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# Topics in Computational Social Choice Theory

## Lecture 03: “EFX: A Simpler Approach and an (Almost) Optimal Guarantee via Rainbow Cycle Number”

Hannaneh Akrami

Saarland Informatics Campus

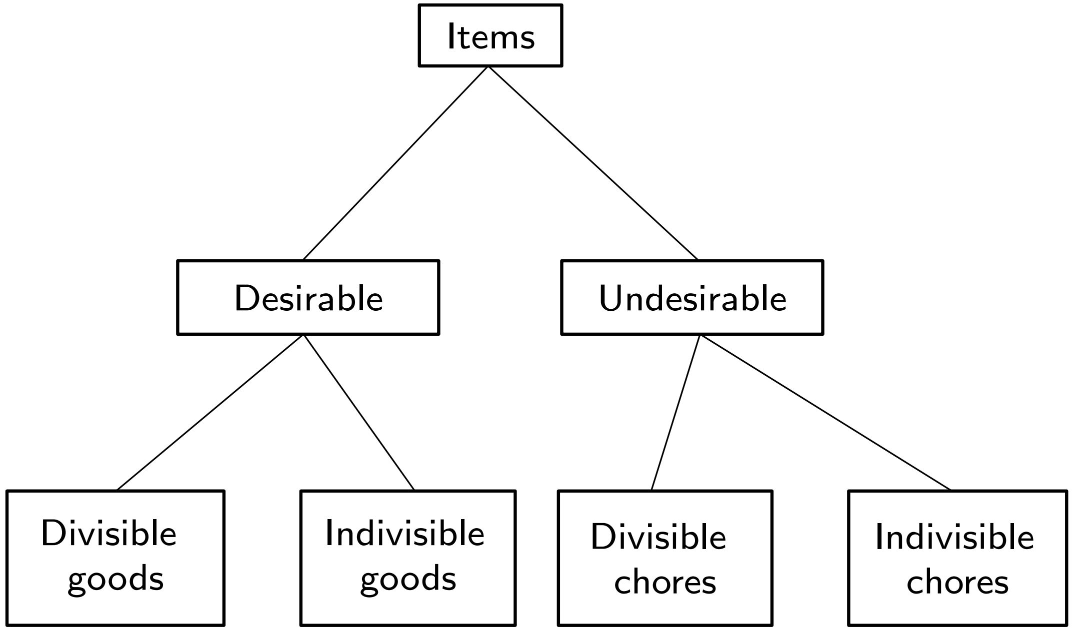
# This Talk

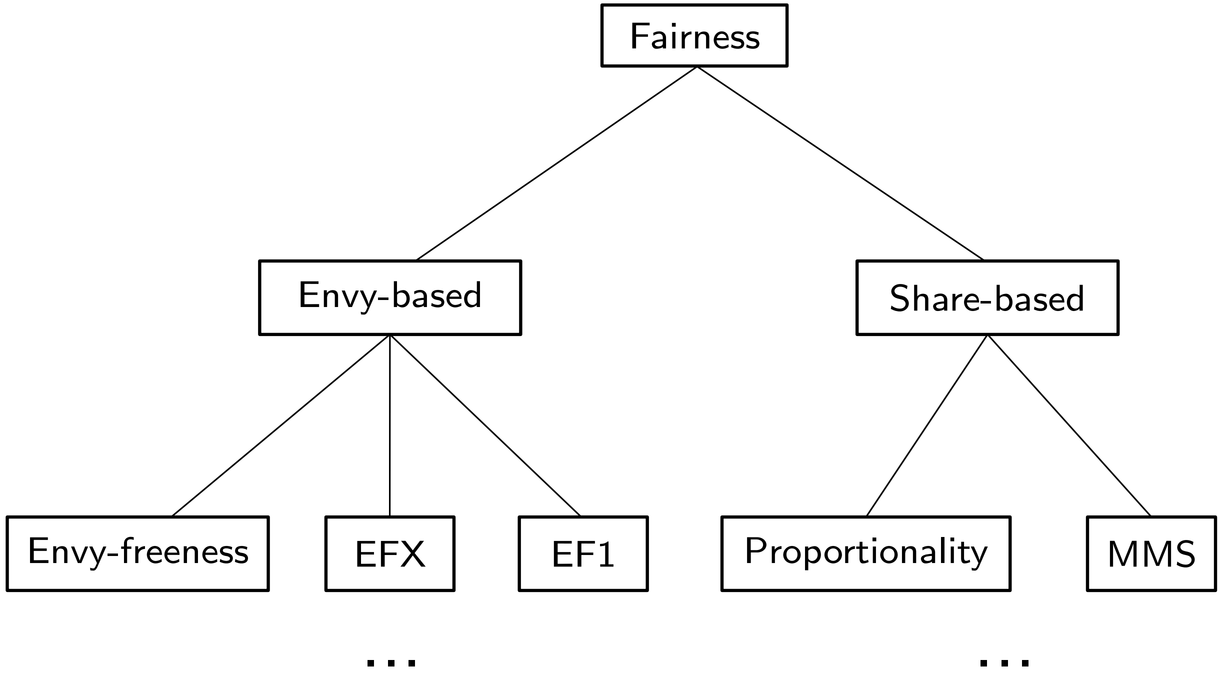
**EFX: A Simpler Approach and an (Almost) Optimal Guarantee via Rainbow Cycle Number.** EC'23


| Hannaneh AKrami | Noga Alon | Bhaskar Ray Chaudhury | Jugal Garg | Kurt Mehlhorn | Ruta Mehta |
|---|---|---|---|---|---|
| MPII | Princeton University | UIUC | UIUC | MPII | UIUC |

# Spectrum of the Problems

Divide **items** among agents in a **fair** manner.



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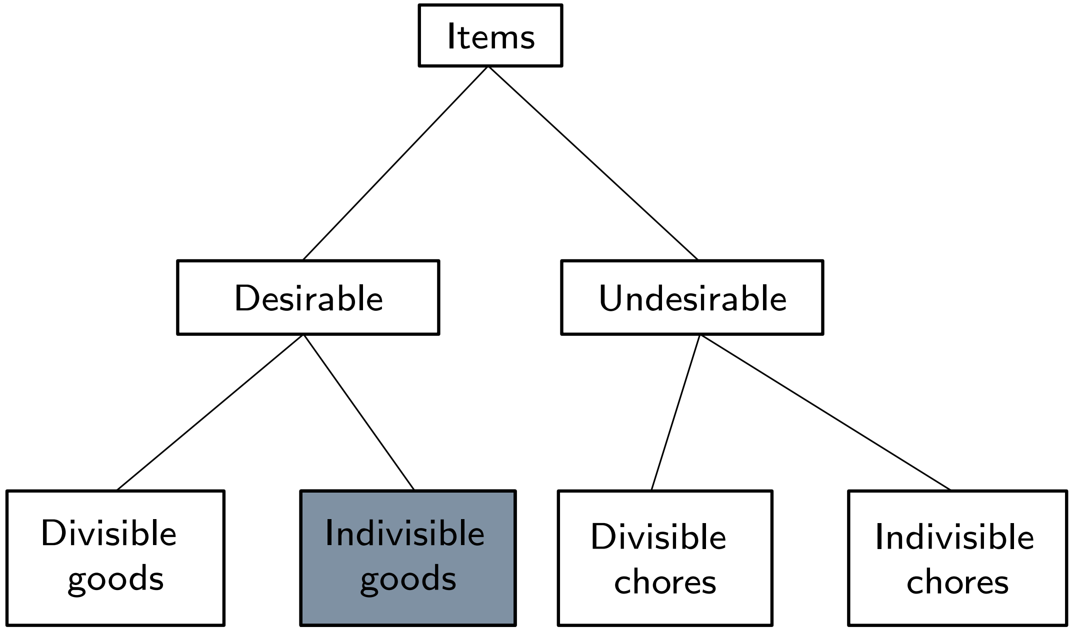


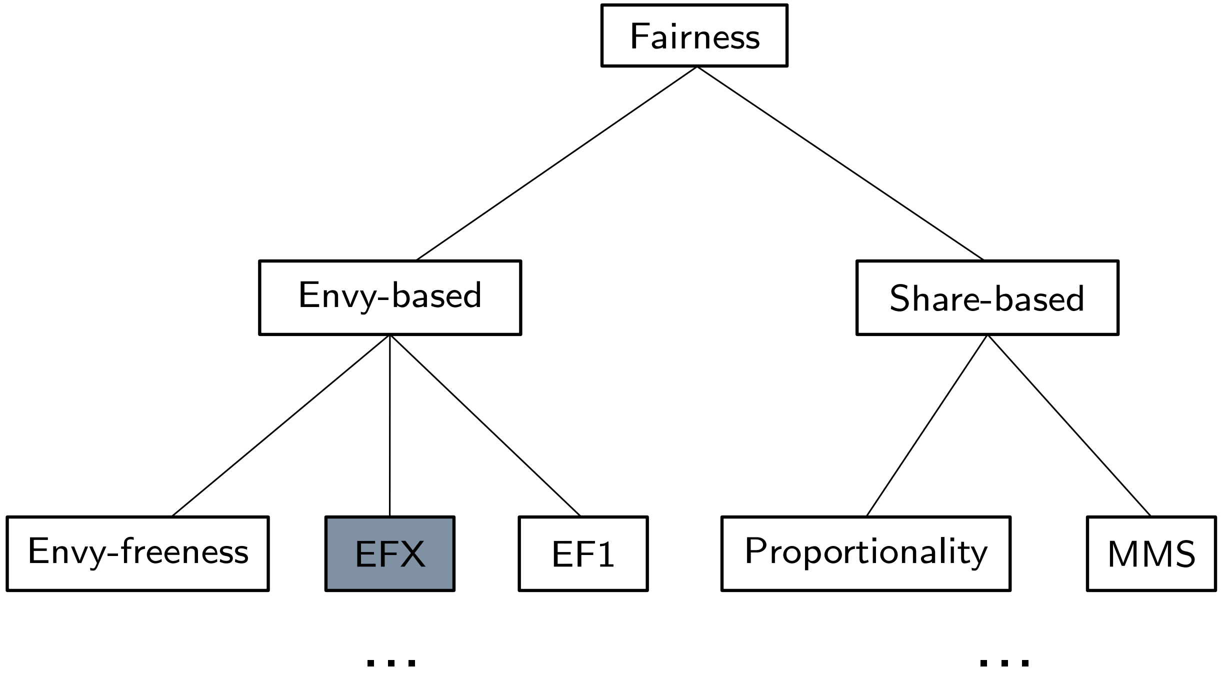

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Given:

- $N$: set of $n$ agents
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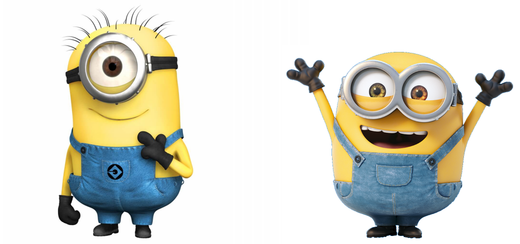

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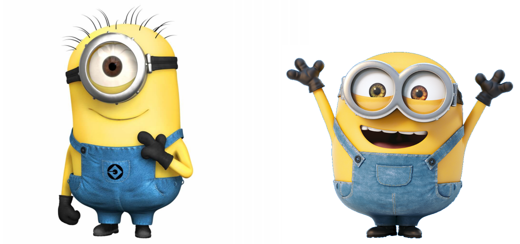

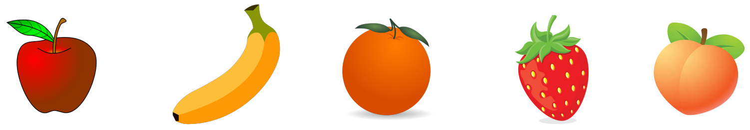

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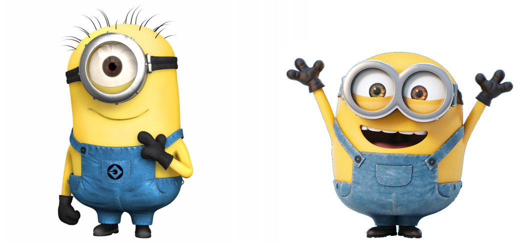

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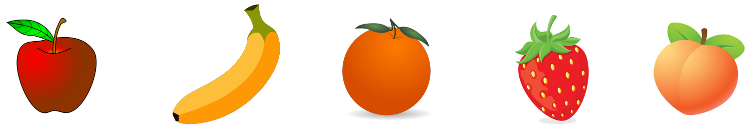

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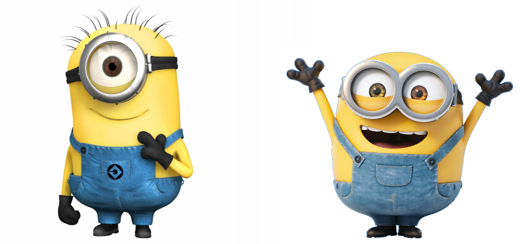

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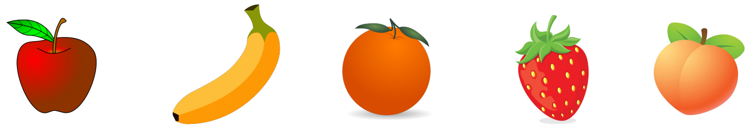

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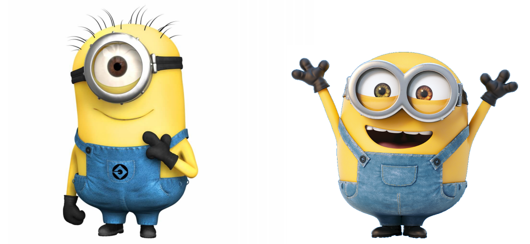

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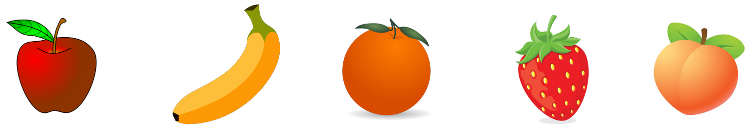

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A partition $X = \langle X_1, X_2, \ldots, X_n \rangle$ of $M$

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Fair division's biggest problem!

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4. Special valuations
   - Identical [Plaut, Roughgarden'18]
   - Binary [Barman, Krishnamurthy, Vaish'18]
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- $n = 3$ with **additive** valuations [Chaudhury, Garg, Mehlhorn'20]

  $v_i(S) = \sum_{g \in S} v_i(\{g\})$

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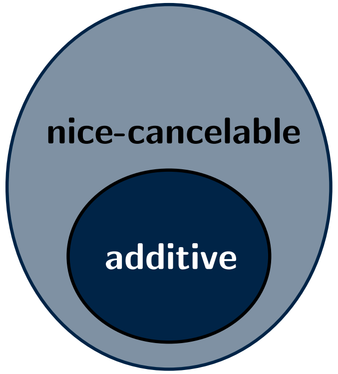
- $n = 2$ with general monotone valuations [Plaut, Roughgarden'18]
- $n = 3$ with **additive** valuations [Chaudhury, Garg, Mehlhorn'20]
- $n = 3$ with **nice-cancelable** valuations [Berger, Cohen, Feldman, Fiat'21]



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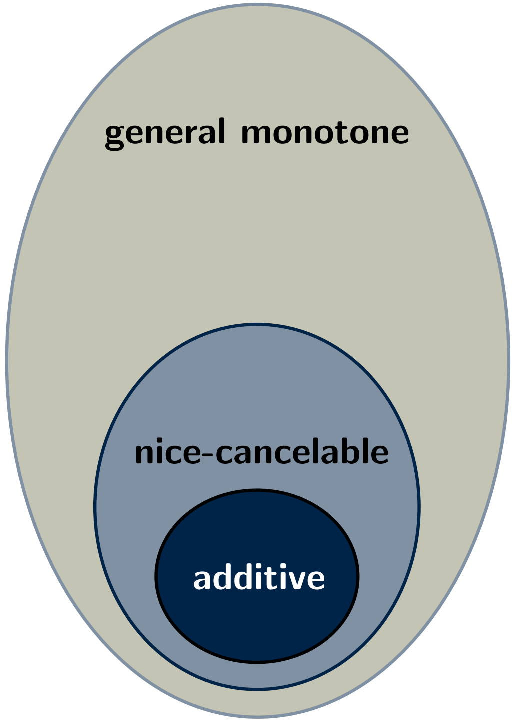


**Theorem 1** [A., Alon, Chaudhury, Garg, Mehlhorn, Mehta]

EFX allocations exists for $n = 3$ when

- one agent has **nice-cancelable** valuation function, and
- two agents have **general monotone** valuation functions.

# Previous Approaches on $n = 3$

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# Previous Approaches: Drawbacks

- Start with the empty allocation.
- Move in the space of **partial EFX** allocations. Some goods might be unallocated.
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1. Fails even if one agent has general monotone valuations.
2. Fails when $n \geq 4$. [Chaudhury, Garg, Mehlhorn'20]

# New Approach

- Move in the space of ~~**partial EFX**~~ **complete** allocations.
- Improve a certain potential function.
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- Move in the space of ~~partial EFX~~ **complete** allocations.
- Improve a certain potential function.
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1. Works even if two agent have general monotone valuations.
2. $n \geq 4$?
3. Add-on: Simpler analysis.

# High Level Idea

# Cake Cutting

How to divide a cake among two agents fairly?

envy-freeness

# Cut and Choose

How to divide a cake among two agents fairly?

