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Topics in Computational Social Choice Theory

Lecture 4: Existence of envy-free cake divisions

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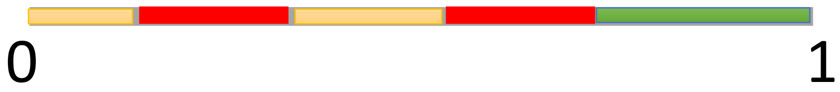
Last Lecture: Introduction to Cake Cutting

- The resource: **Cake [0,1]** (heterogeneous and divisible)
- Set of **agents**: {1,2, …, n}
- **Piece** of a cake: finite union of subintervals of [0,1]
- Valuation function v_i : Agent i values piece X at $v_i(X) \geq 0$

Fairness Notions

Allocation:

A partition $A = (A_1, A_2, \dots, A_n)$ of the cake $[0,1]$ where piece A_i belongs to agent i



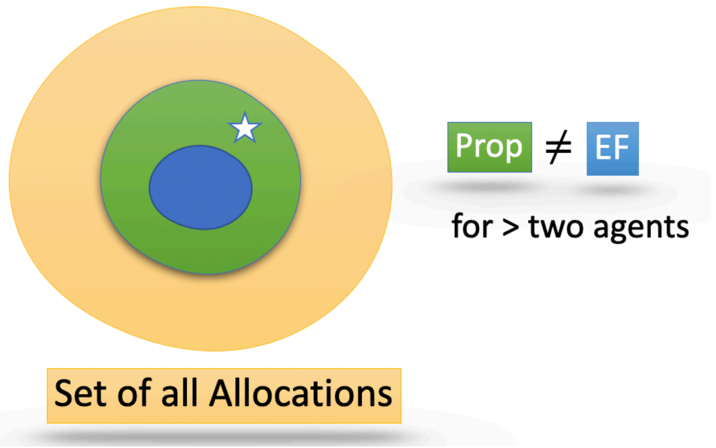
- **Proportionality:** for each agent $i \in [n]$, we have $v_i(A_i) \geq 1/n$

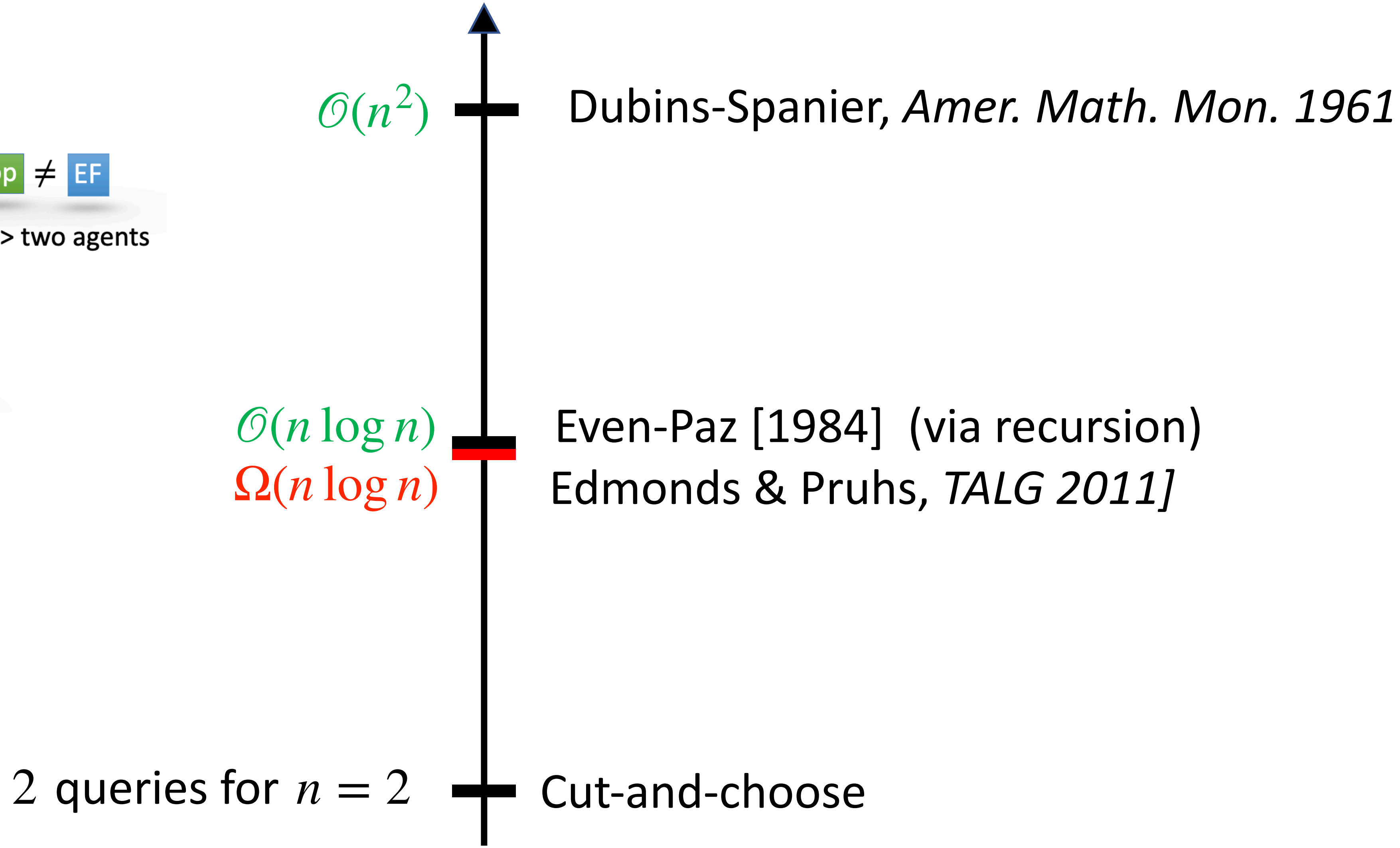
  [Steinhaus, 1948]

- **Envy-freeness:** for every pair $i, j \in [n]$ of agents, we have $v_i(A_i) \geq v_i(A_j)$

  [Foley 1967]

Query Complexity of Proportionality





Source: 'Computational Social Choice' Lecture Slides by Rohit Vaish, IITD

Existence of Envy-free Cake Divisions

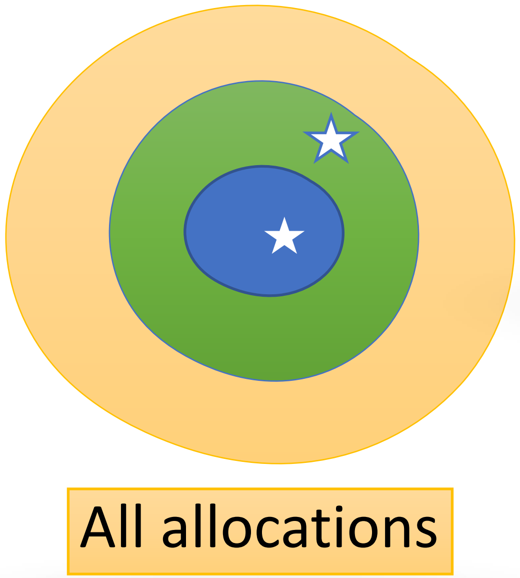
- Computing an envy-free cake division:

- **Cut-and-choose:** between two agents using 2 queries
- **Selfridge-Conway:** among three agents using 8 queries

non-contiguous pieces

What about $n \geq 4$ agents?

