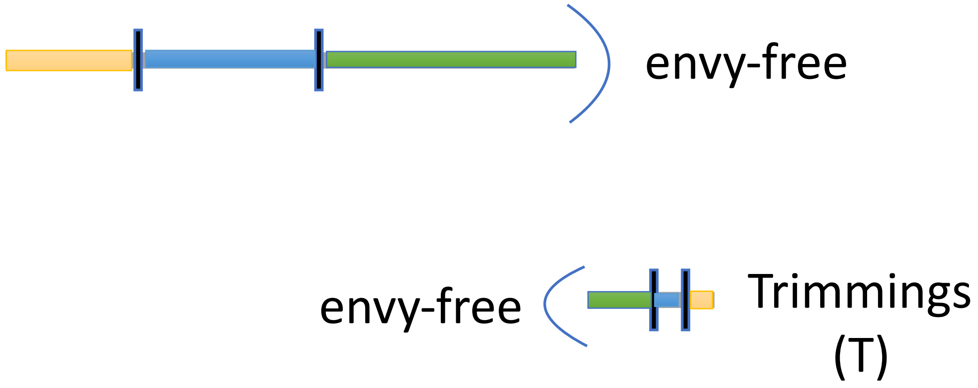
I equi-divided hence EF

Envy-free Protocol for 3 agents



Hence, we find an envy-free cake division

Envy-free Protocol for 3 agents



Selfridge-Conway protocol finds an EF cake division among *three* agents using $\mathcal{O}(1)$ queries

Existence of Envy-free Cake Divisions

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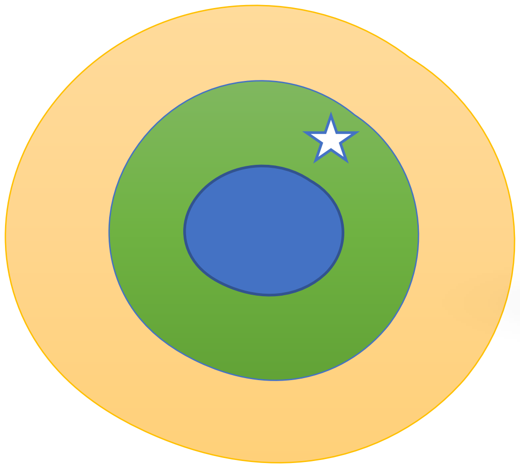


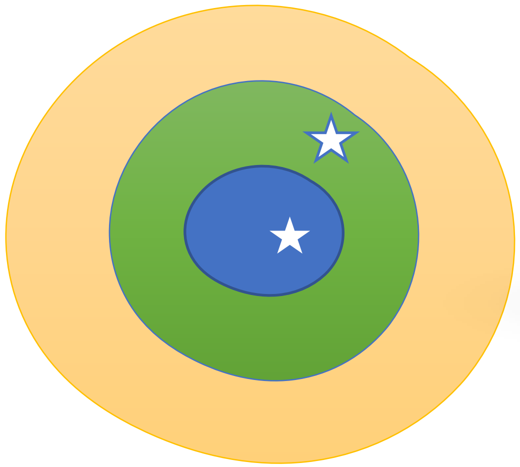
All allocations

EF $\Rightarrow$ Prop, EF $\not\Leftarrow$ Prop

Stromquist [1980], Su [1999]

Envy-free cake division exist for any number of agents (Lecture 04)

Existence of Envy-free Cake Divisions



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Query Complexity of Envy-freeness

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2 queries for $n = 2$ Cut-and-choose