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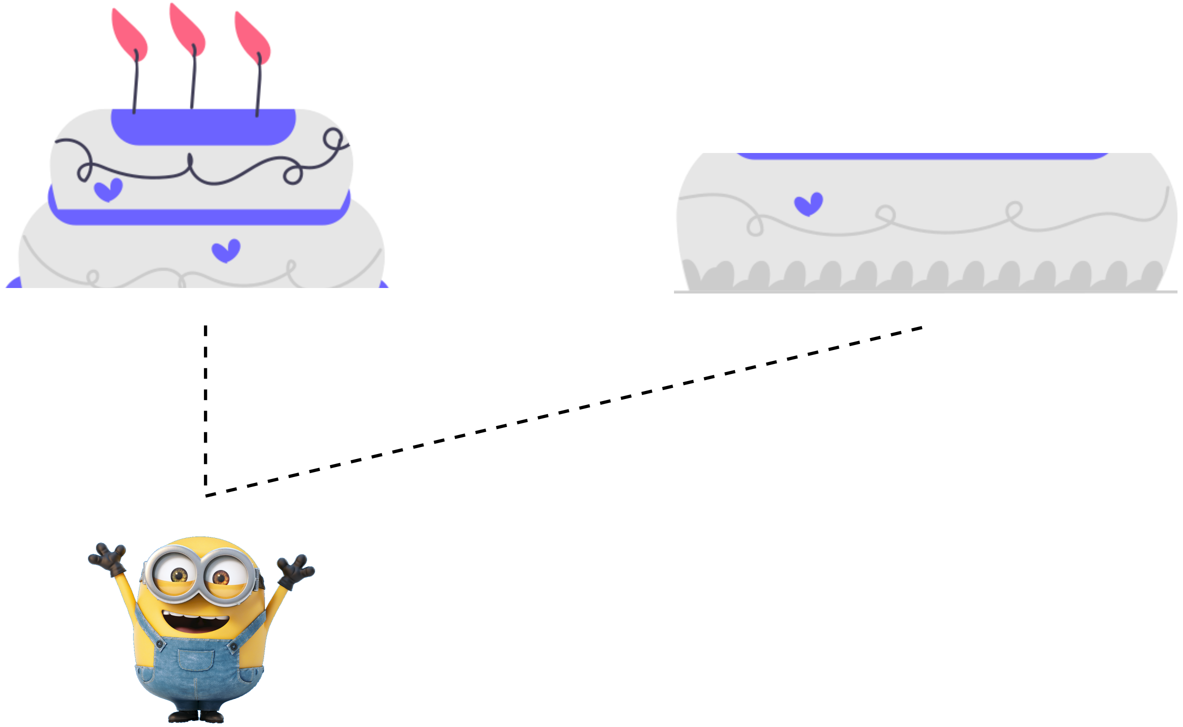
- Agent 1 cuts.



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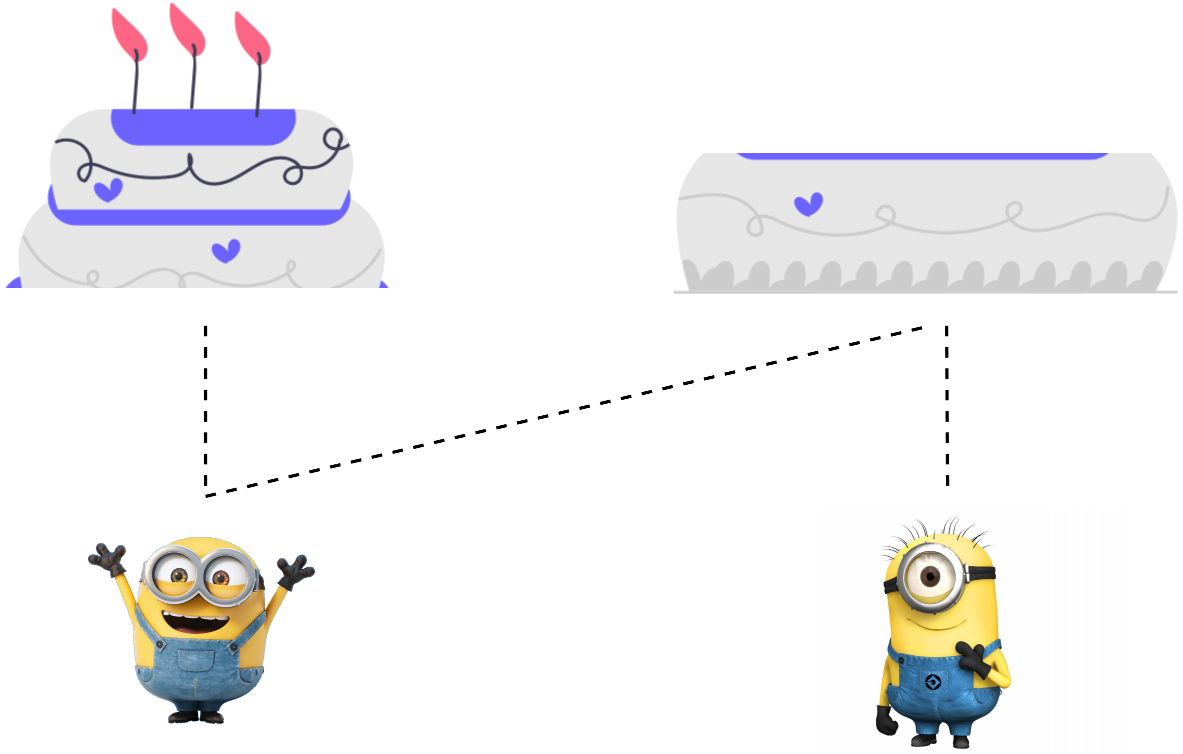

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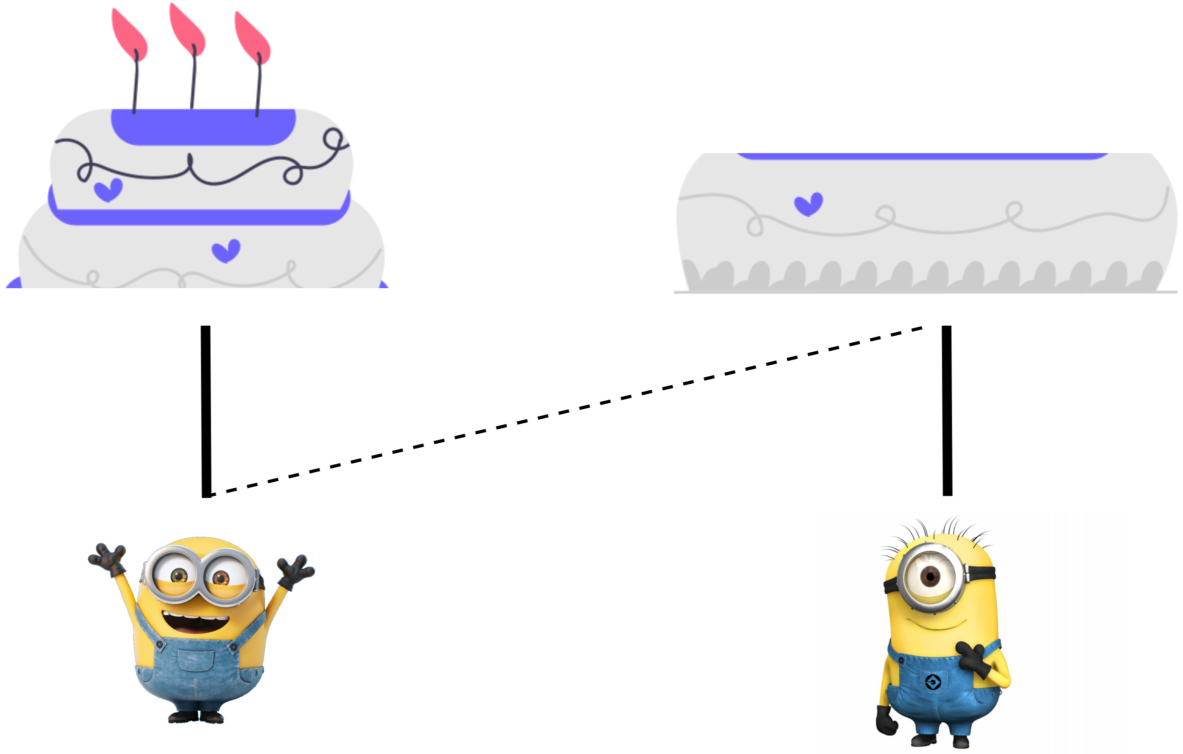

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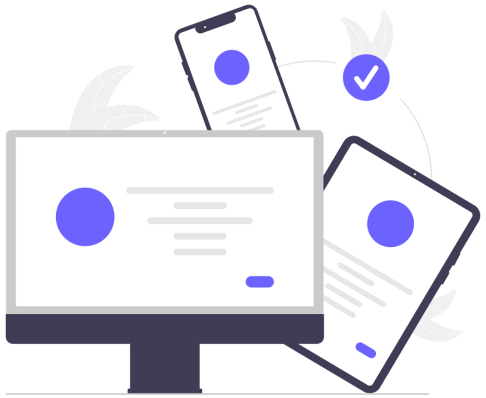

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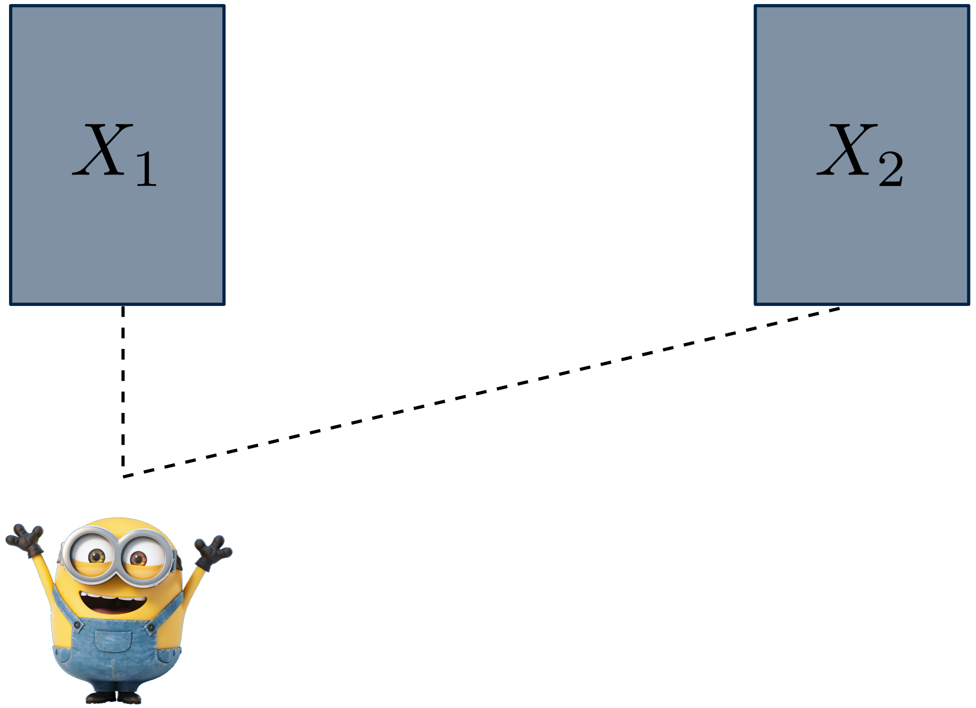

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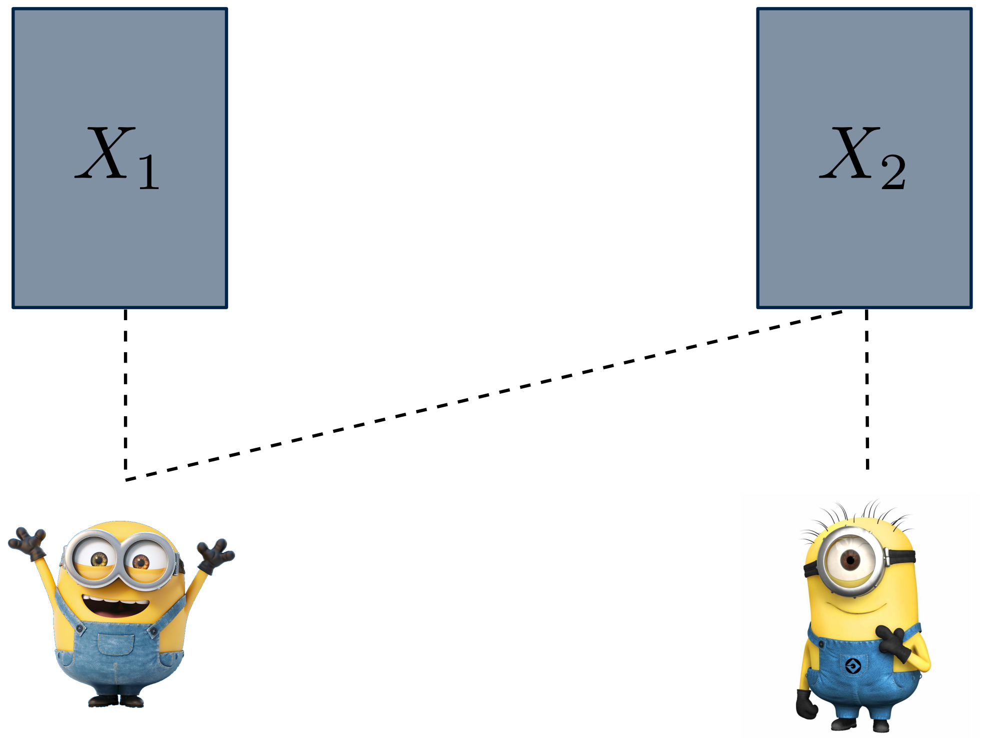

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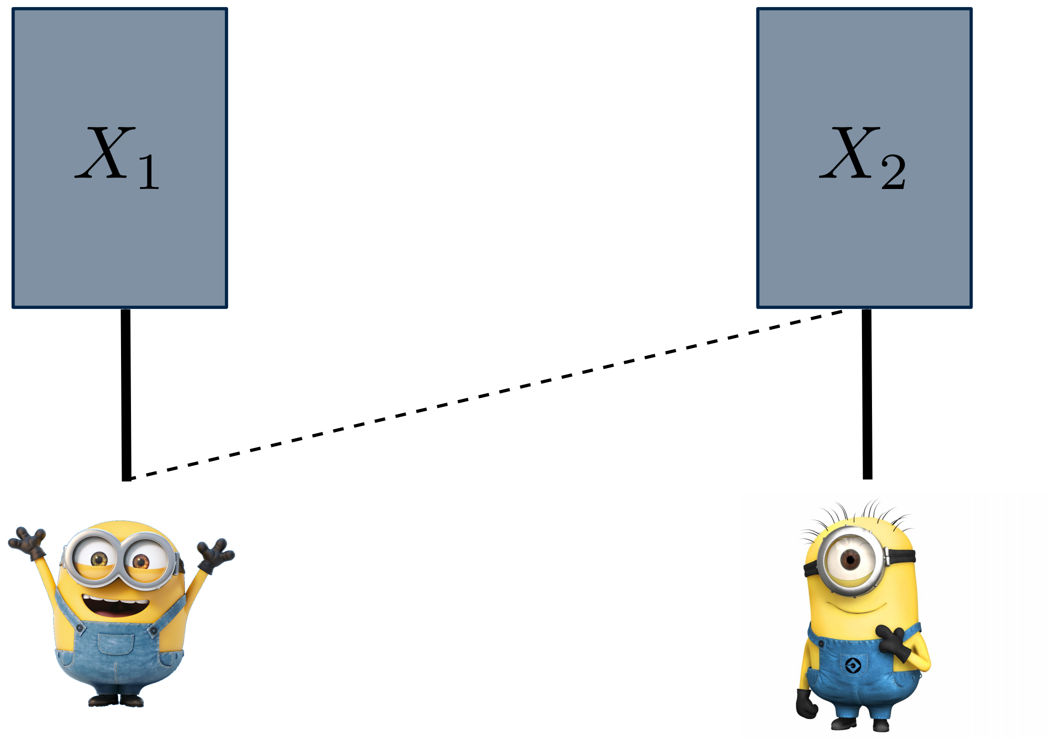


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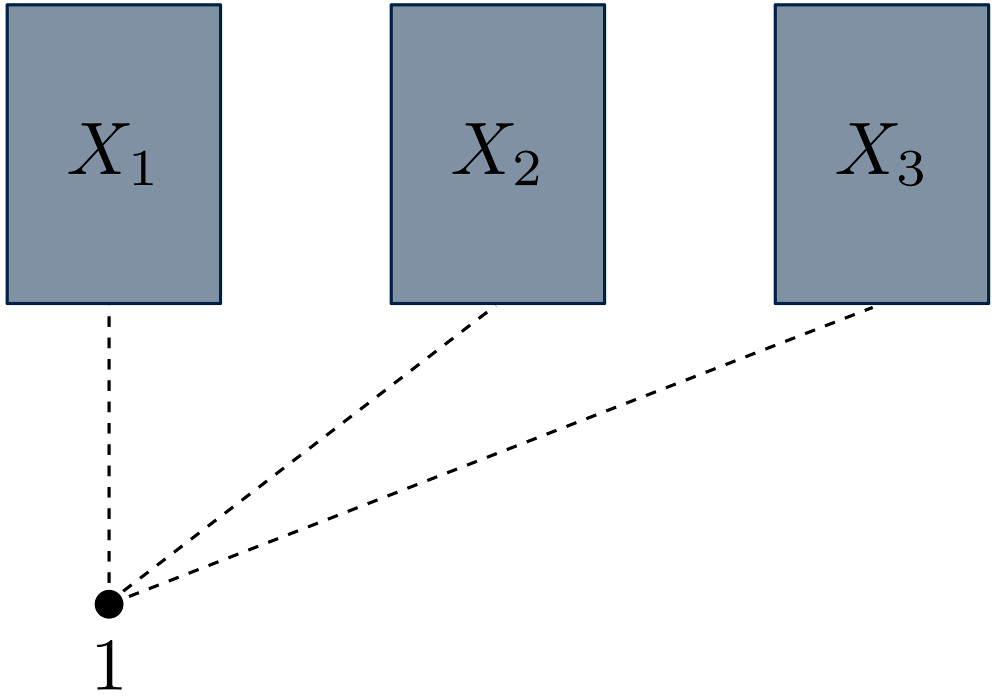

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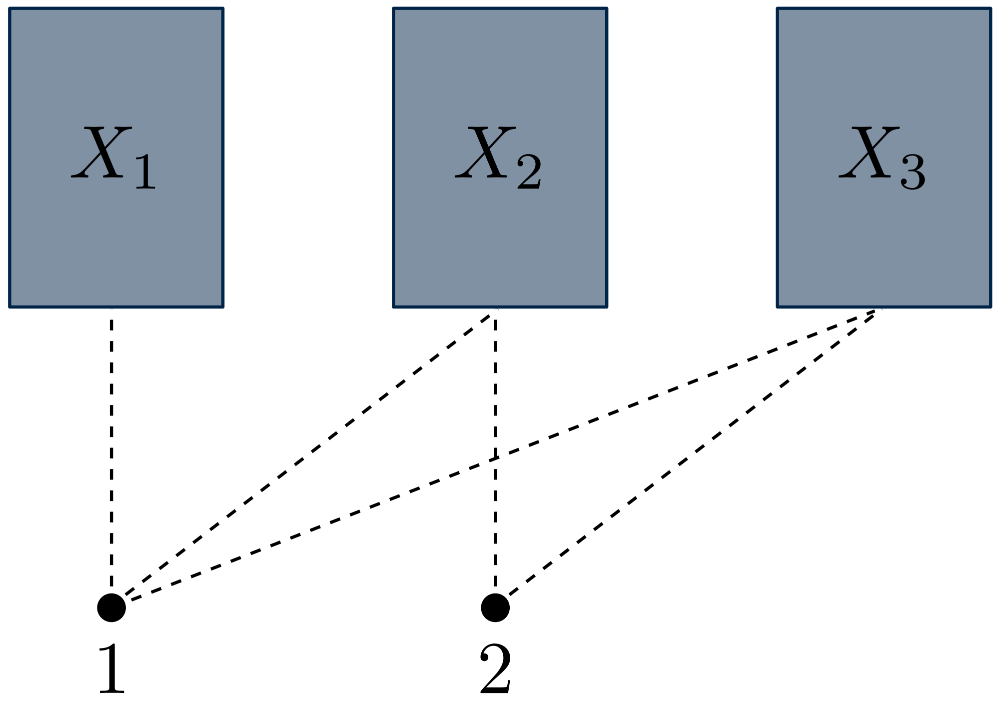