Existence of Envy-free Cake Divisions



EF $\Rightarrow$ Prop

EF $\not\Leftarrow$ Prop

Stromquist [1980], Su [1999] **connected pieces**

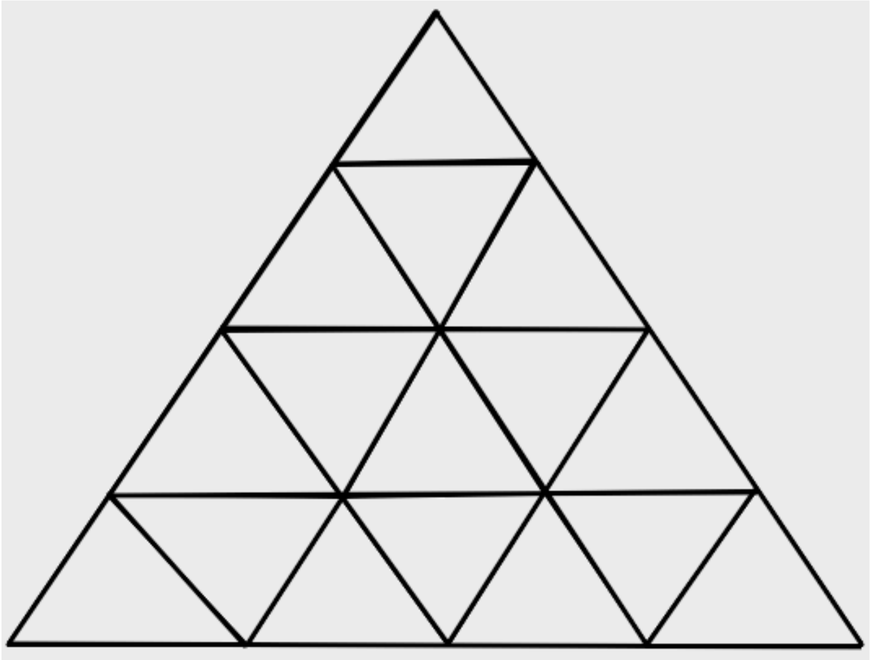
Envy-free cake division exist for any number of agents

Sperner's Lemma

Sperner's Lemma

A **beautiful lemma** that, on the face of it, has nothing to do with **cake division**

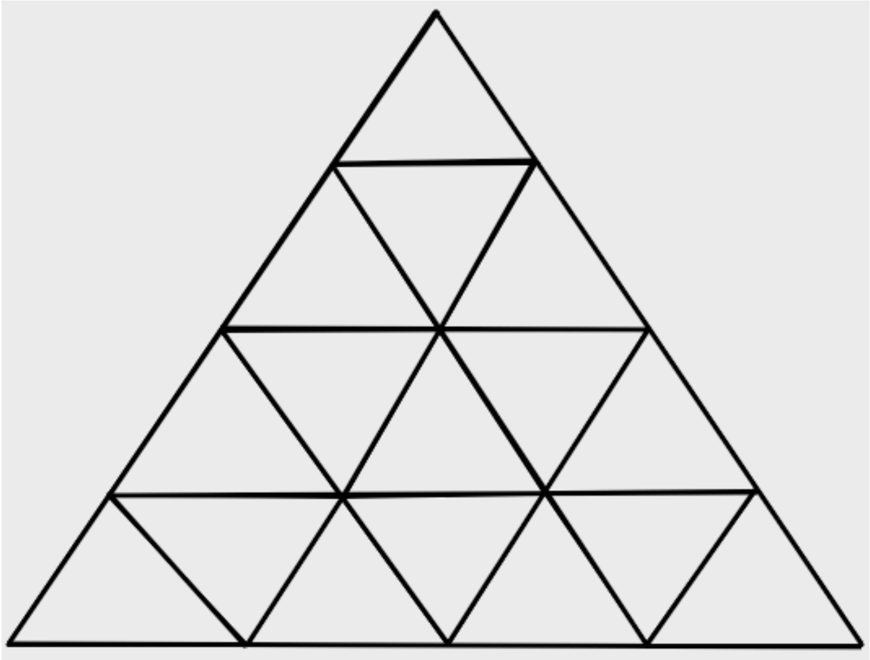
Sperner’s Lemma



Sperner's Lemma

Ingredients:

1) A **triangle** that is *subdivided* into smaller triangles
   (Formal terms: simplex and its triangulation)



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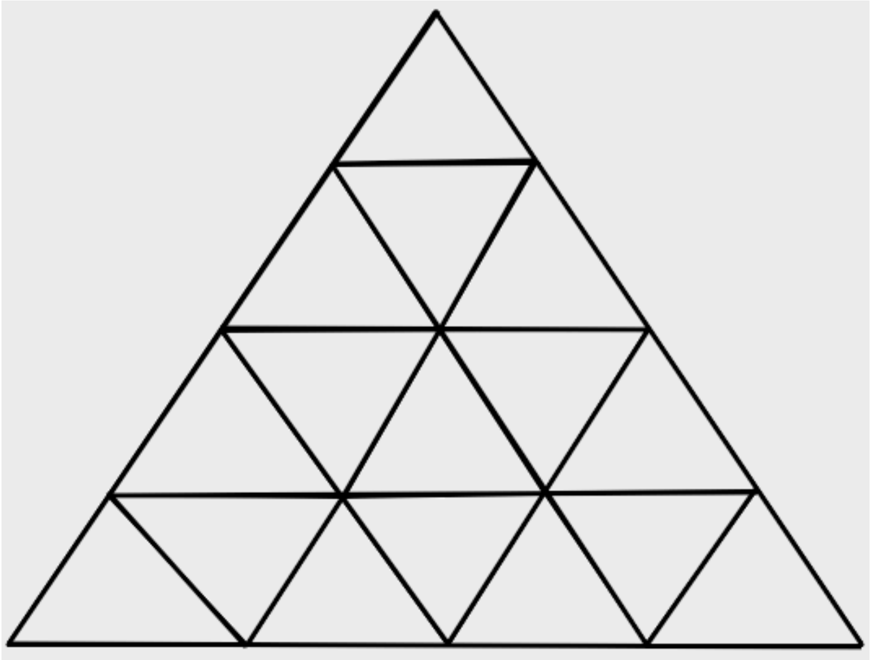

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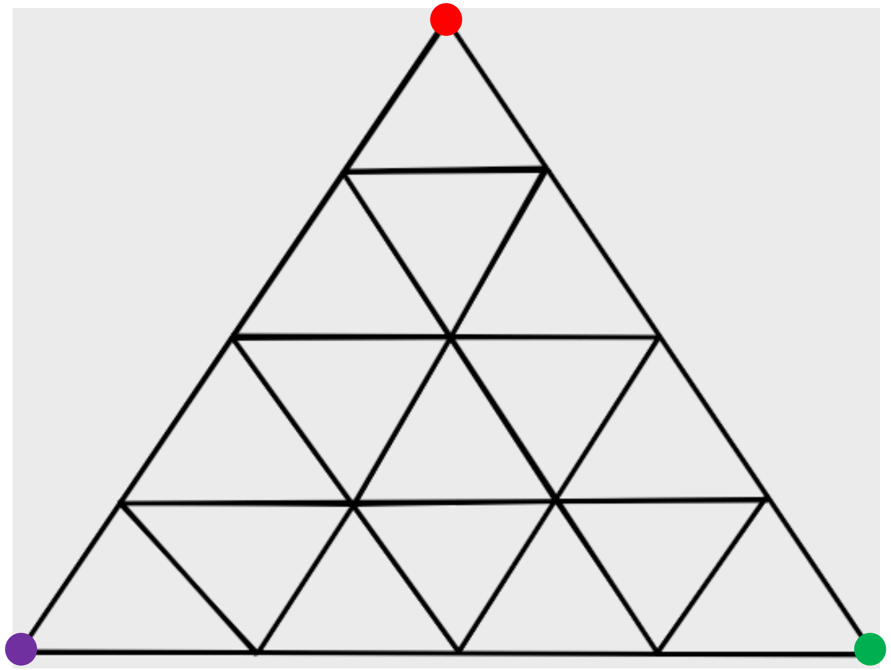
   - *Main vertices* have ***distinct*** colors