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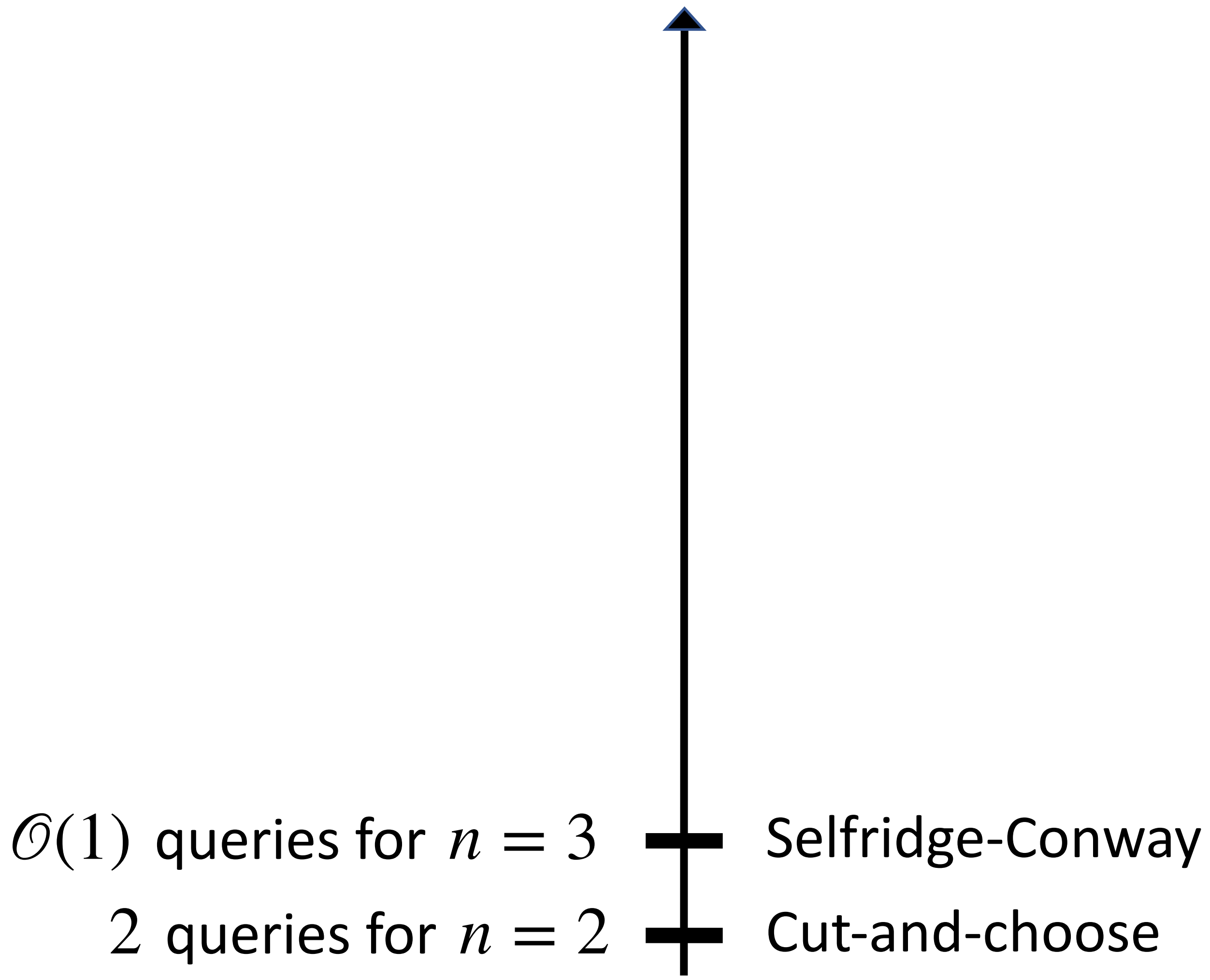
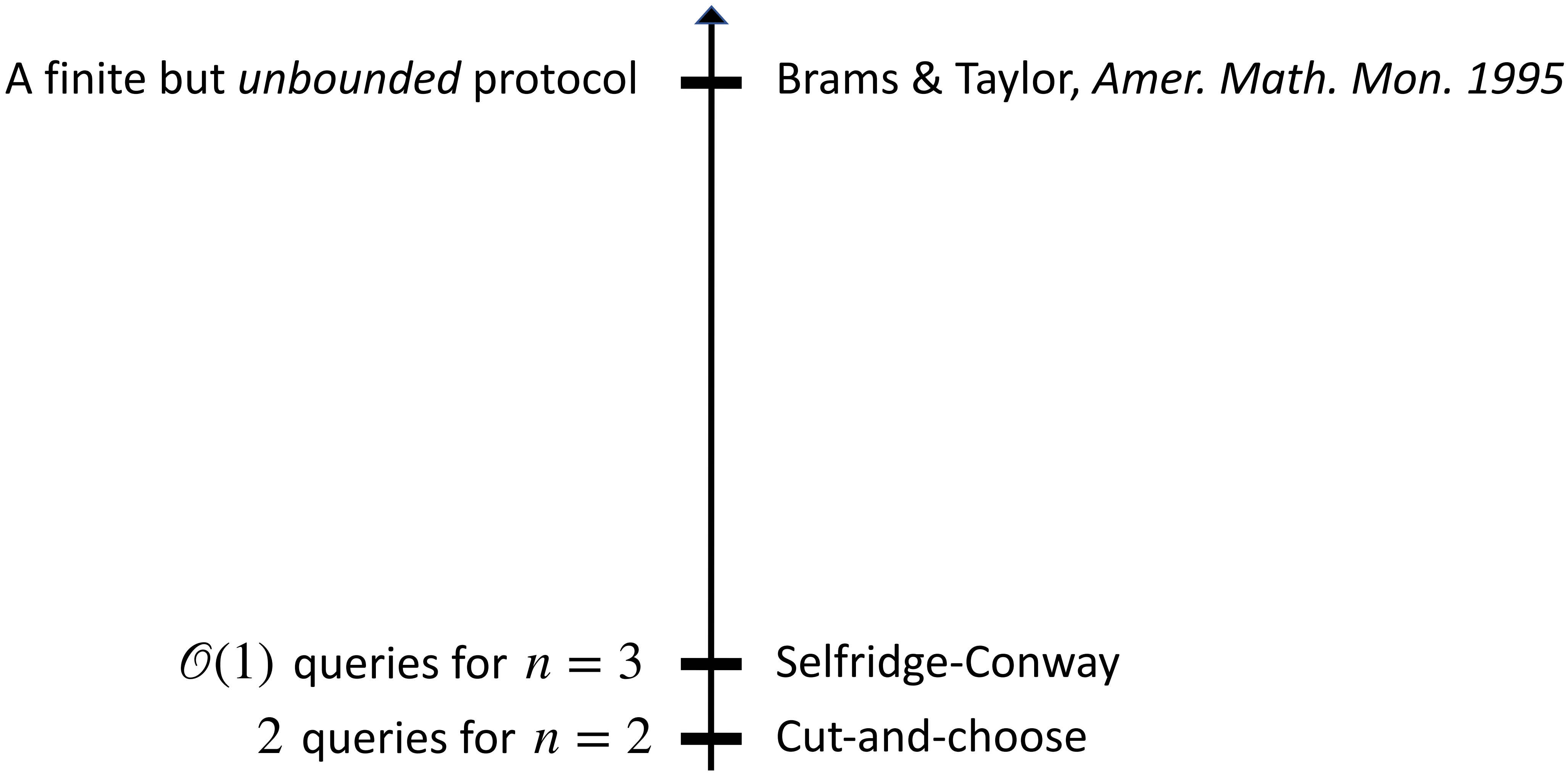