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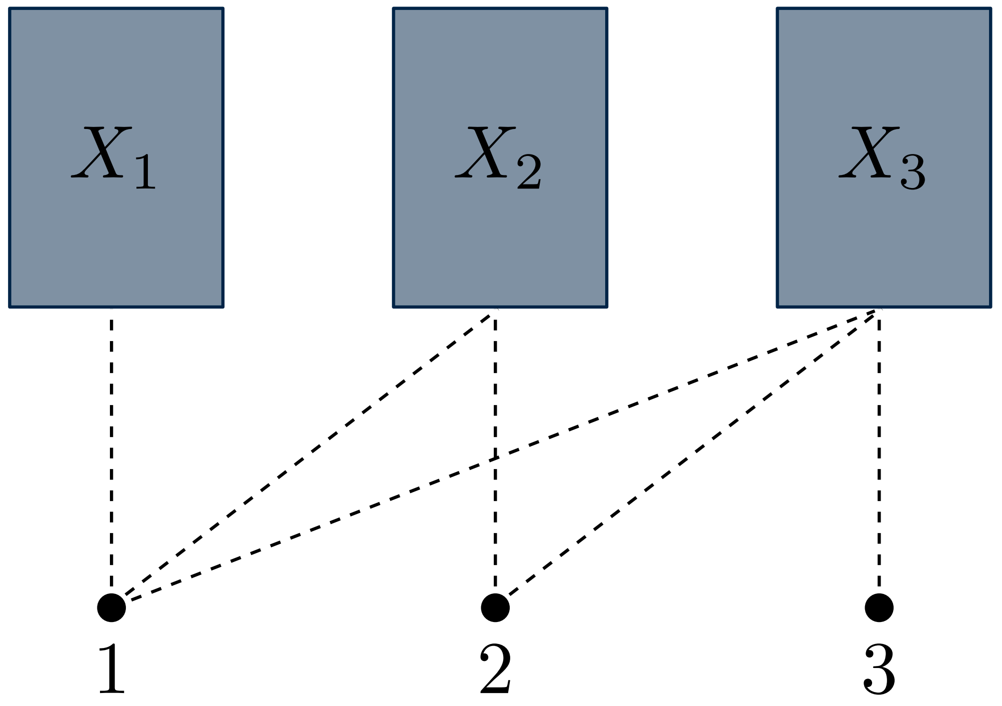

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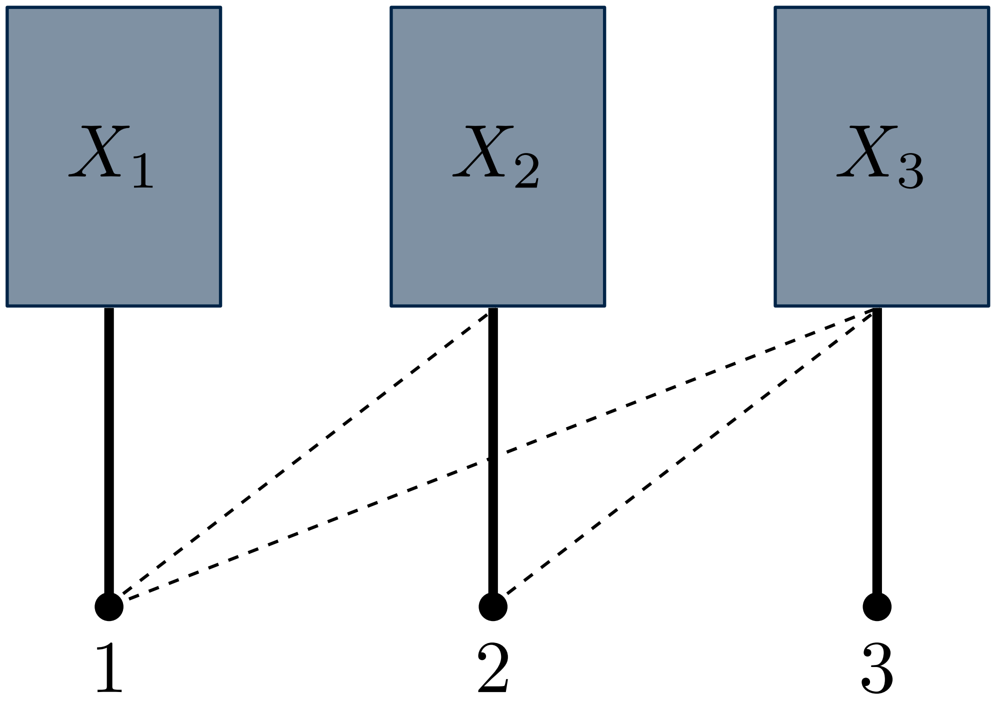

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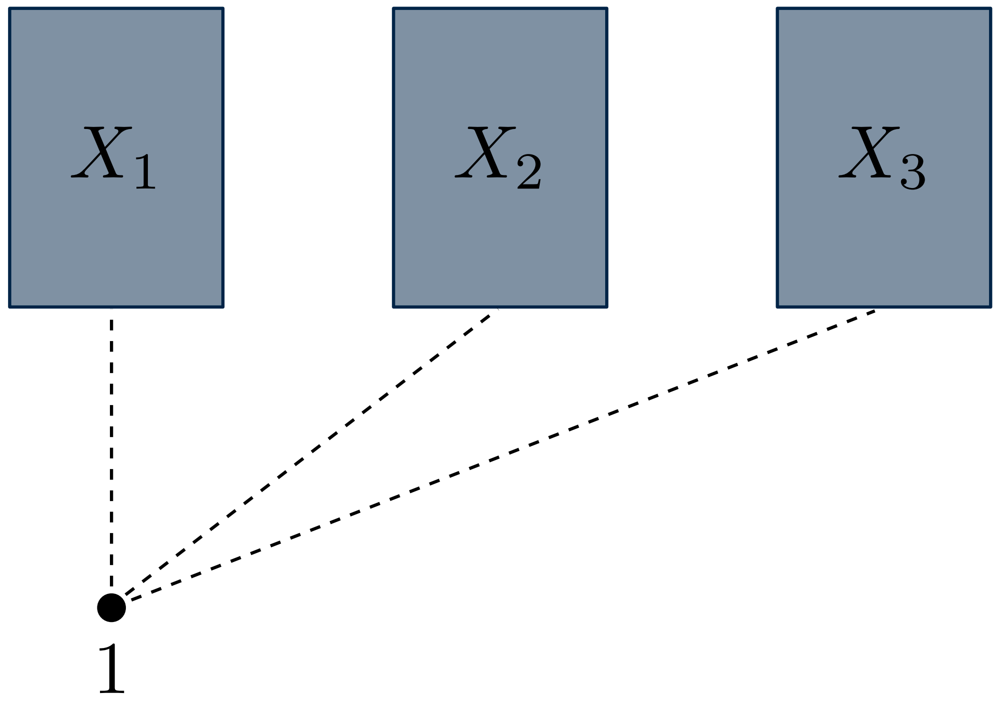

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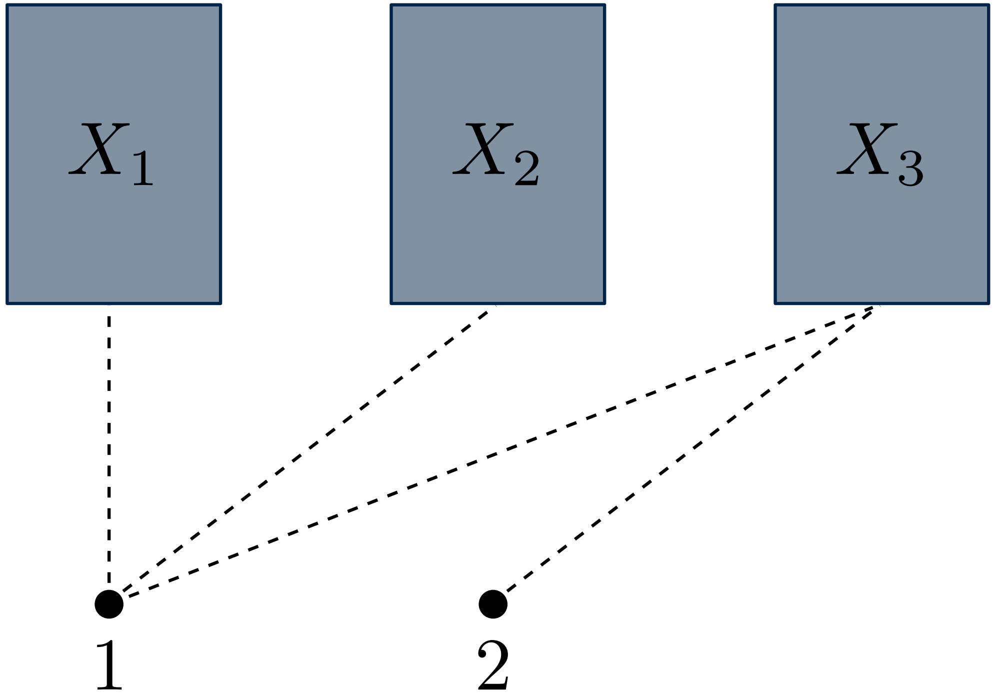

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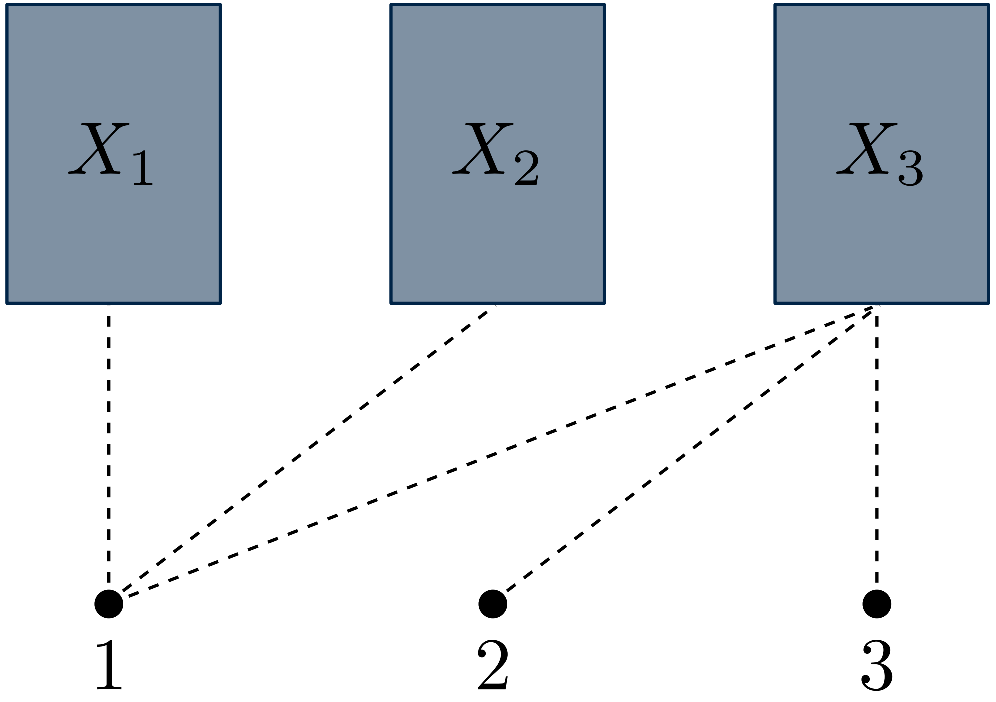

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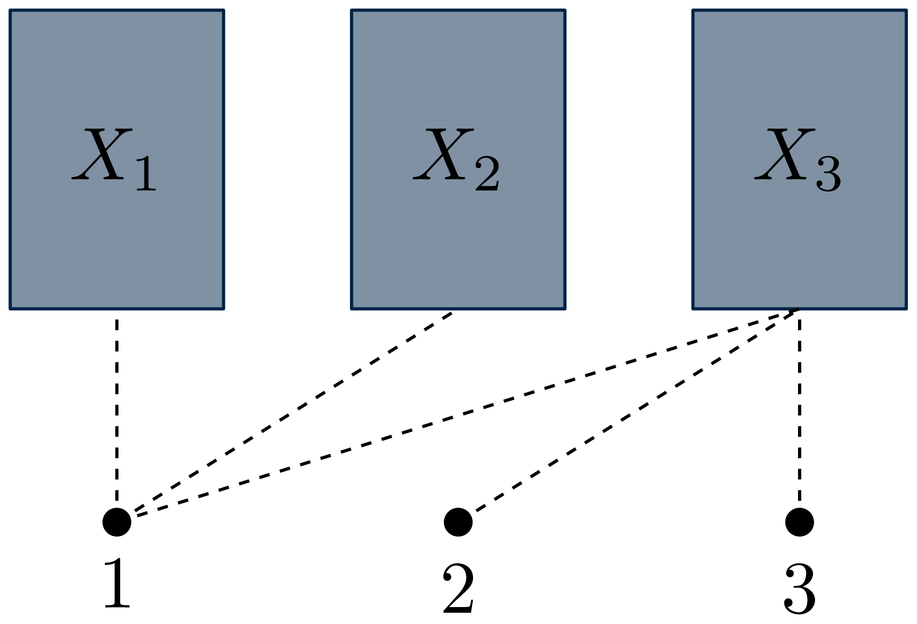

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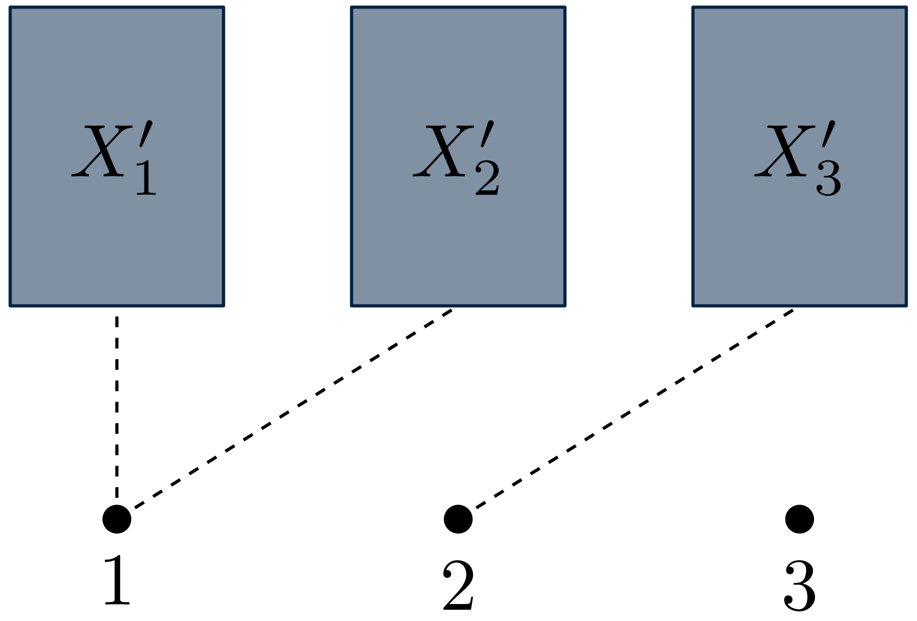

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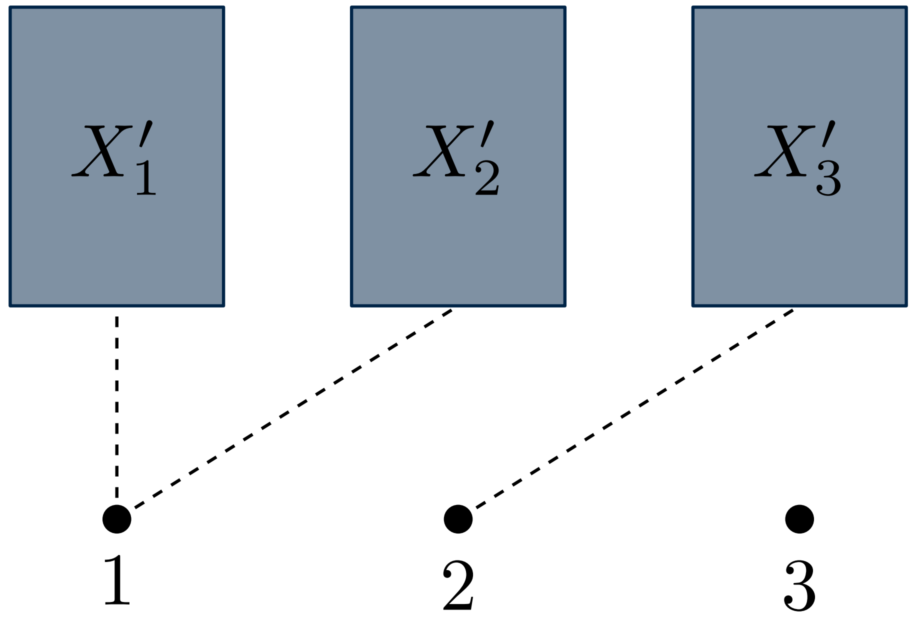