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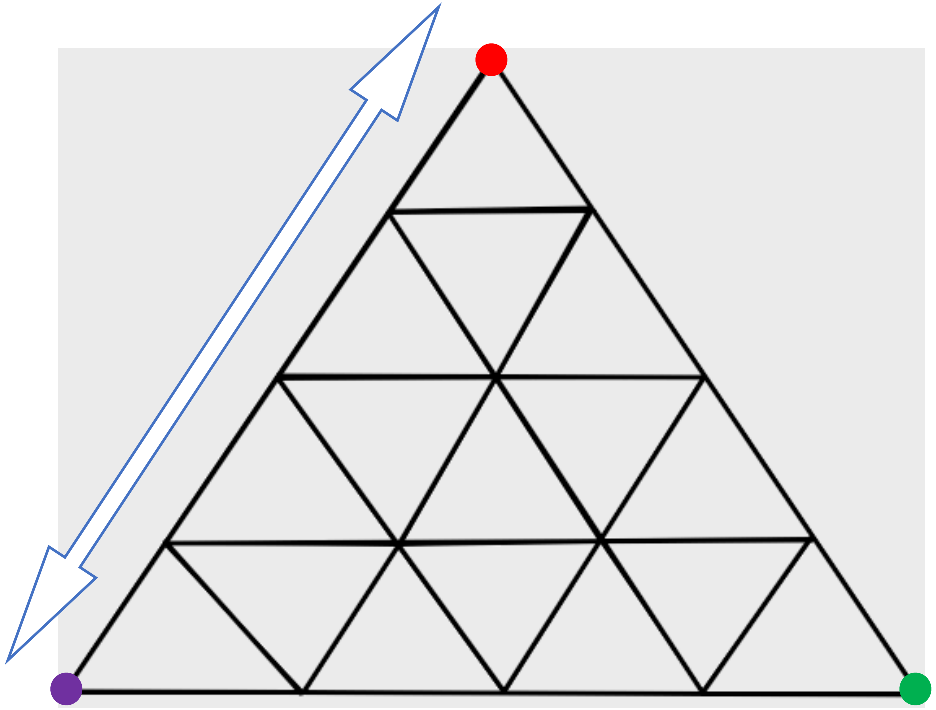

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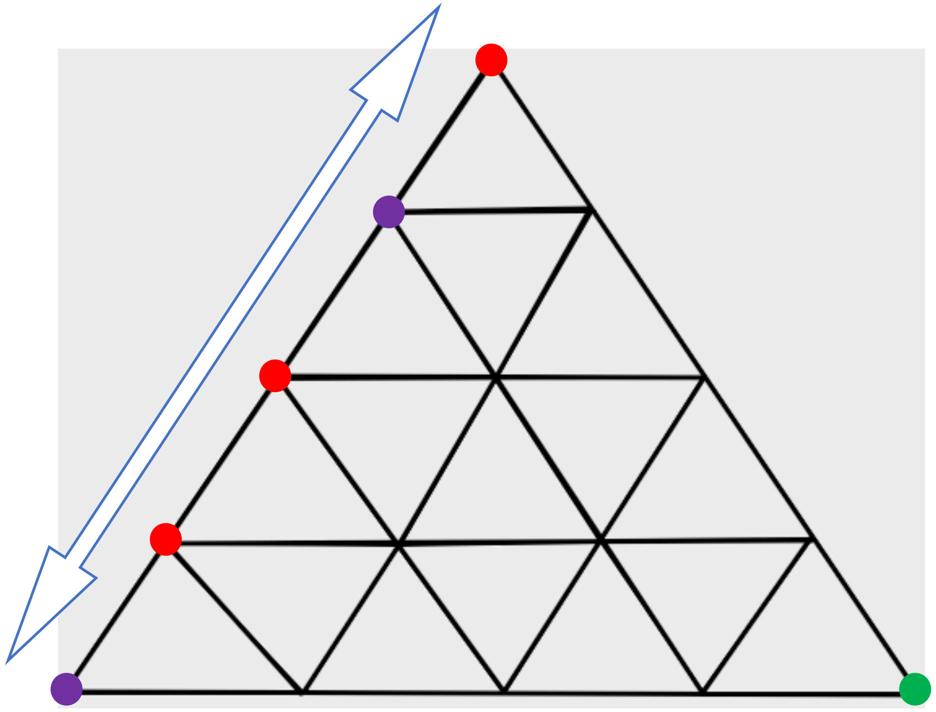

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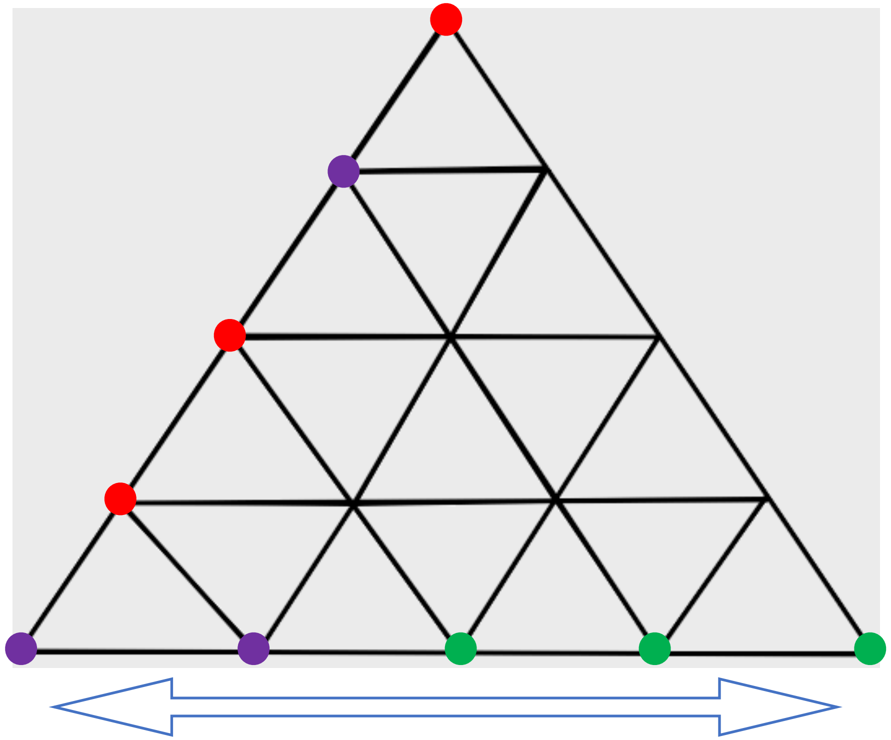

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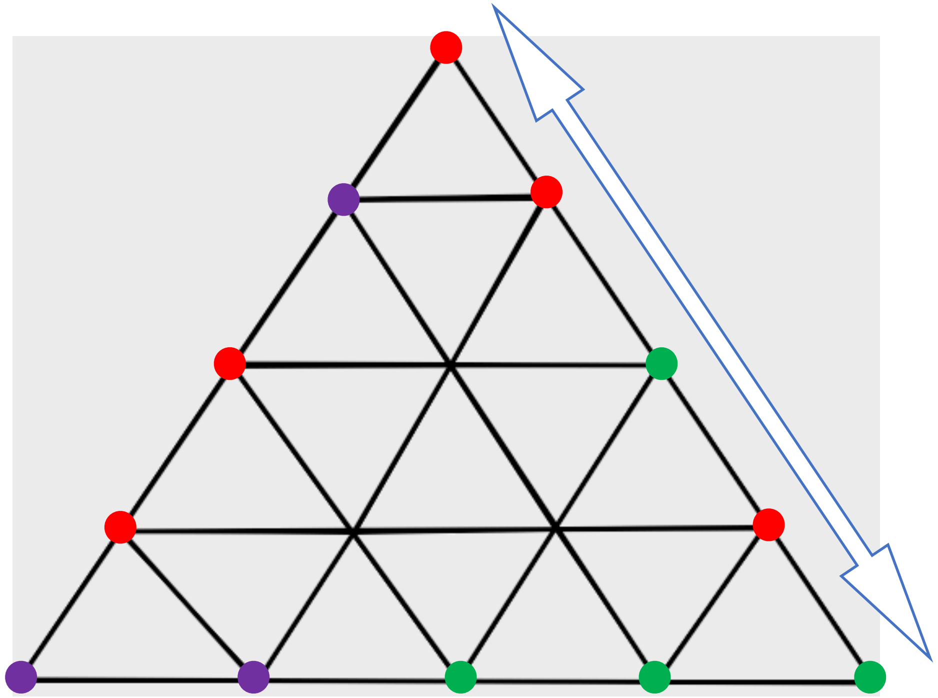