Query Complexity of Envy-freeness



Query Complexity of Envy-freeness

A finite but *unbounded* protocol — Brams & Taylor, *Amer. Math. Mon. 1995*

$\mathcal{O}(n^{n^{n^{n^{n^{n}}}}})$ — Aziz & Mackenzie, *FOCS 2016*

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Source: Lecture slides of Rohit Vaish, IIT Delhi

Query Complexity of Envy-freeness

What happens when every agent wishes to have a ***contiguous*** piece of the cake?

Query Complexity of Envy-freeness

What happens when every agent wishes to have a ***contiguous*** piece of the cake?

Stromquist [1980], Su [1999]

Envy-free cake division exists for any number of agents

(4th Lecture)

Query Complexity of Envy-freeness

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Stromquist [1980], Su [1999] **connected pieces**

Envy-free cake division exists for any number of agents (4th Lecture)

Query Complexity of Envy-freeness

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No finite-query protocol exists for connected EF cake division

Query Complexity of Envy-freeness

Stromquist [1980], Su [1999] **connected pieces**

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(30 April)

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[ABK**R**] *WINE 2019* (Fair and Efficient Cake Division with Connected Pieces)

An efficient algorithm: **1/2-EF + 1/3-NSW** allocation for *connected* EF cake division

(28 May)

Don't forget!

Send us your preferred list of the student papers by April 30th.