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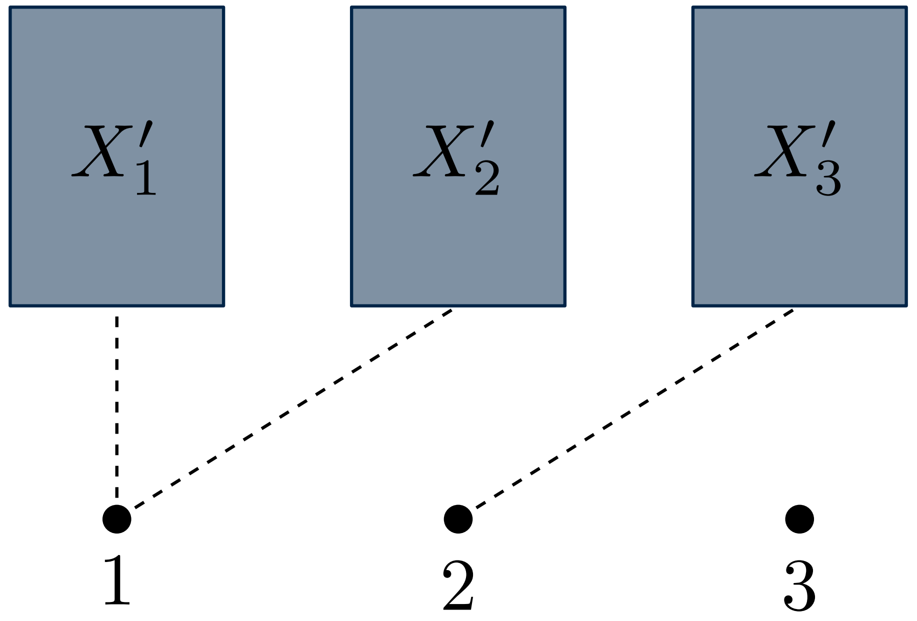

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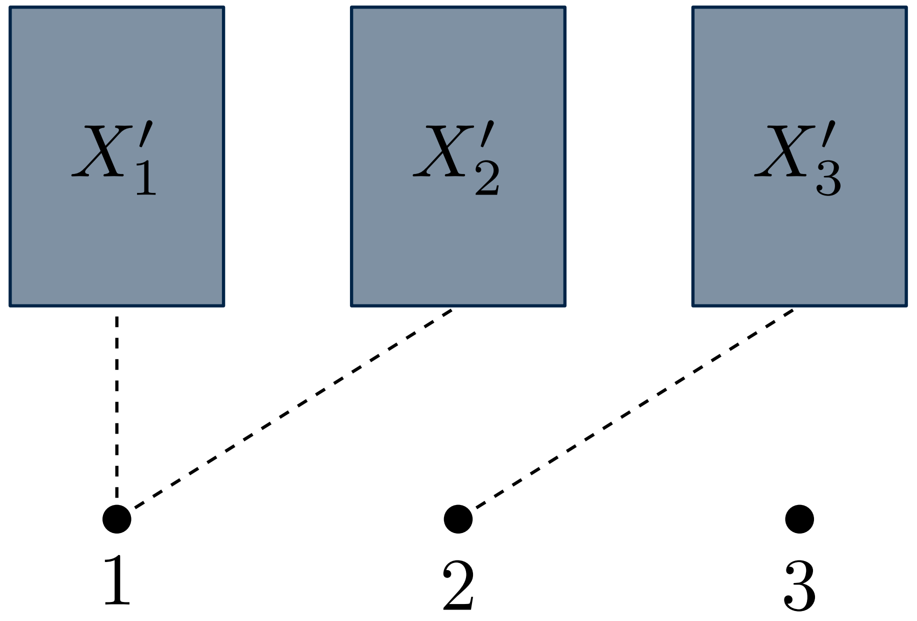


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Why does the algorithm terminate? Potential argument.

# Algorithm

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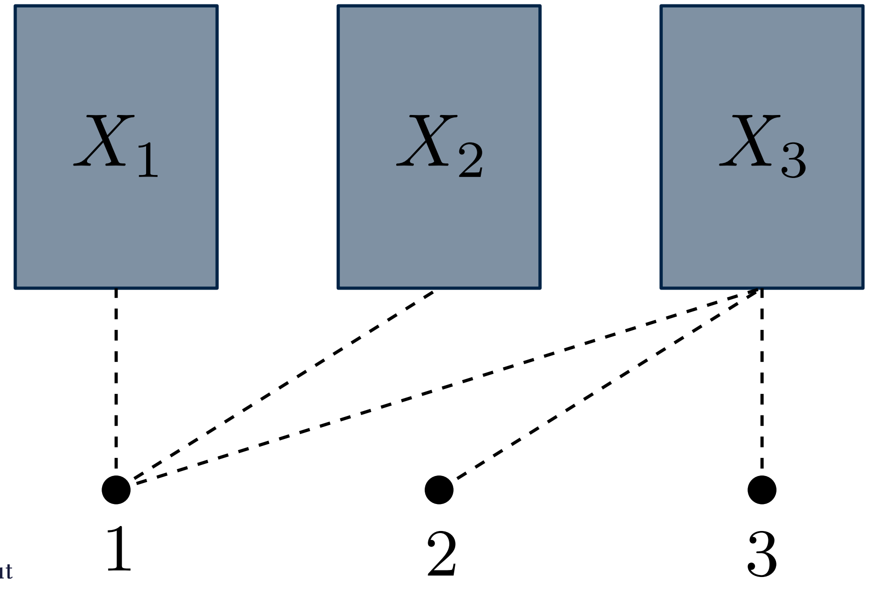

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**Strong envy**: Given allocation $A$, agent $i$ strongly envies agent $j$, if there exists a good $g \in A_j$ s.t. $A_j \setminus \{g\} >_i A_i$.

**EFX-feasibility**: Given partition $X$ of the goods, $X_j$ is EFX-feasible to agent $i$, if upon receiving $X_j$, $i$ does not strongly envy anyone.
I.e., $X_j \geq_i X_{j'} \setminus \{g\}$ for all $j'$ and all $g \in X_{j'}$.

- Assume agent $3$ has additive valuation.

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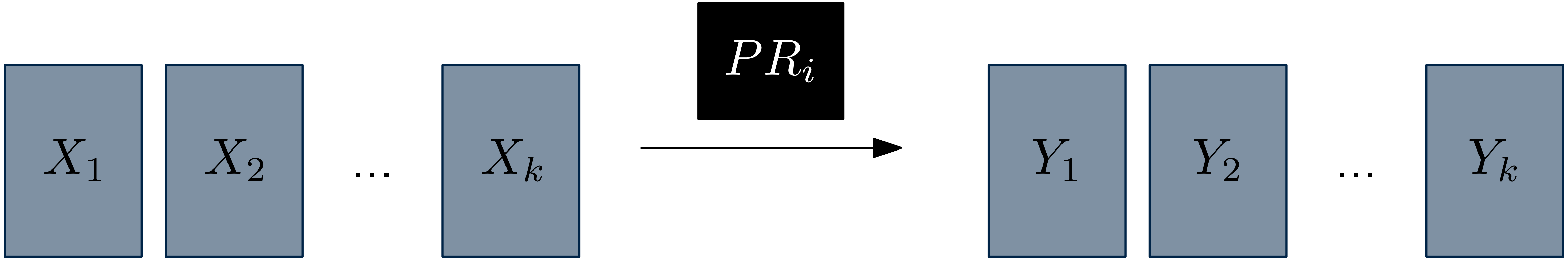
- Assume agent $3$ has additive valuation.

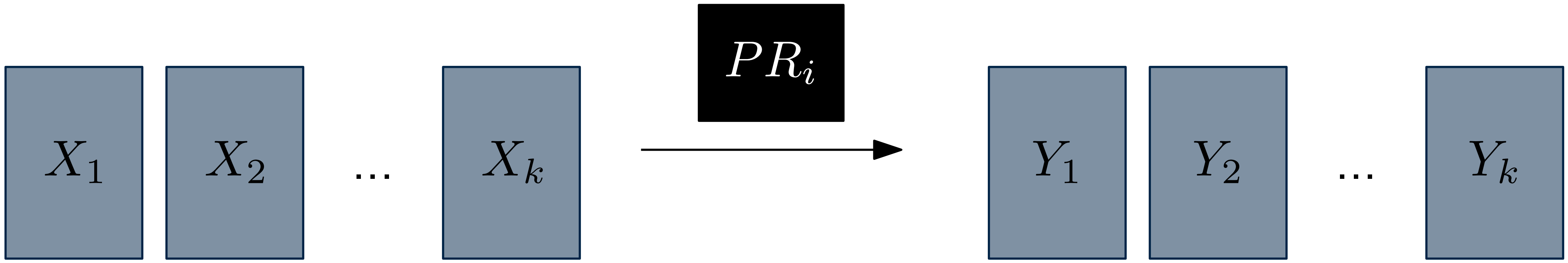
- Non-degeneracy: For all bundles $A \neq B$, $v_i(A) \neq v_i(B)$ for all agents $i$.

# PR Algorithm




- $(Y_1, \ldots, Y_k) = PR_i(X_1, \ldots, X_k)$

# PR Algorithm




- $(Y_1, \ldots, Y_k) = PR_i(X_1, \ldots, X_k)$
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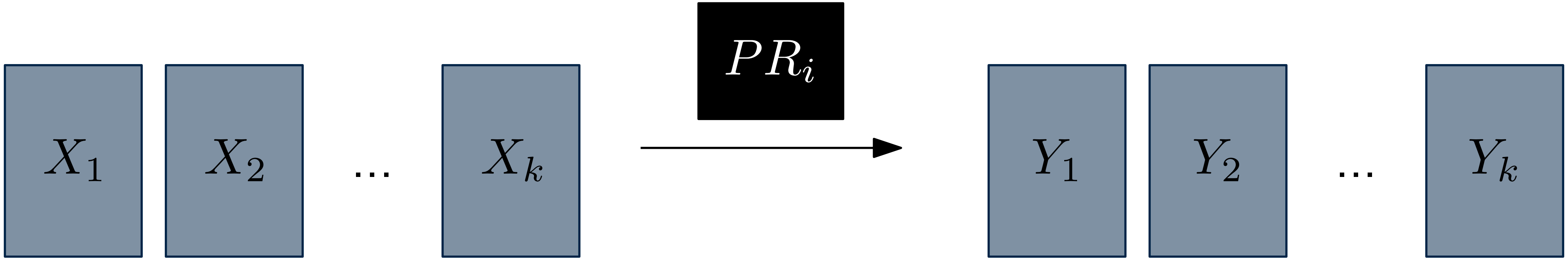
# PR Algorithm




- $(Y_1, \ldots, Y_k) = PR_i(X_1, \ldots, X_k)$
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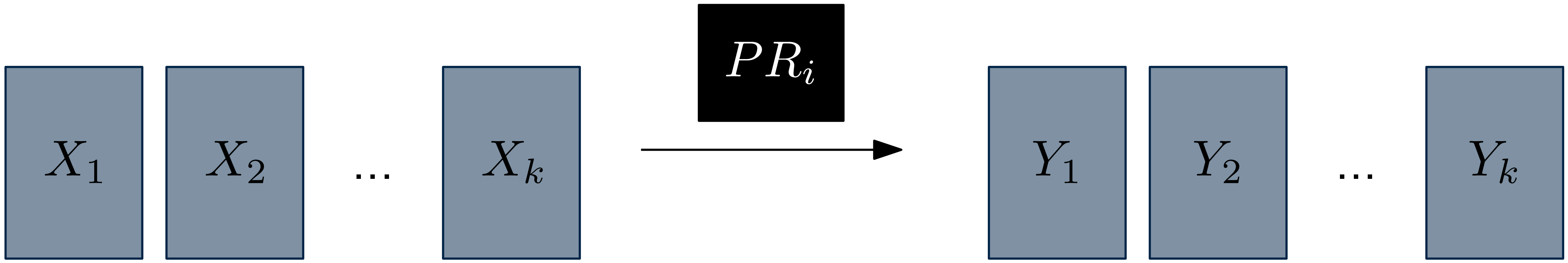
- $\min_j v_i(Y_j) \geq \min_j v_i(X_j)$.

# PR Algorithm

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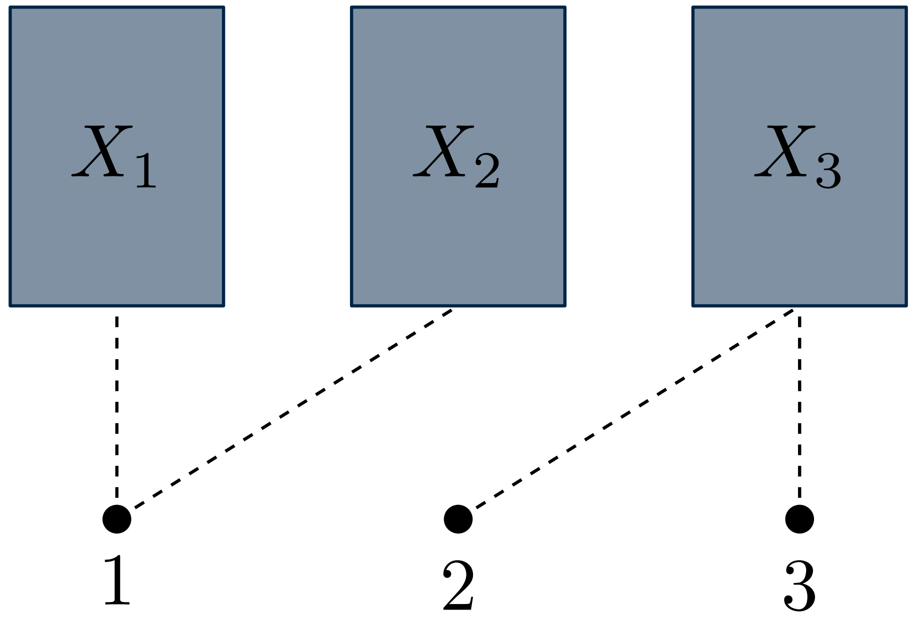
- $\min_j v_i(Y_j) > \min_j v_i(X_j)$ if $Y \neq X$.

# Algorithm

Invariants:

- $X_1$ and $X_2$ are **EFX-feasible** to agent $1$.
- $X_3$ is **EFX-feasible** to agent $2$ or $3$.



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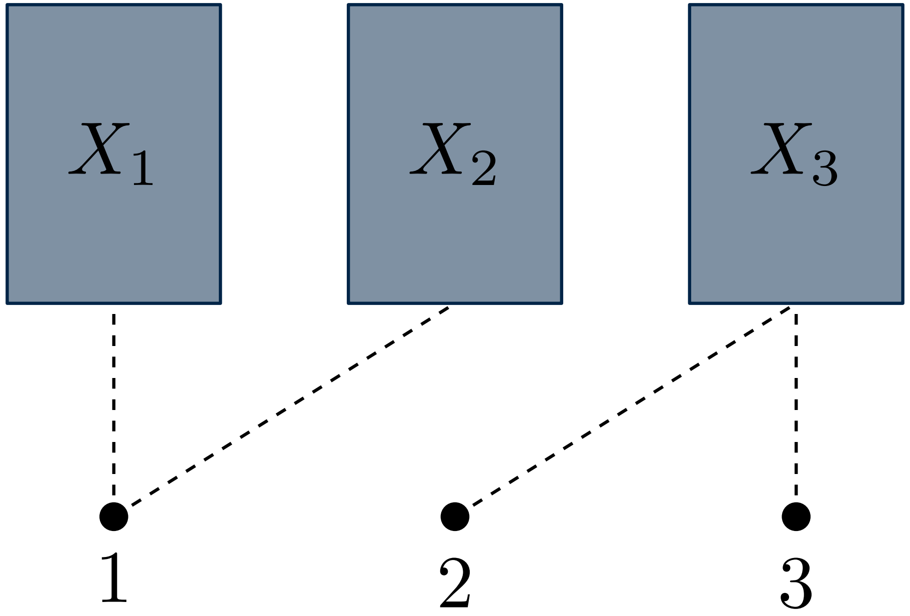


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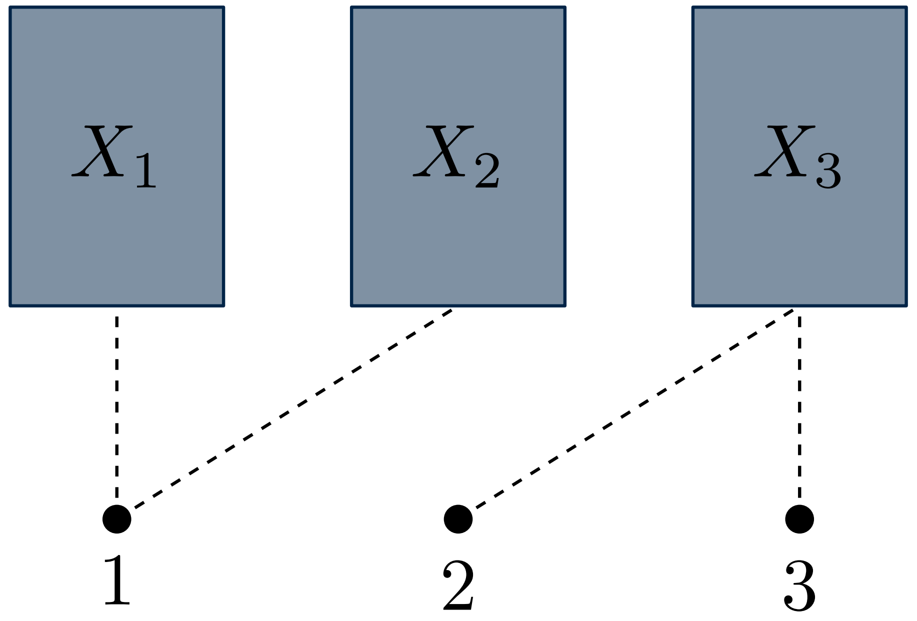


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Make $X_3$ less desirable by moving goods from $X_3$ to $X_1$ and $X_2$.

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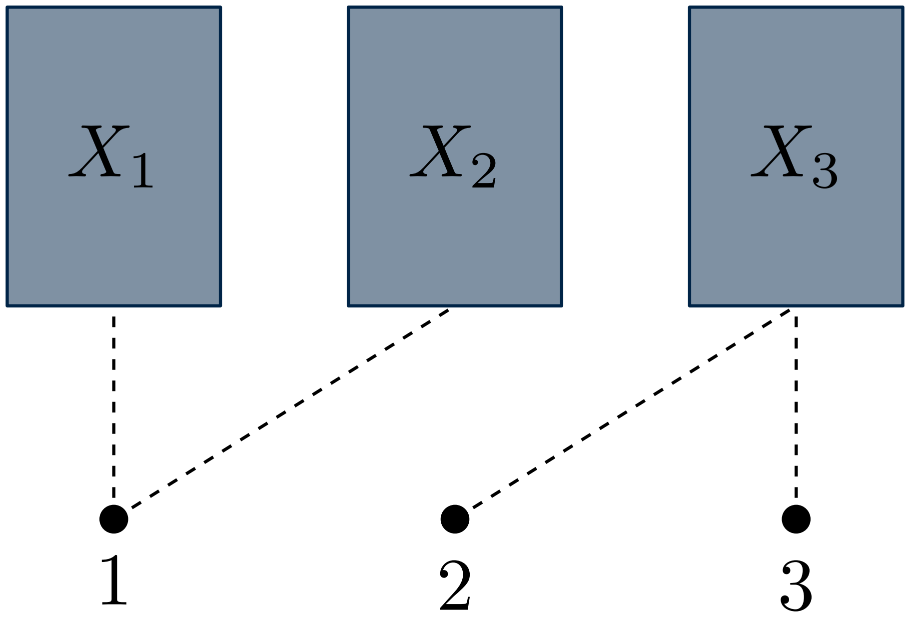



How to proceed?