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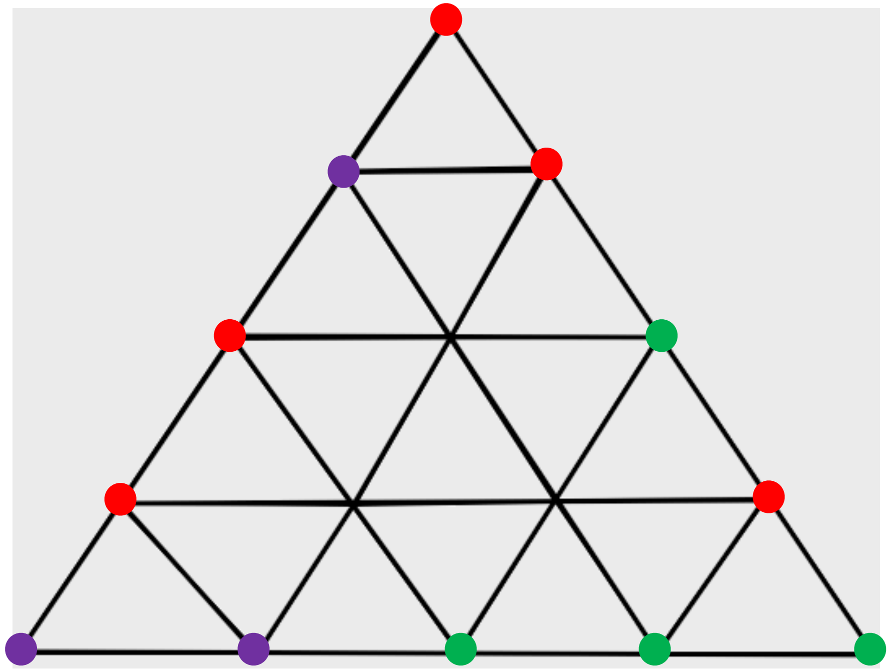

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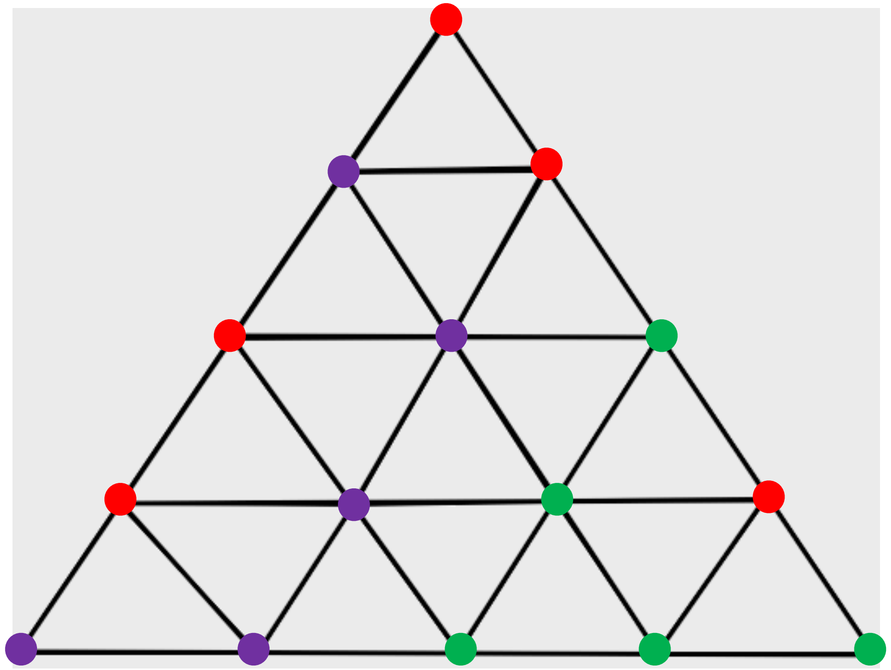

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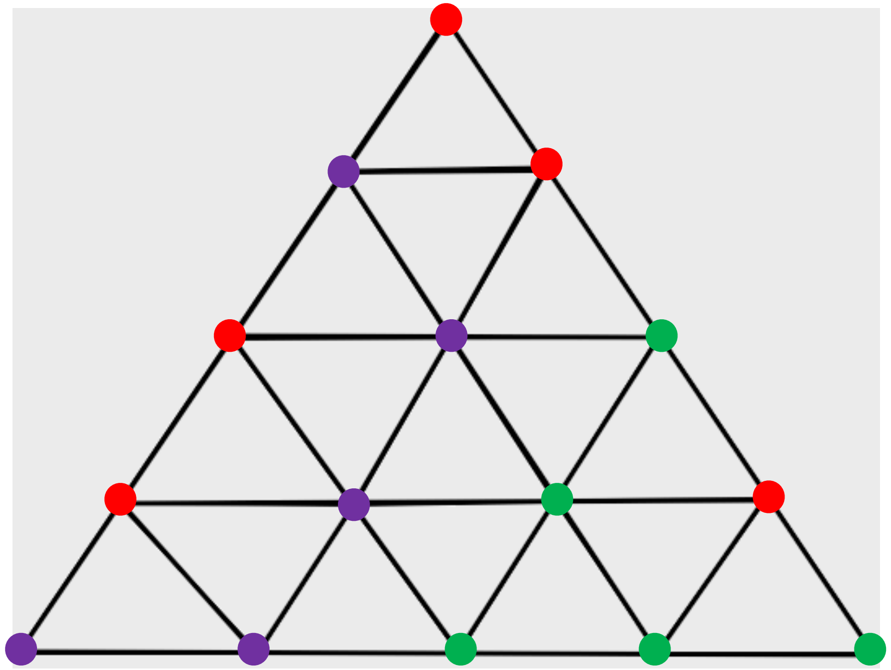

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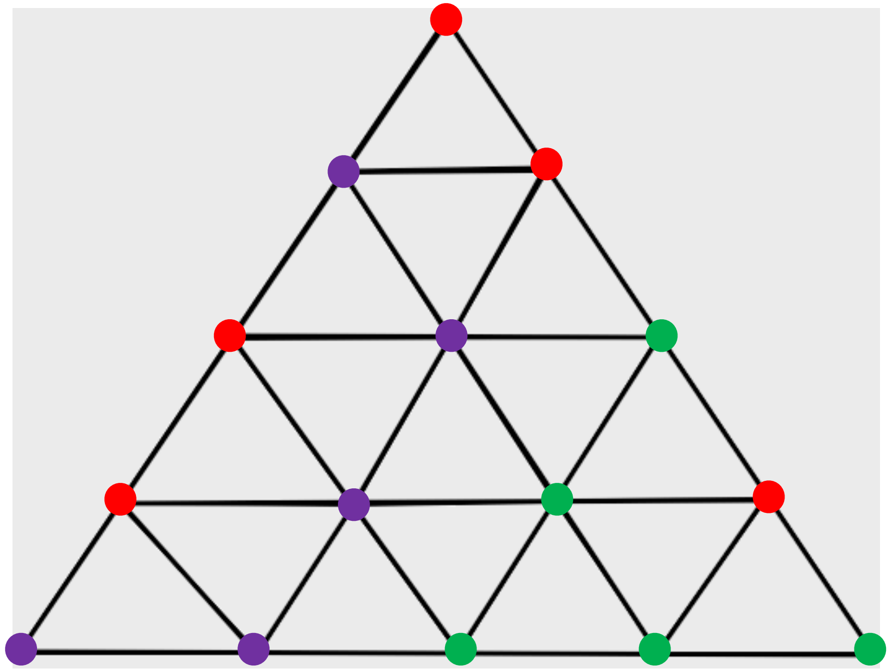