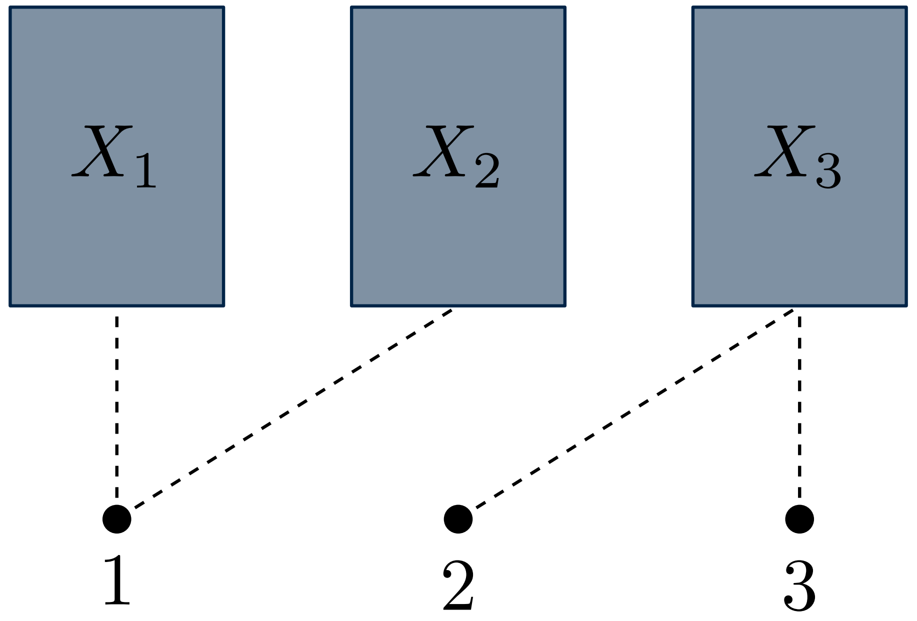
Make $X_3$ less desirable by moving goods from $X_3$ to $X_1$ and $X_2$.

**Potential function:**

$$\Phi(X) = \min(v_1(X_1), v_1(X_2))$$

# Algorithm

$X_3$ is the only EFX-feasible bundle for agent $2$.



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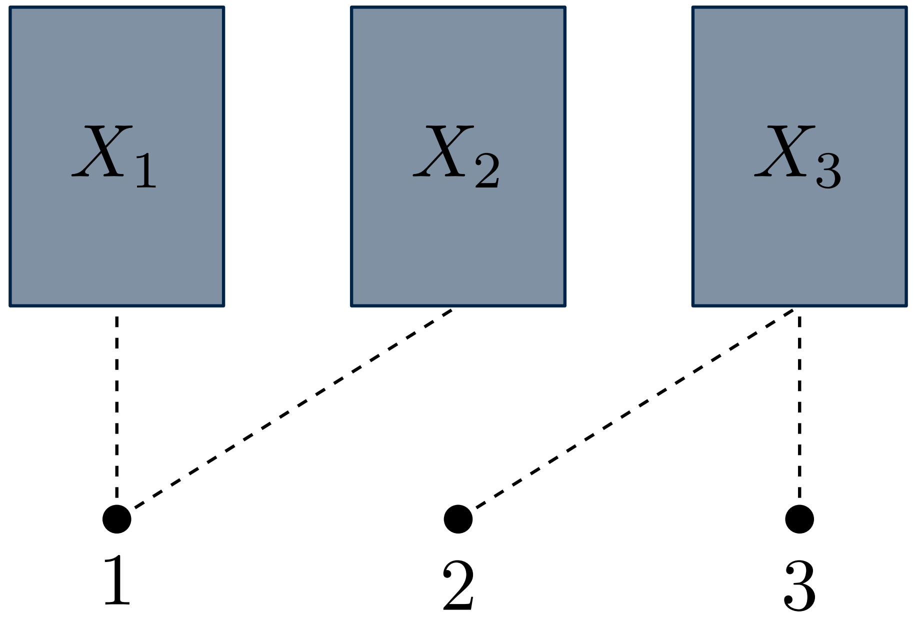
- $X_3 >_2 X_1$ and $X_3 >_2 X_2$



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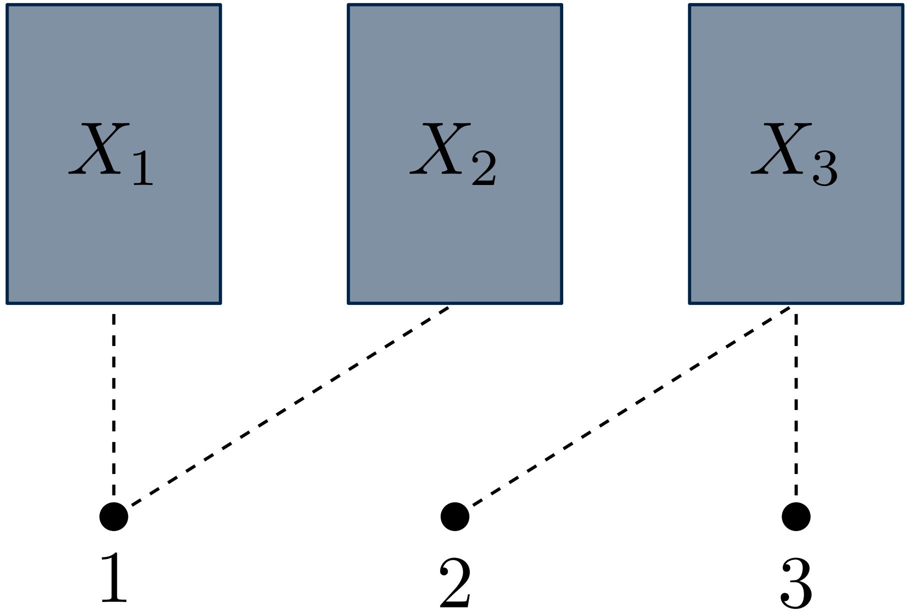

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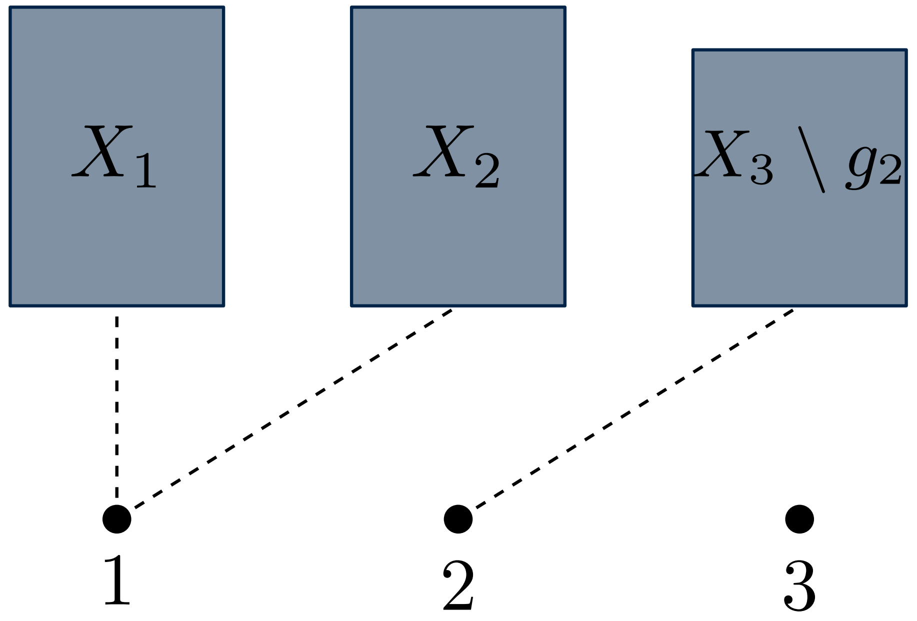

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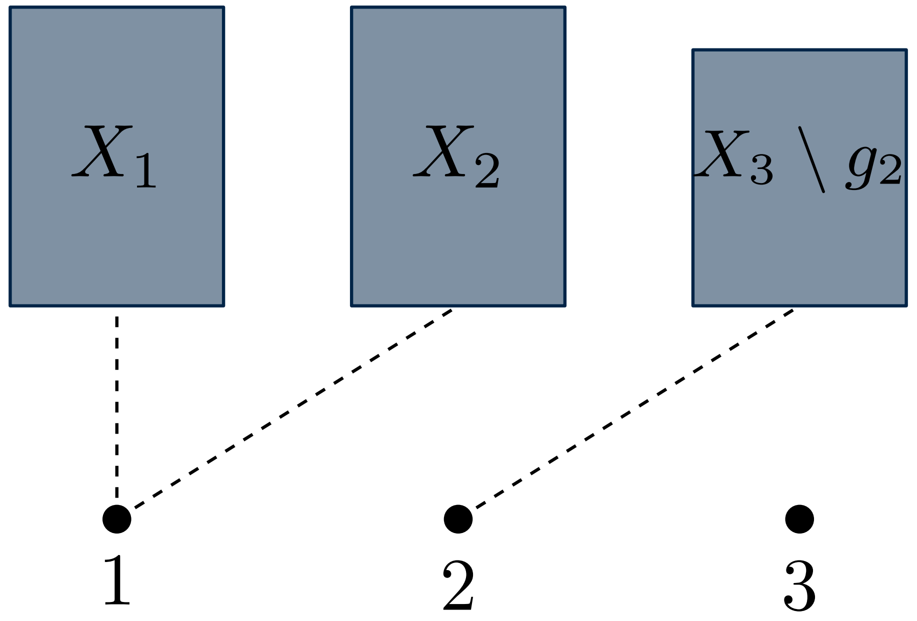


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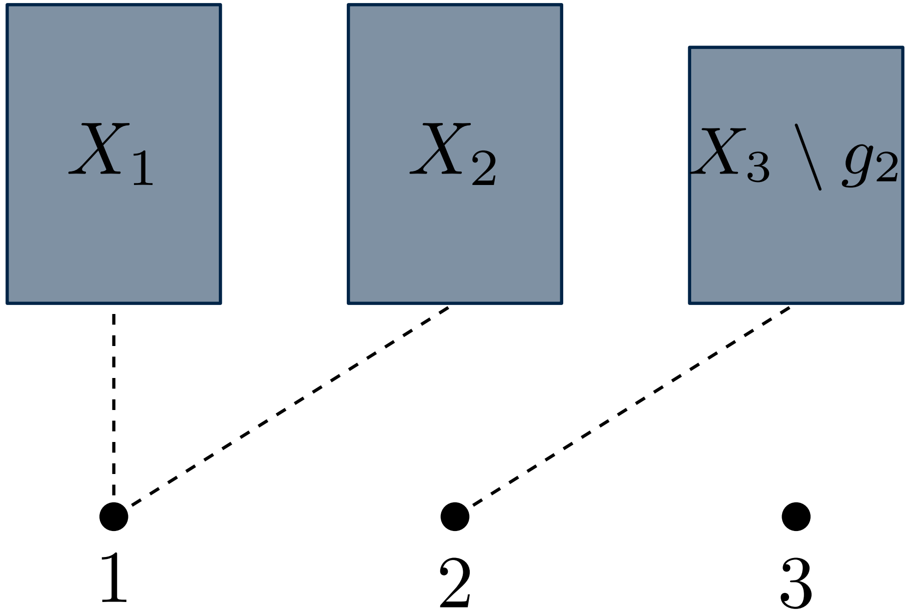


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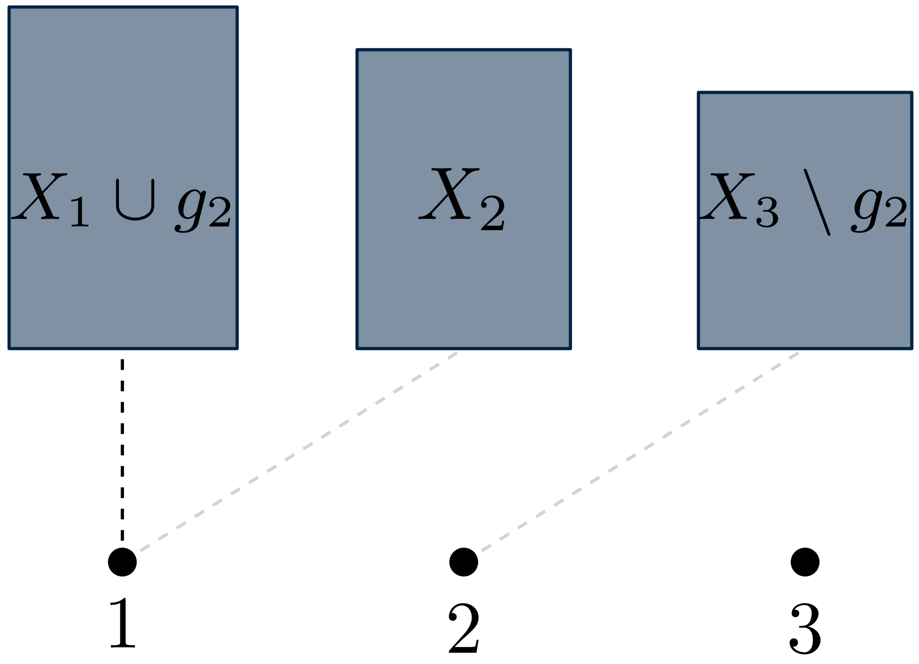


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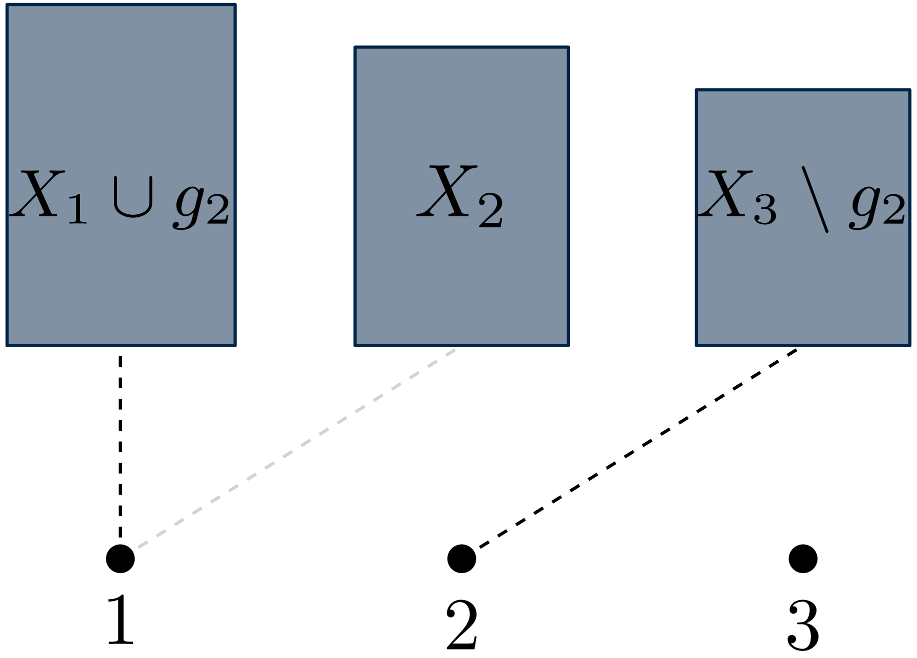
# Case Analysis

**Case 1:**

$$\boldsymbol{X_3 \setminus \{g_2\} >_2 X_1 \cup \{g_2\}}$$



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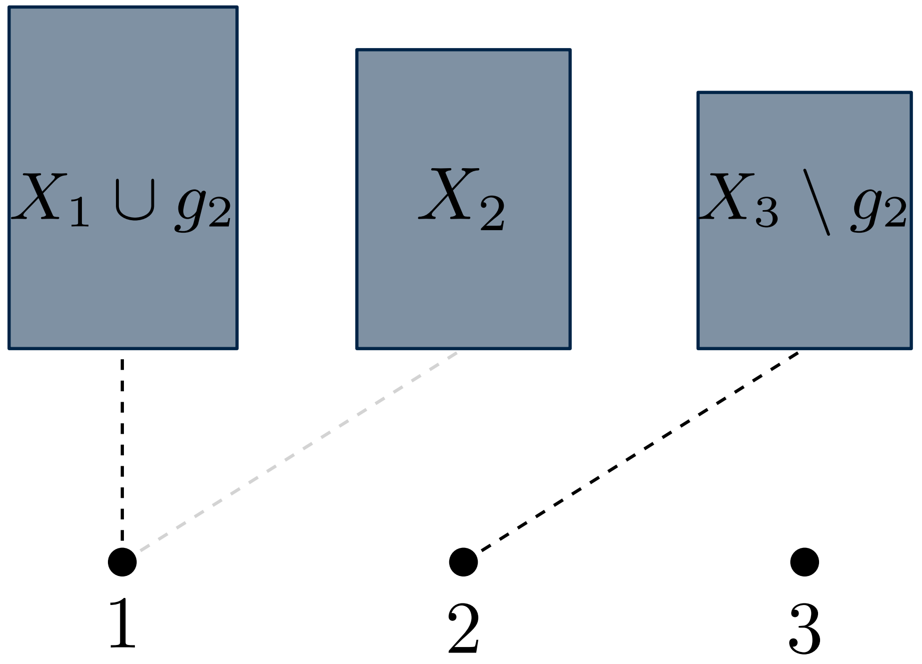

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$$X_2 <_2 \boldsymbol{X_3 \setminus \{g_2\} >_2 X_1 \cup \{g_2\}}$$

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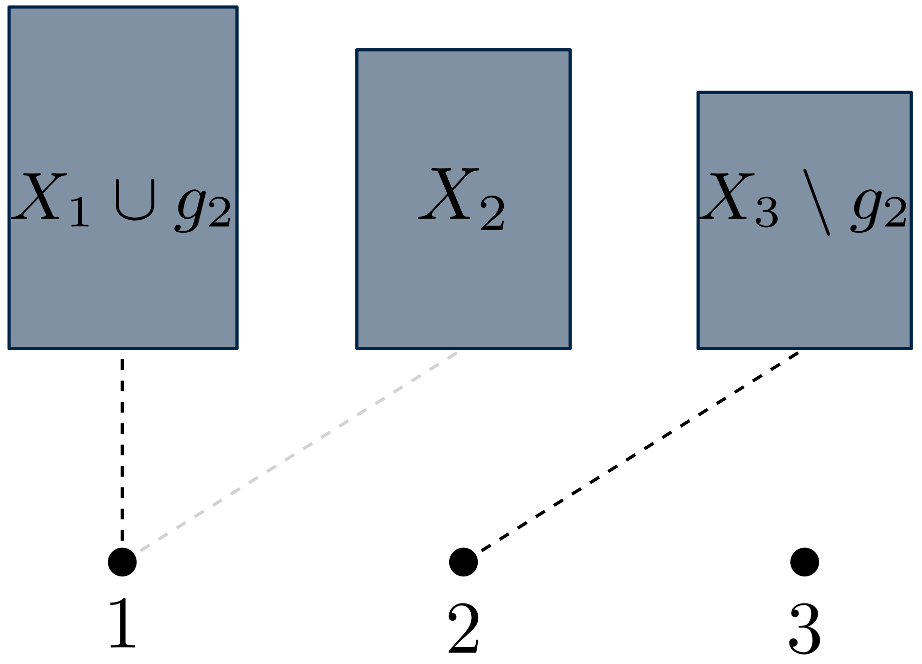

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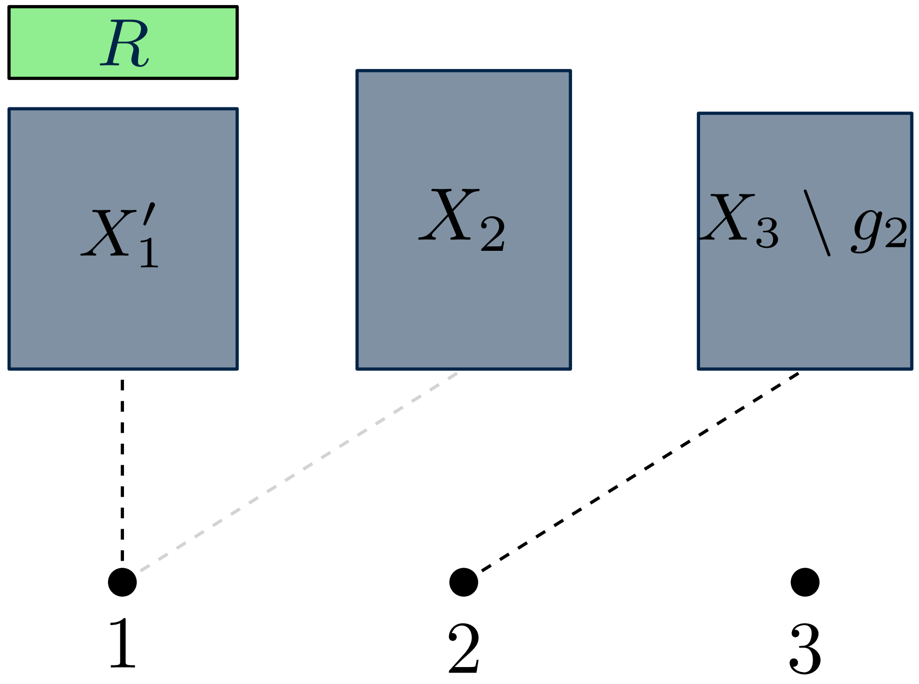

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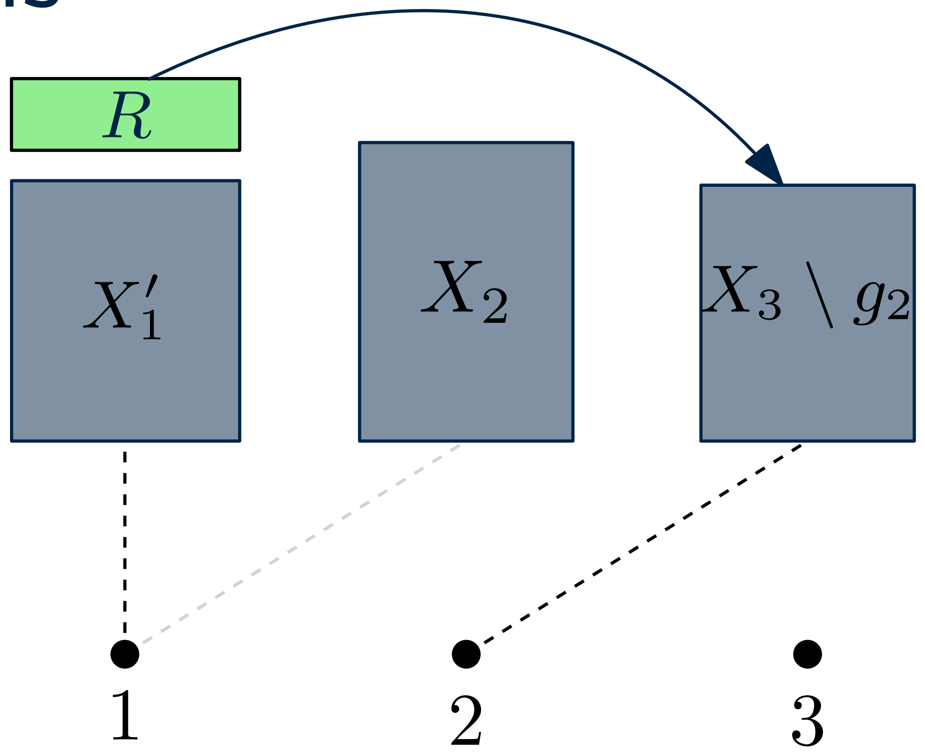

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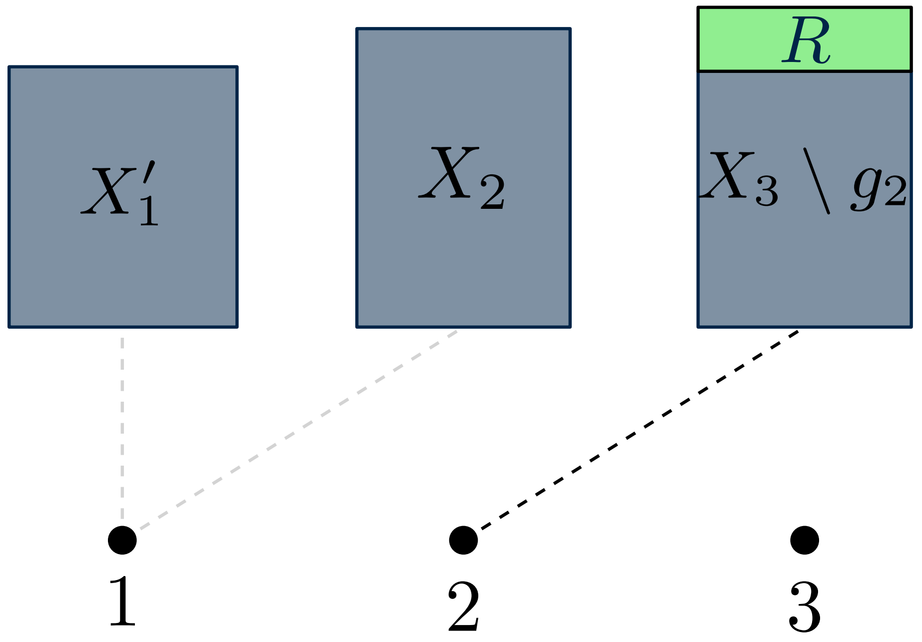

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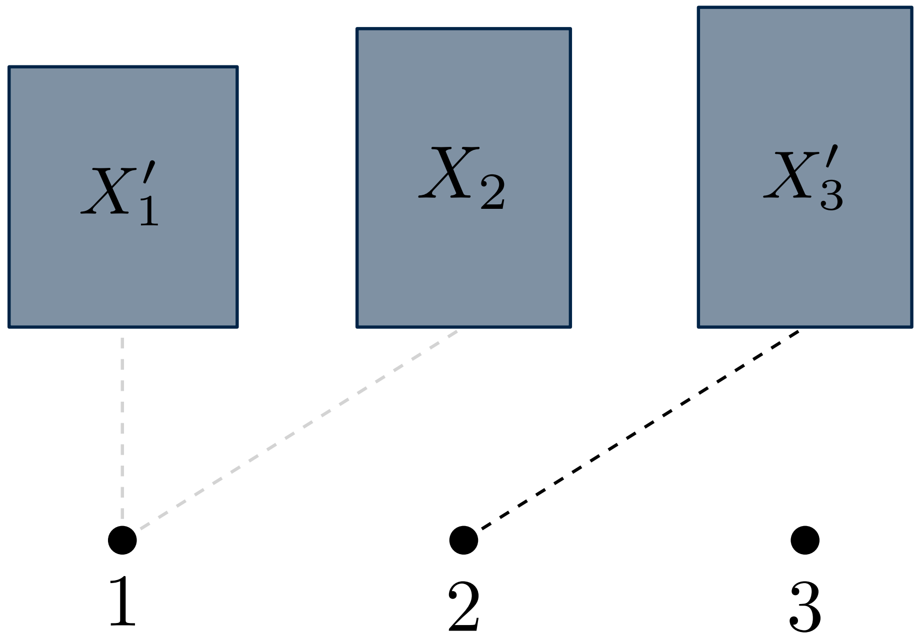

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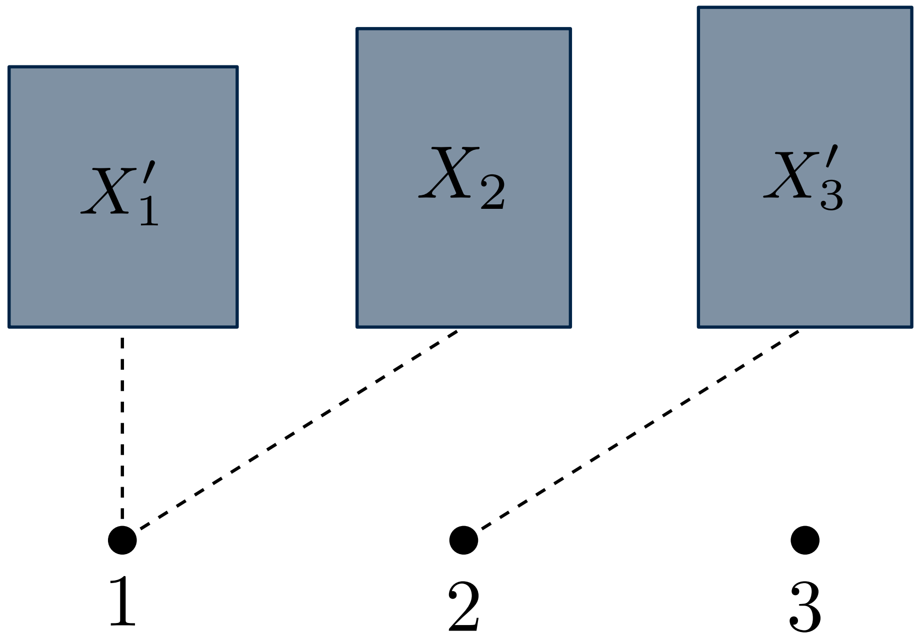


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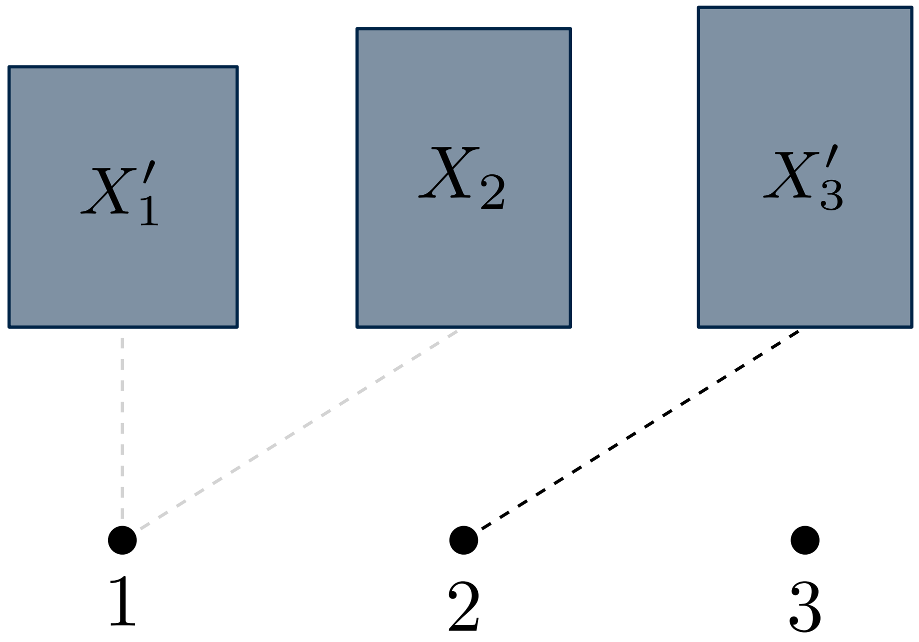


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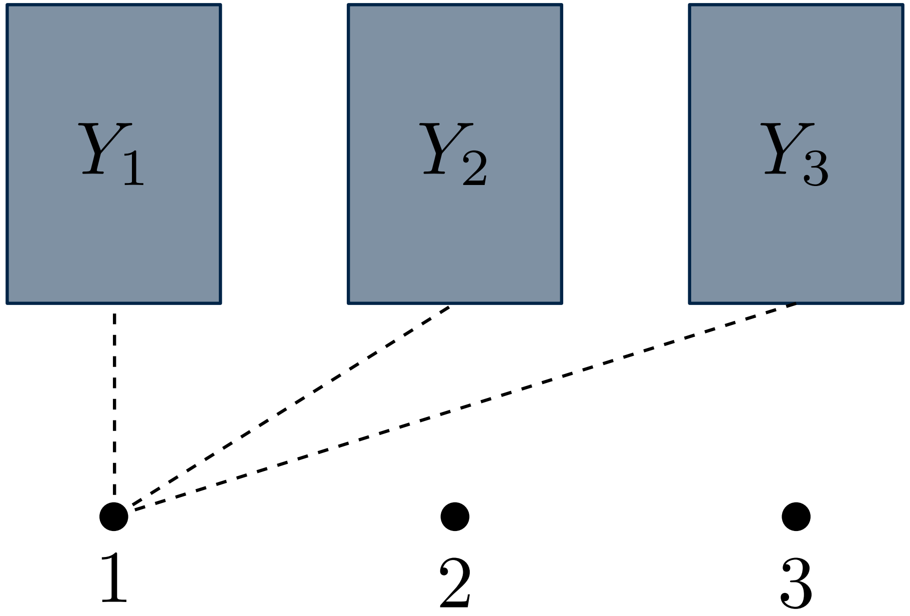


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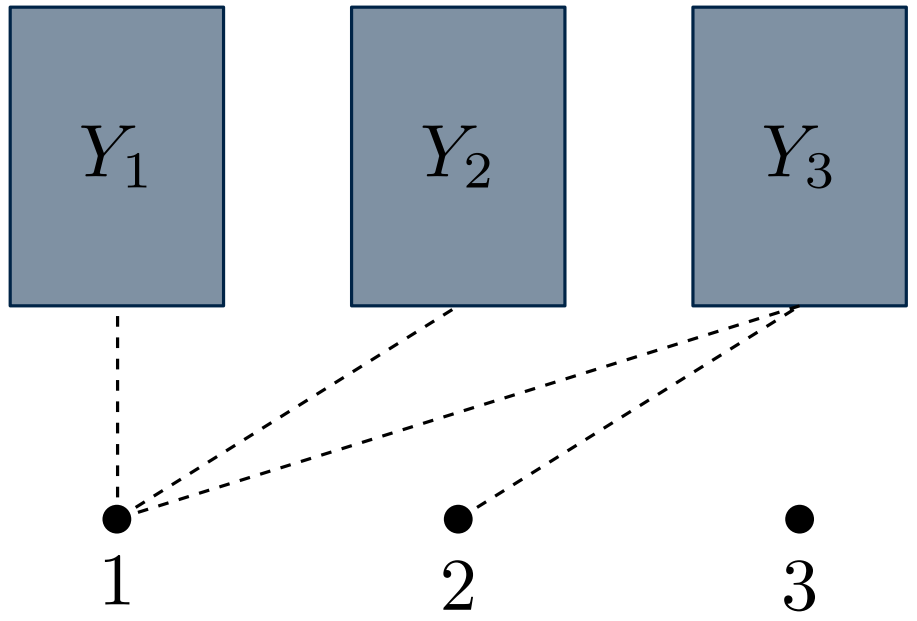


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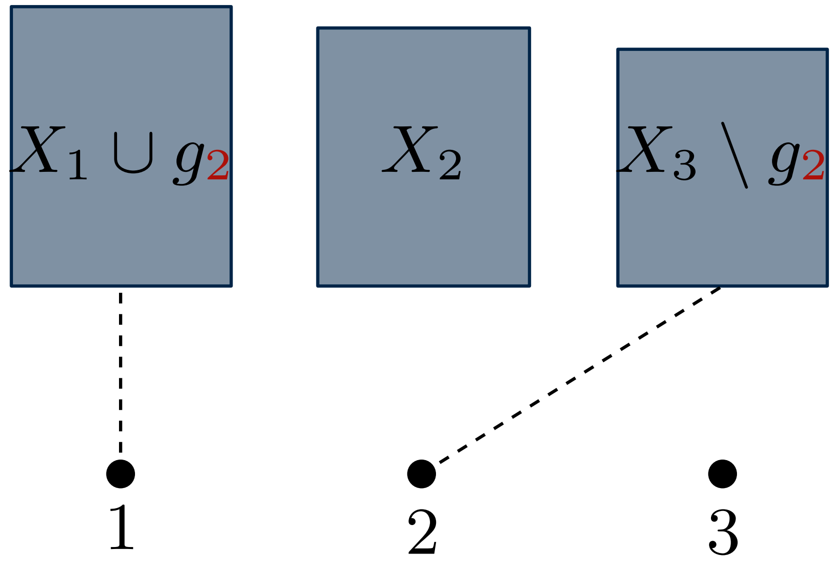