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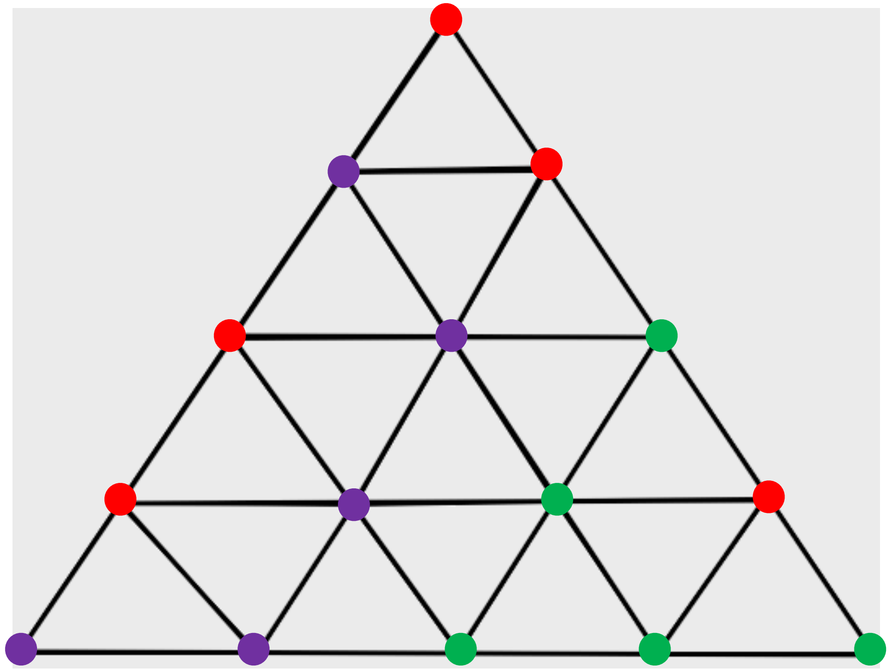

Any Sperner colored triangulation has at least **one fully colored baby triangle**

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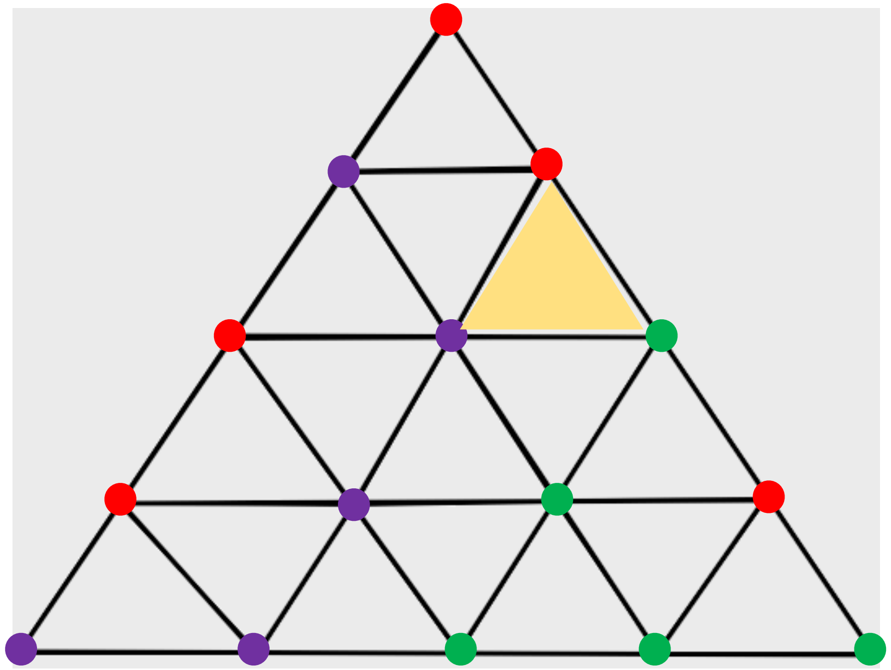

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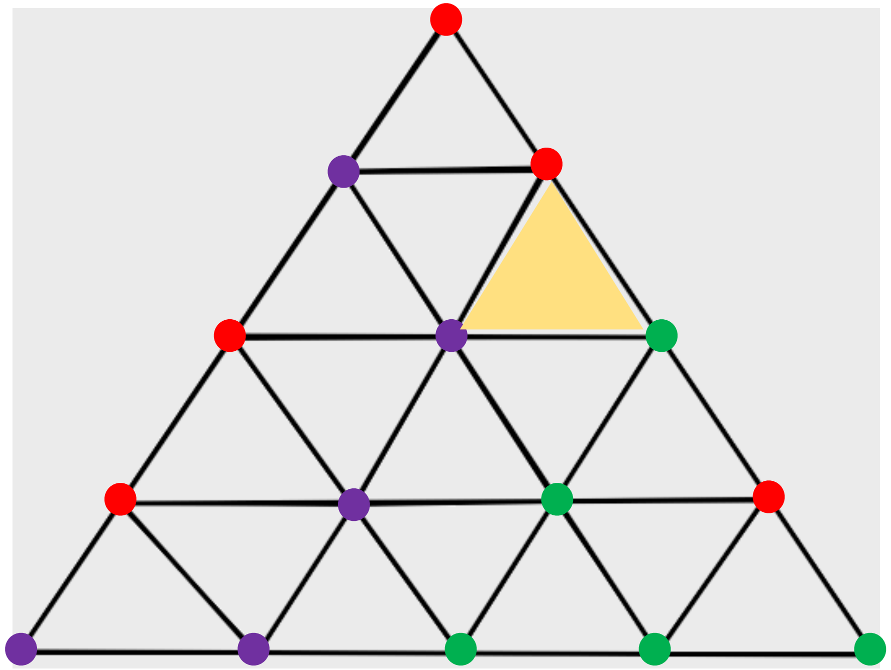

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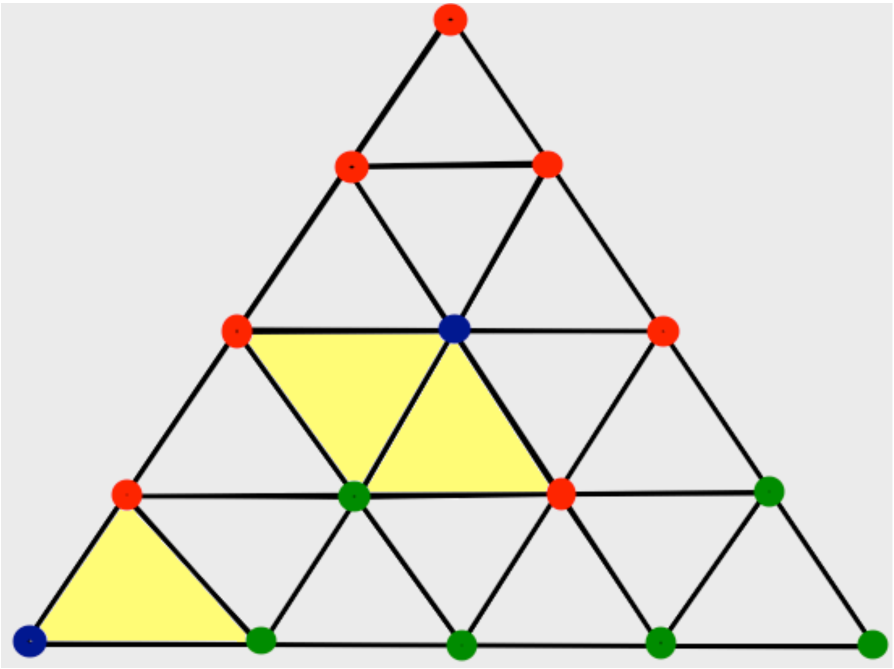