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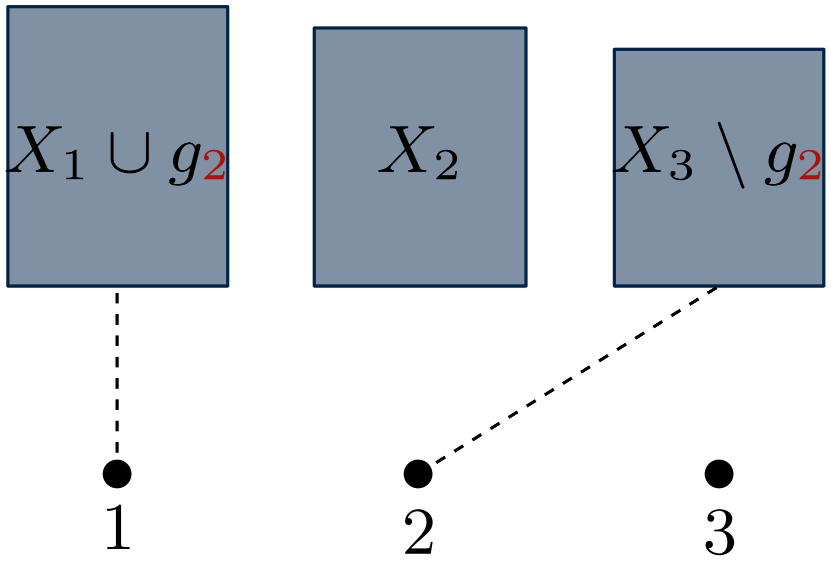


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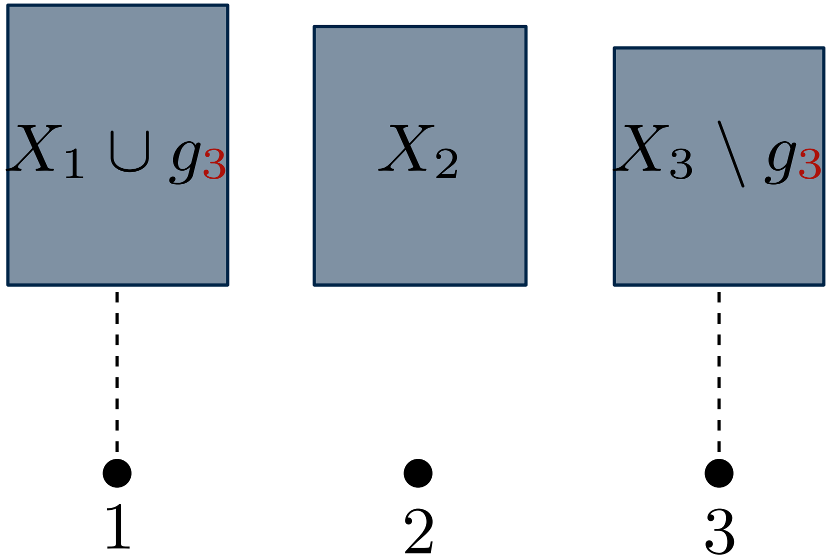


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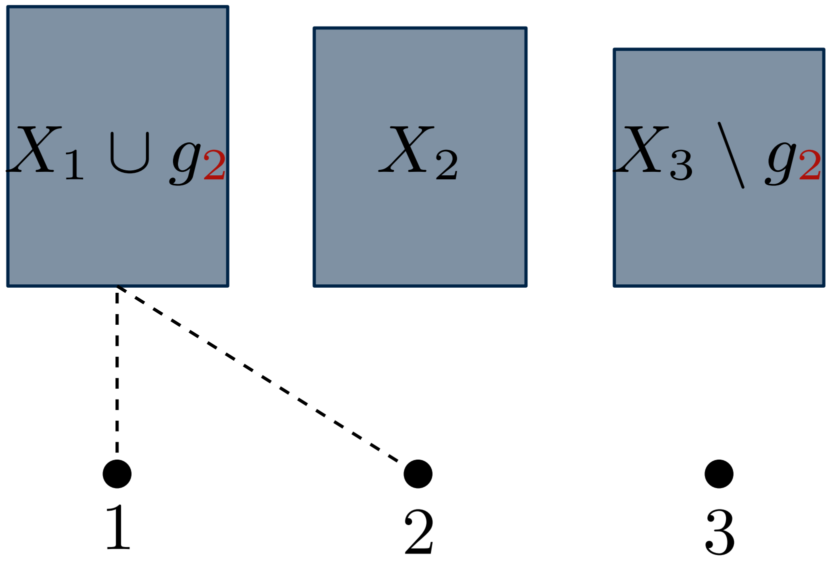
# Case Analysis

**Case 2:**

$$X_2 <_2 \boldsymbol{X_3 \setminus \{g_2\} <_2 X_1 \cup \{g_2\}}$$

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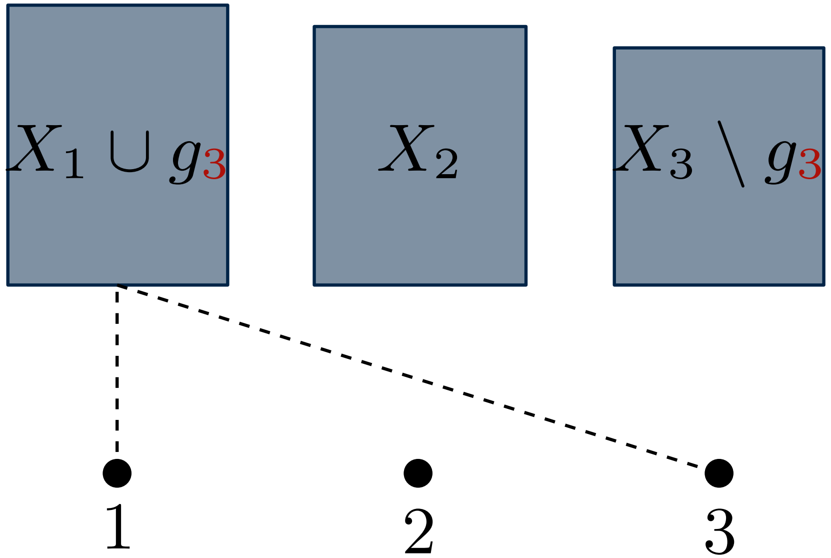

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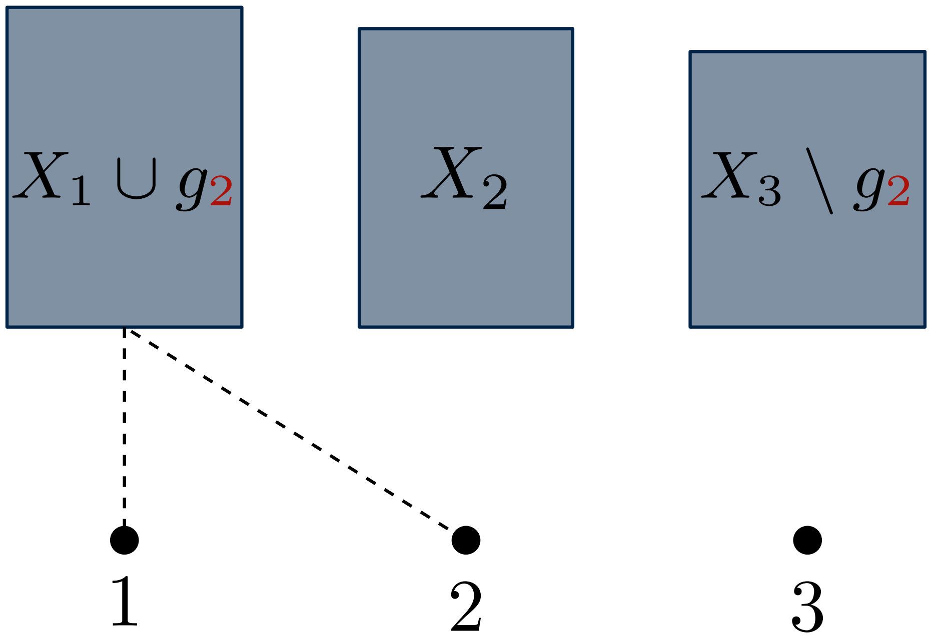

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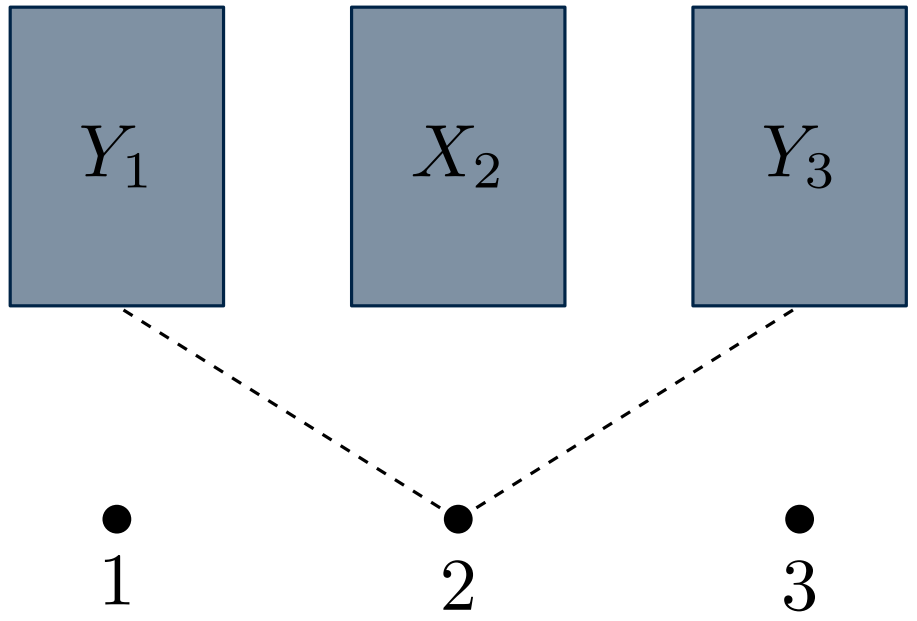

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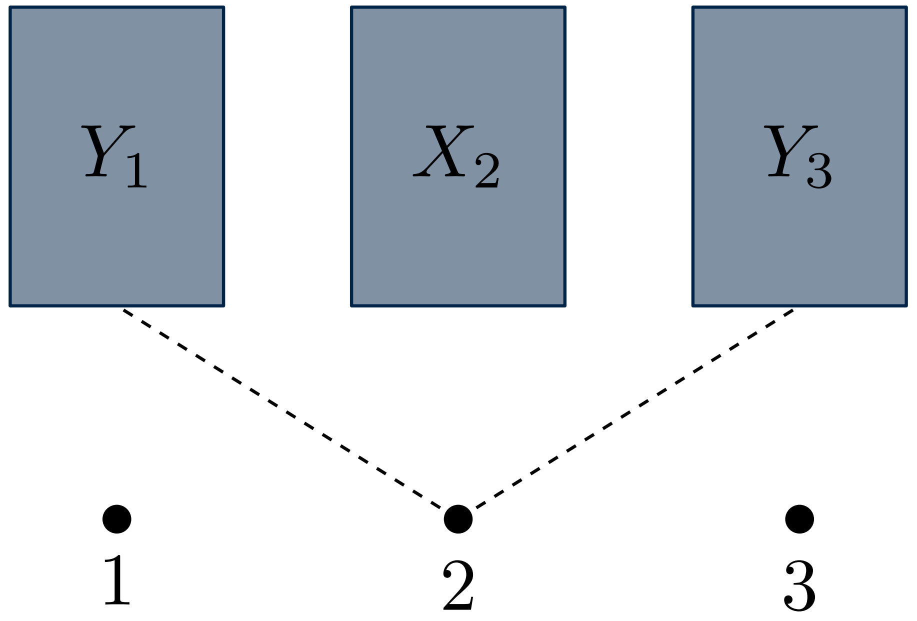

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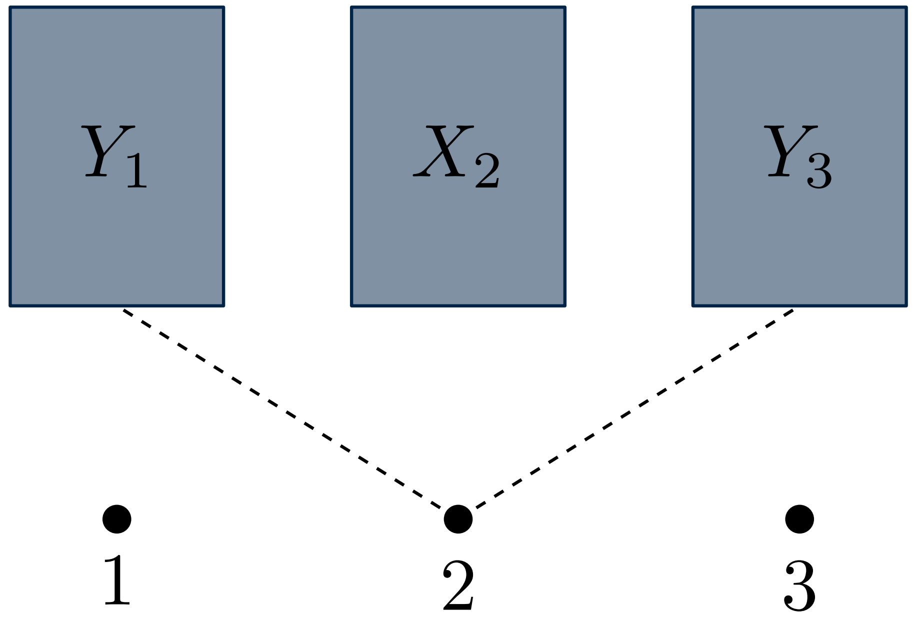

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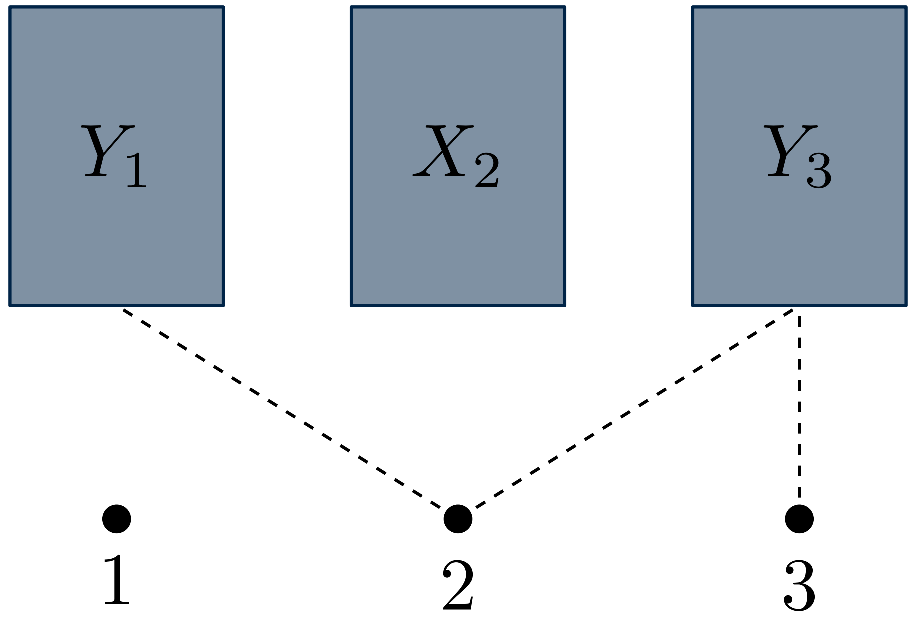

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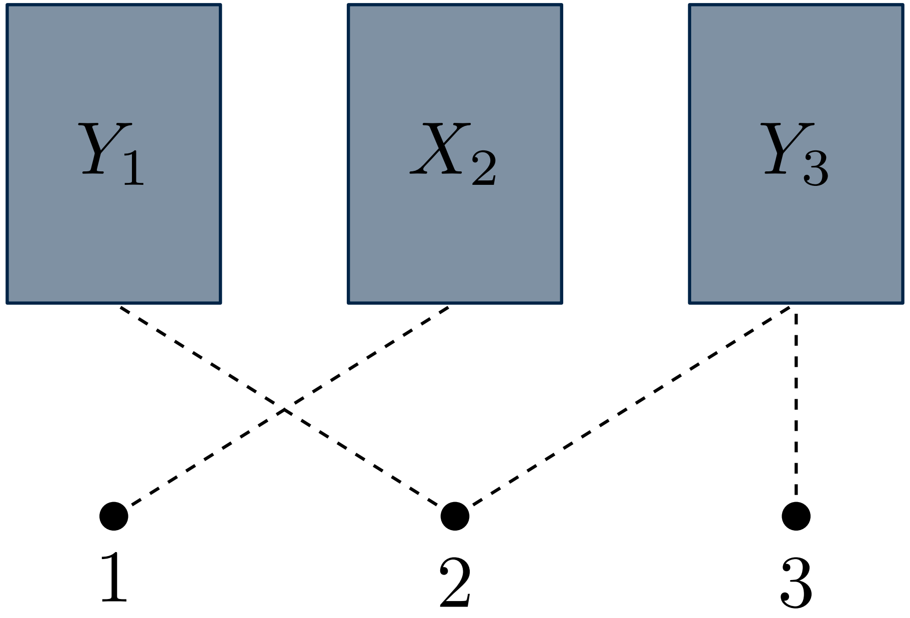

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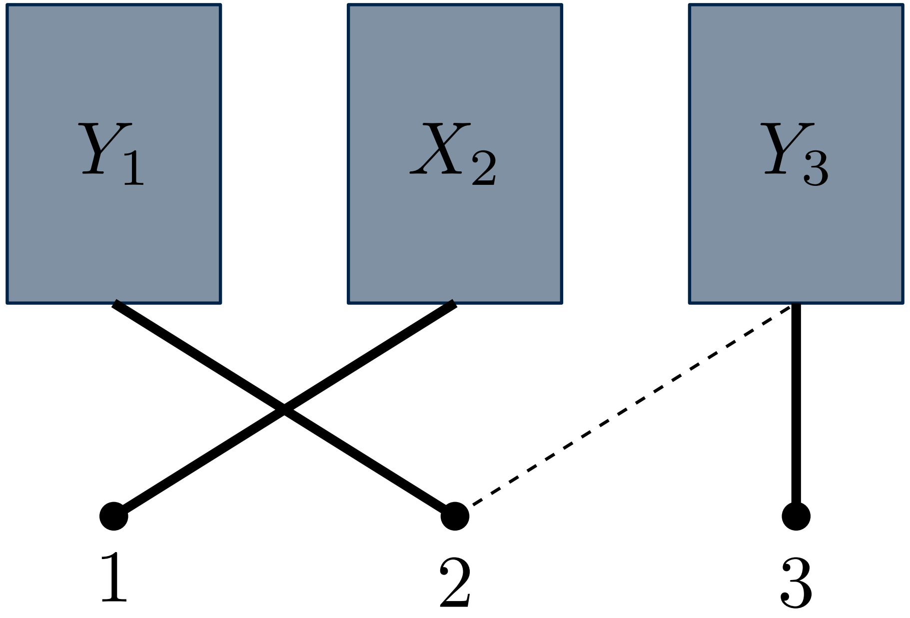