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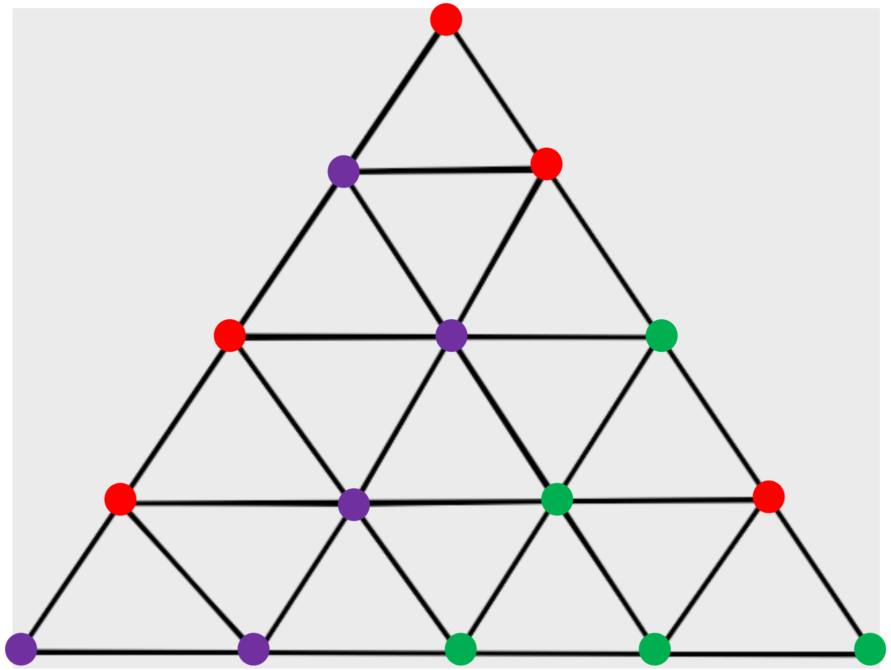

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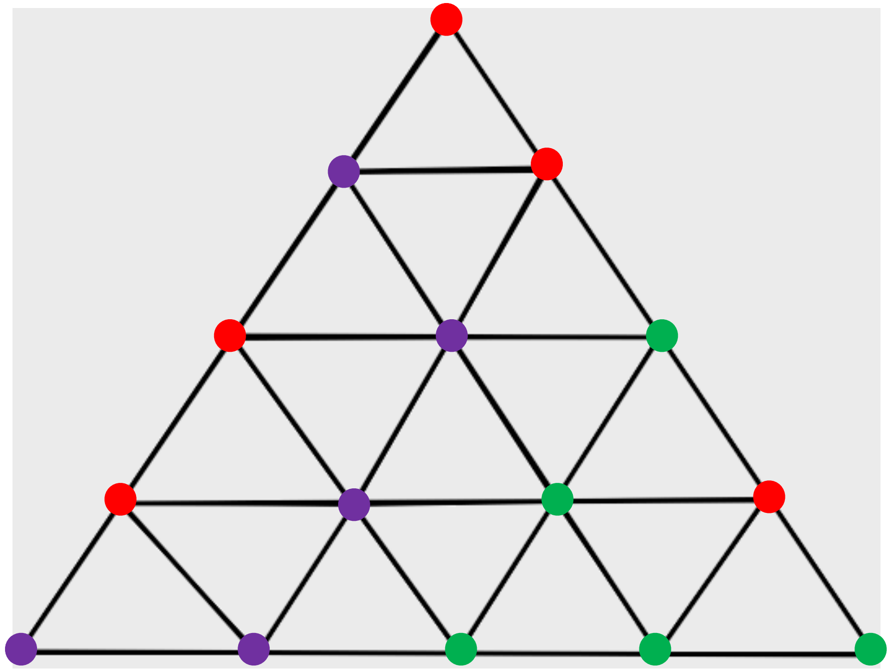
Sperner's Lemma

- Entire main triangle: HOUSE
- Baby triangles: ROOMS
-  : DOOR



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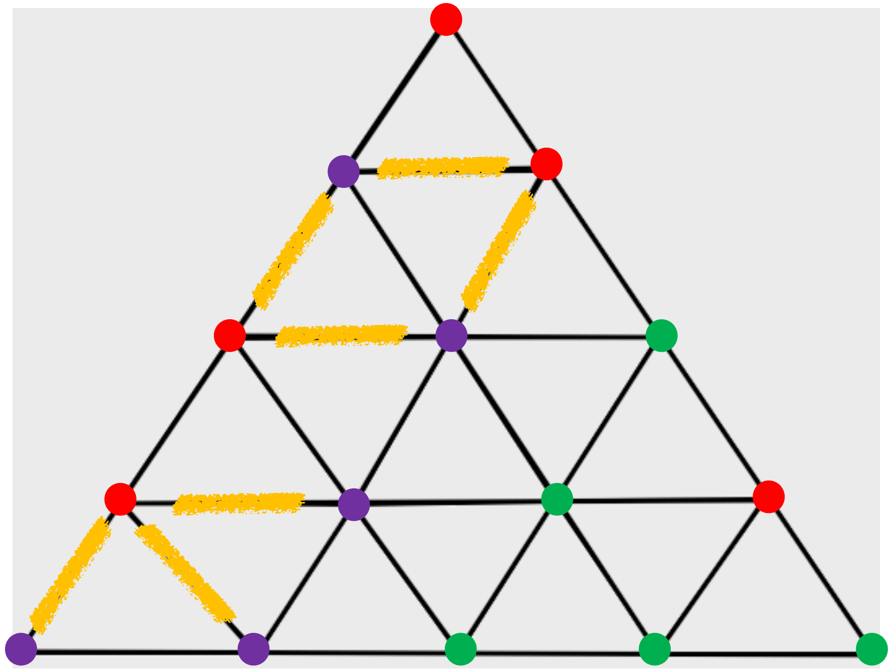

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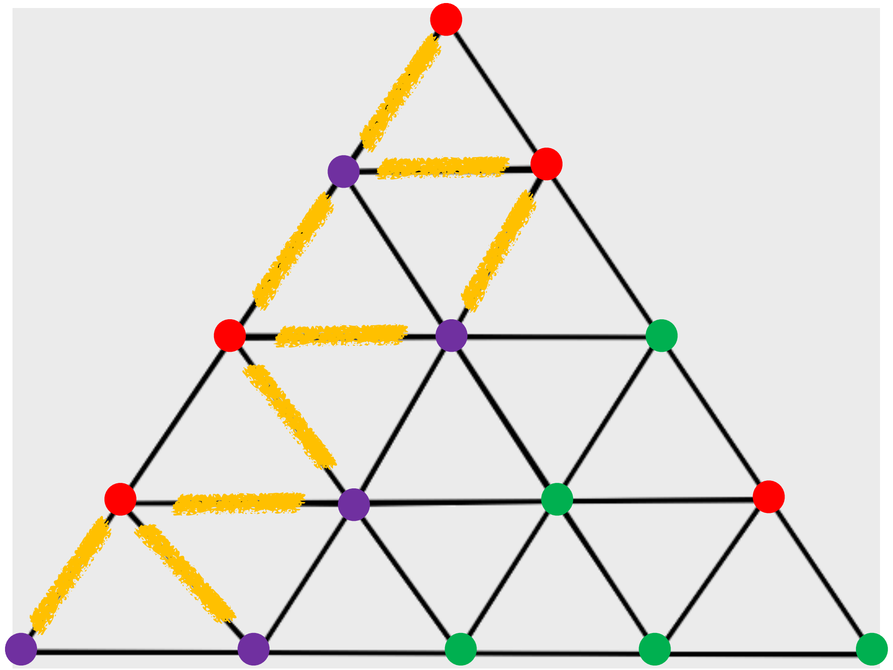

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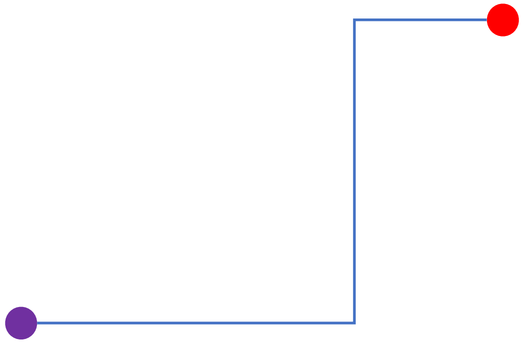
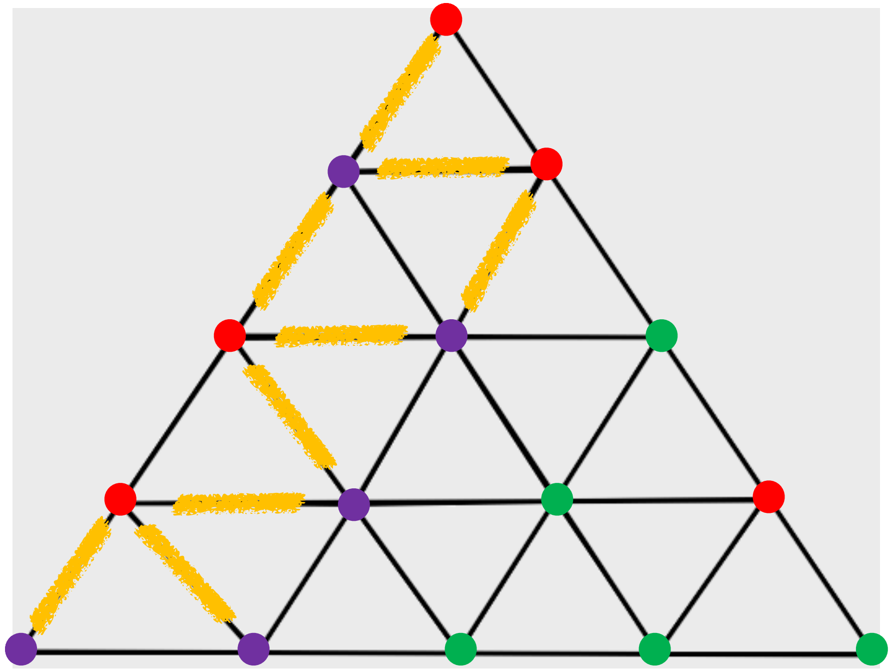
Observation 1:

Number of doors on the boundary is **ODD**





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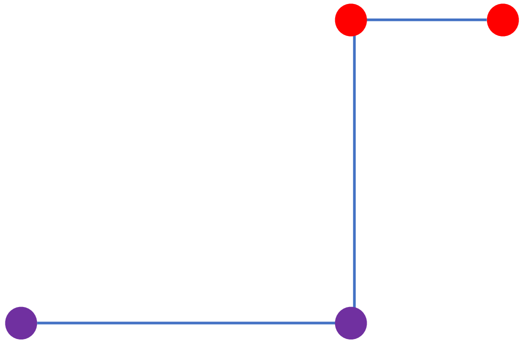

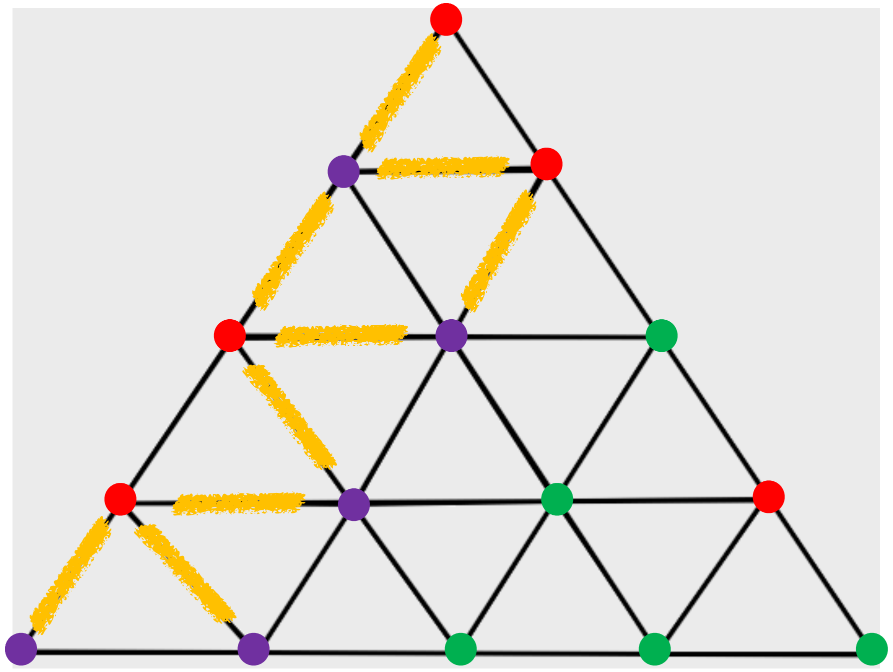

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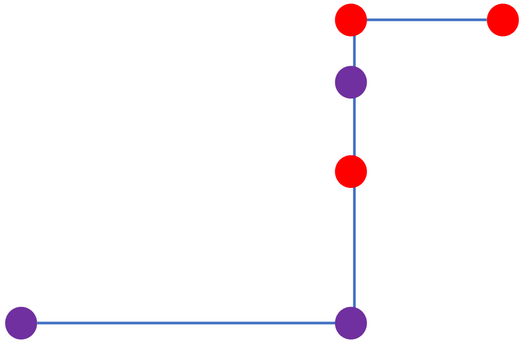

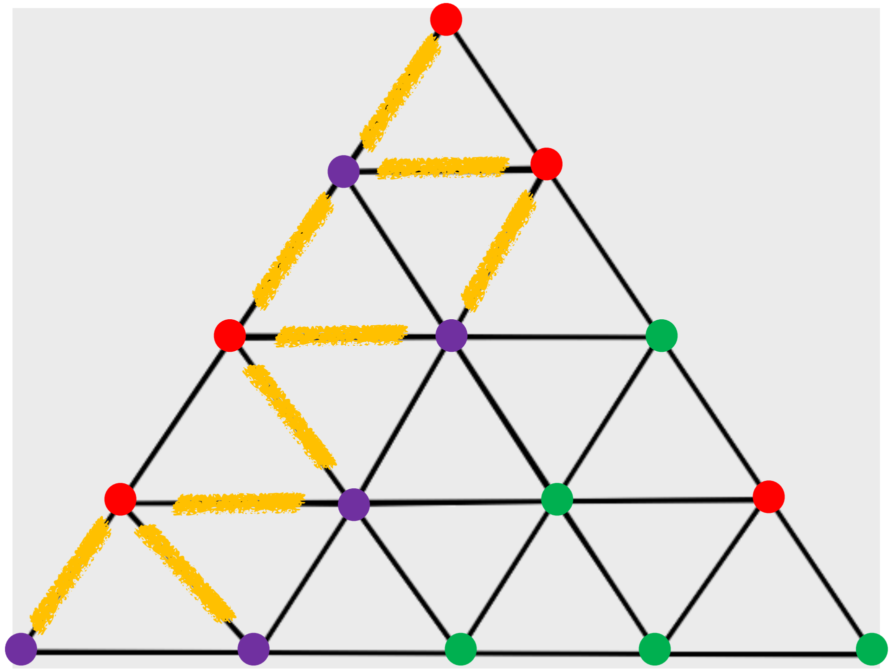

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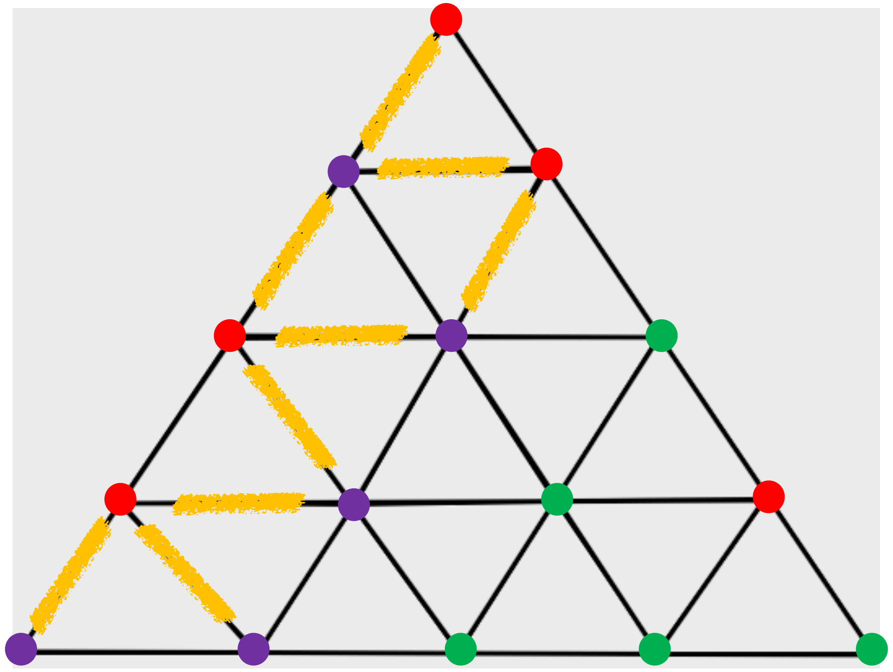
Sperner's Lemma