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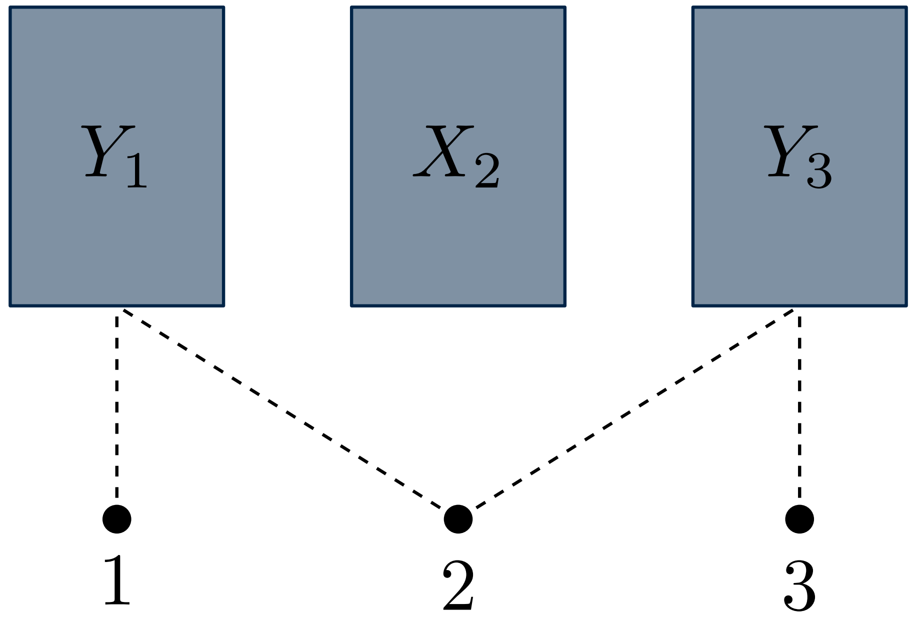

# Case Analysis

**Case 2:**

$$\boldsymbol{X_2 <_2 X_3 \setminus \{g_2\} <_2 X_1 \cup \{g_2\}}$$

- $(Y_1, Y_3) \leftarrow PR_2(X_1 \cup \{g_2\}, X_3 \setminus \{g_2\})$

$$\boldsymbol{X_2 <_3 X_3 \setminus \{g_3\} <_3 X_1 \cup \{g_3\}}$$

$$\implies \max_3(Y_1, Y_3) >_3 X_2$$

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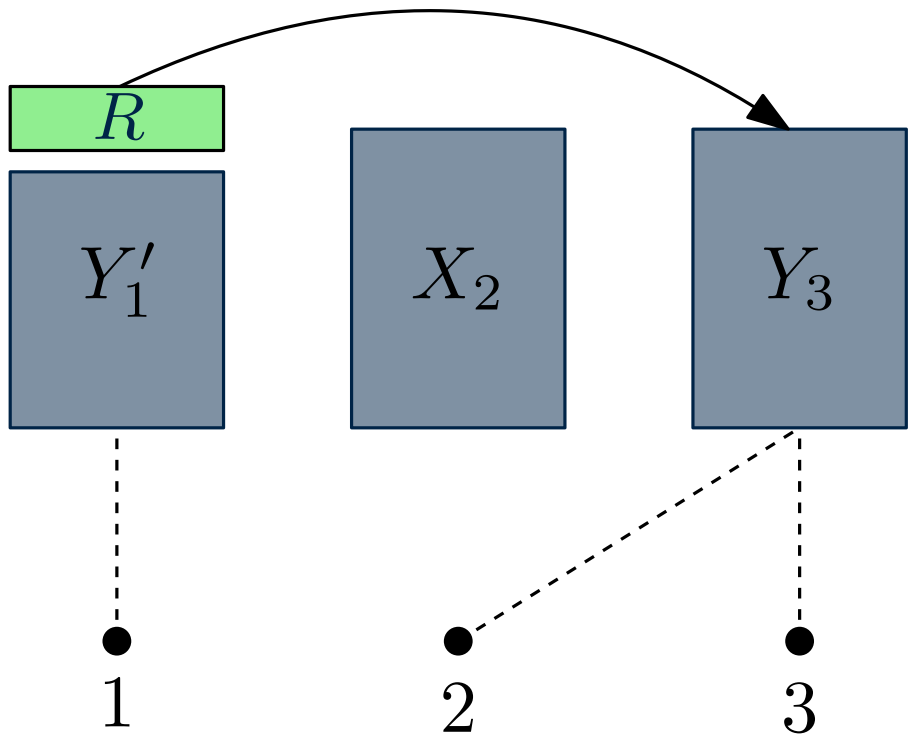

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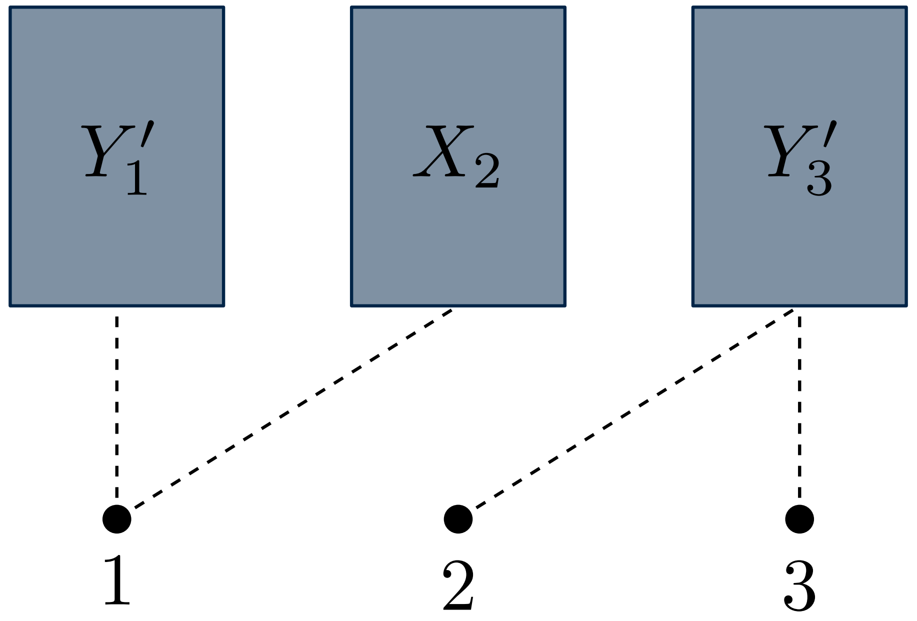
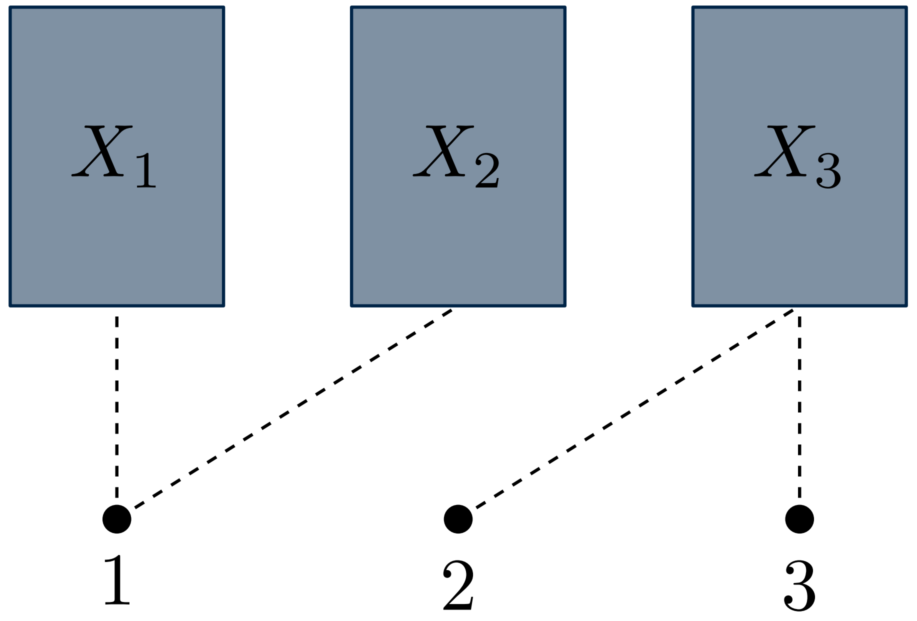


# Recap

Invariants:

- $X_1$ and $X_2$ are **EFX-feasible** to agent $1$.
- $X_3$ is **EFX-feasible** to agent $2$ or $3$.

$$\Phi(X) = \min(v_1(X_1), v_1(X_2))$$



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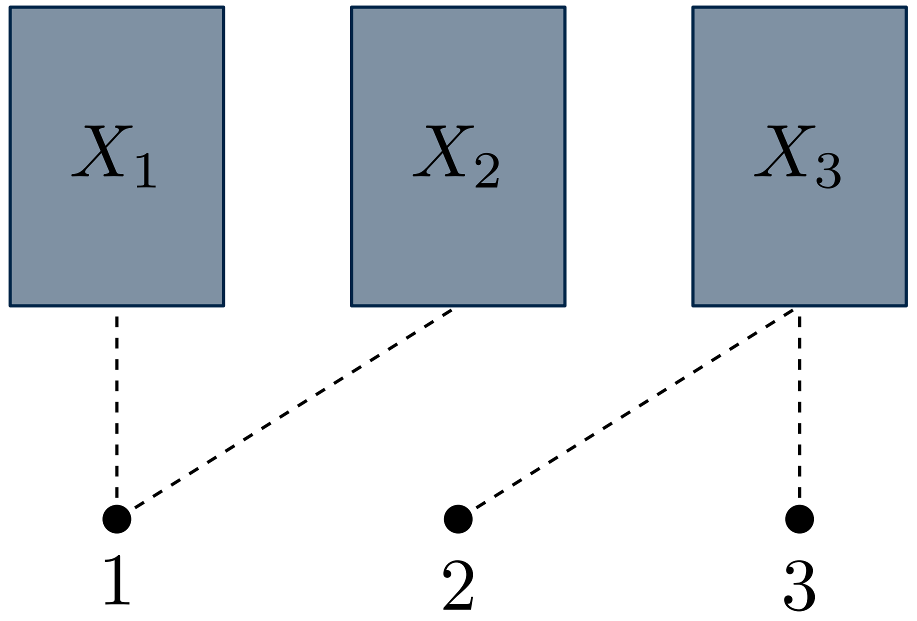
- Make $X_3$ less desirable!



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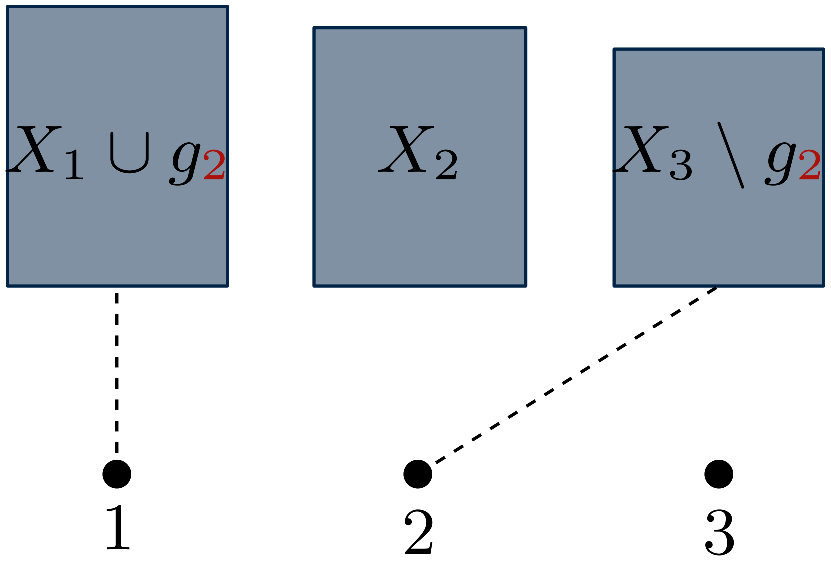


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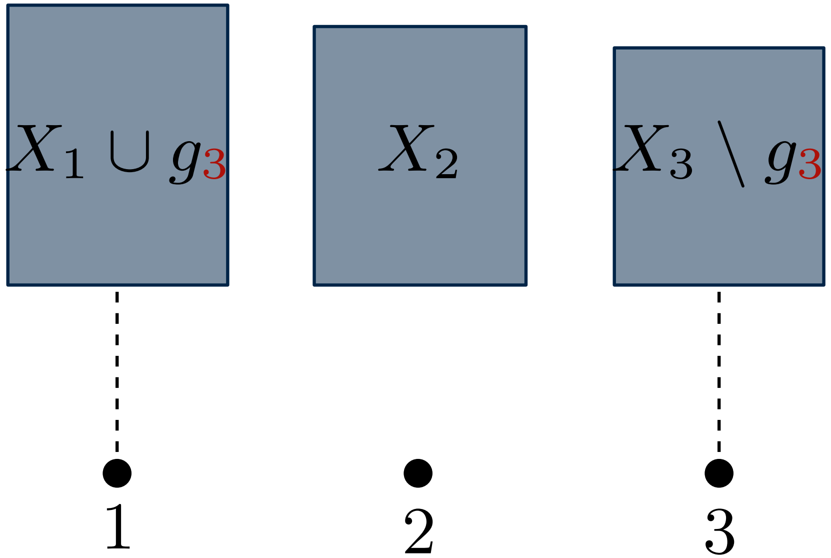

# Recap

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AND

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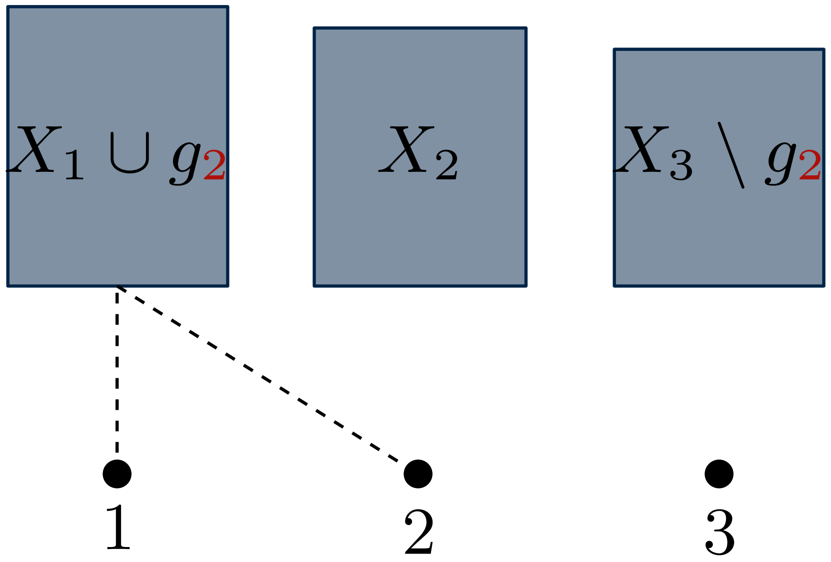


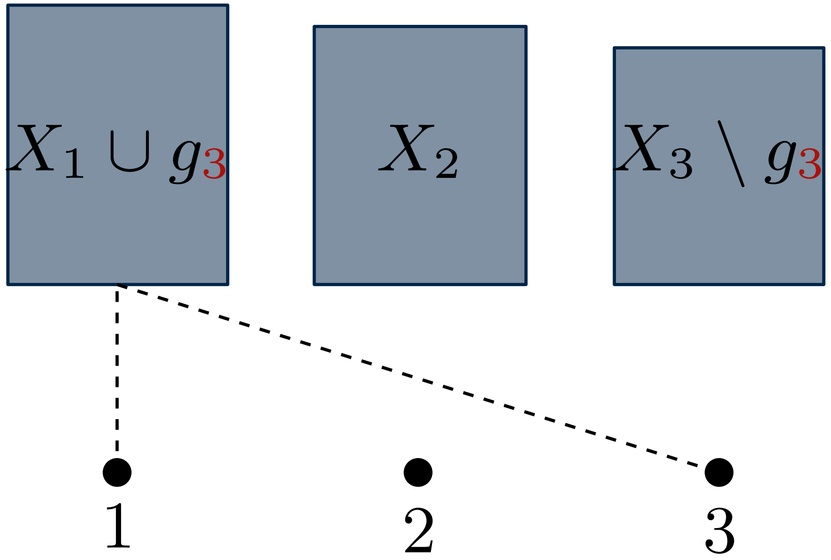

# EFX with Charity

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- $\frac{1}{2}$-NSW [Caragiannis, Gravin, Huang'19]
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- $(1-\epsilon)$-EFX allocation with $\mathcal{O}((n/\epsilon)^{\frac{4}{5}})$ unallocated goods [Chaudhury, Garg, Mehlhorn, Mehta, Misra'21]

$v_i(X_i) \geq (1-\epsilon) \cdot v_i(X_j \setminus \{g\})$ for all agents $i, j$ and all goods $g \in X_j$.

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# Our Results

1. Assume $n$ is small.

**Theorem 1** [A., Alon, Chaudhury, Garg, Mehlhorn, Mehta]

EFX allocations exists for $n = 3$ when

- one agent has **nice-cancelable** valuation function, and
- two agents have **general monotone** valuation functions.

2. EFX with charity: Allow a small subset of goods to remain unallocated.

**Theorem 2** [A., Alon, Chaudhury, Garg, Mehlhorn, Mehta]

$(1-\epsilon)$-EFX allocations exist with $\tilde{\mathcal{O}}((n/\epsilon)^{\frac{1}{2}})$ unallocated goods.

# Future Directions

- EFX for $3$ agents with general monotone valuations?
- EFX for $4$ agents?
- ...

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