Observation 2:

A room can have 0, 1, or 2 doors 



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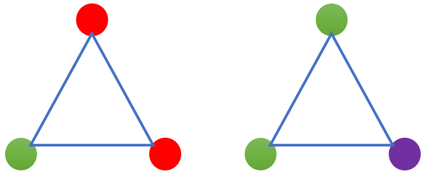
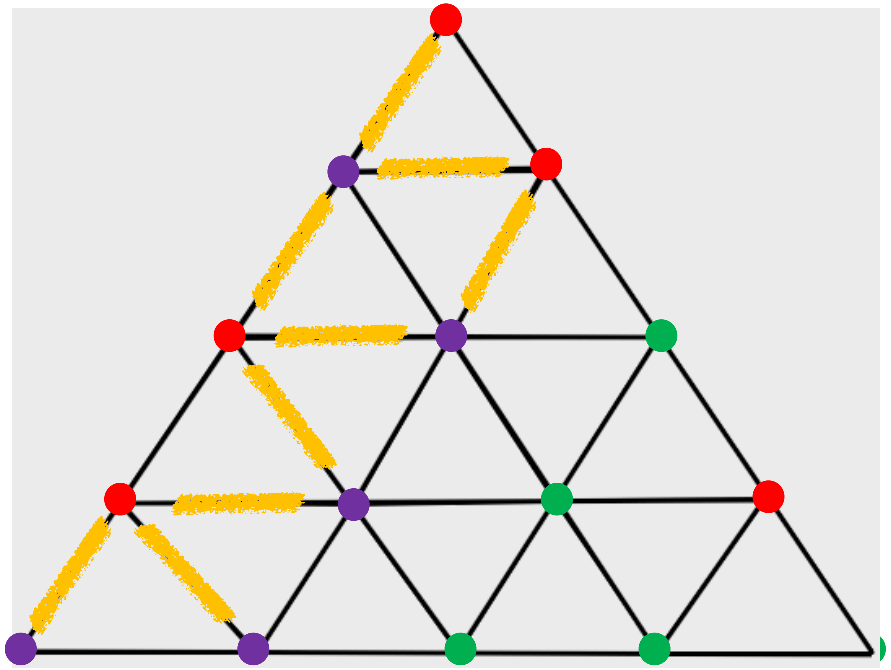


0 door



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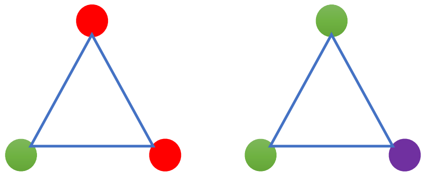

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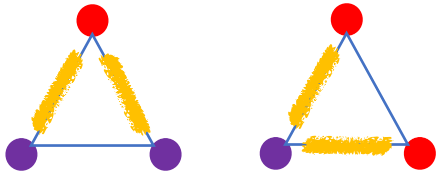
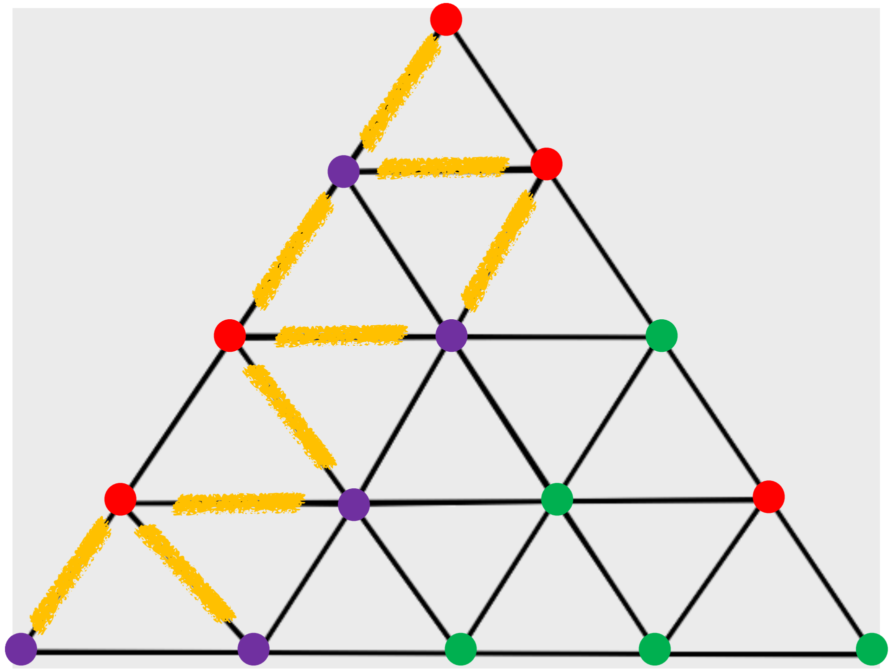


2 doors



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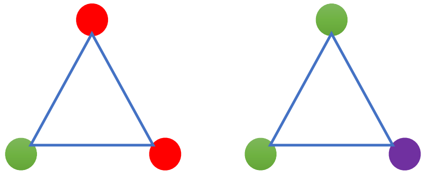

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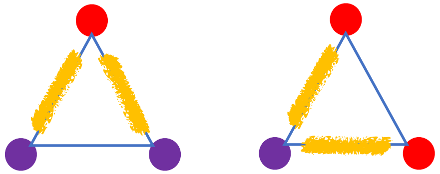

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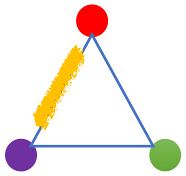
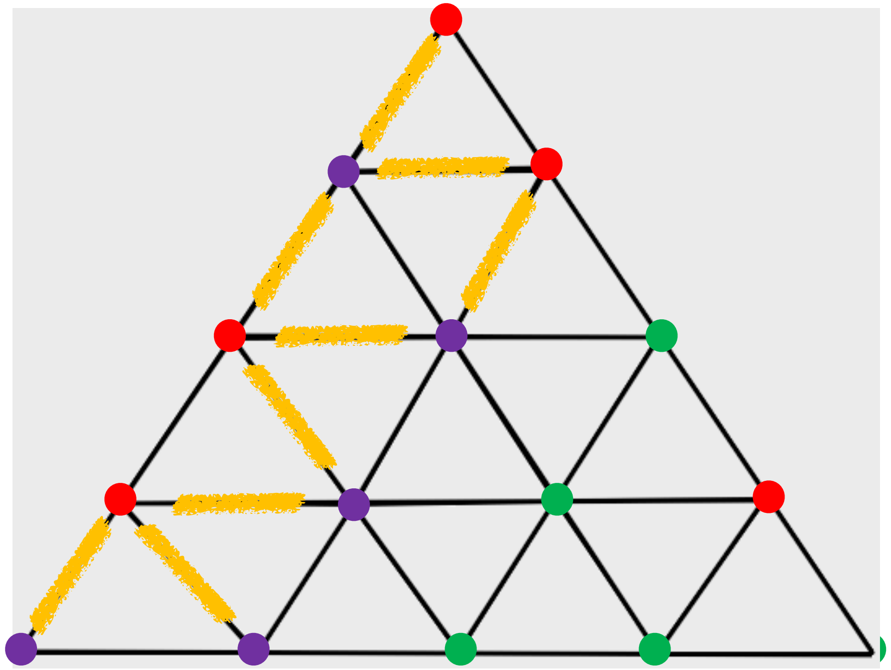


1 door



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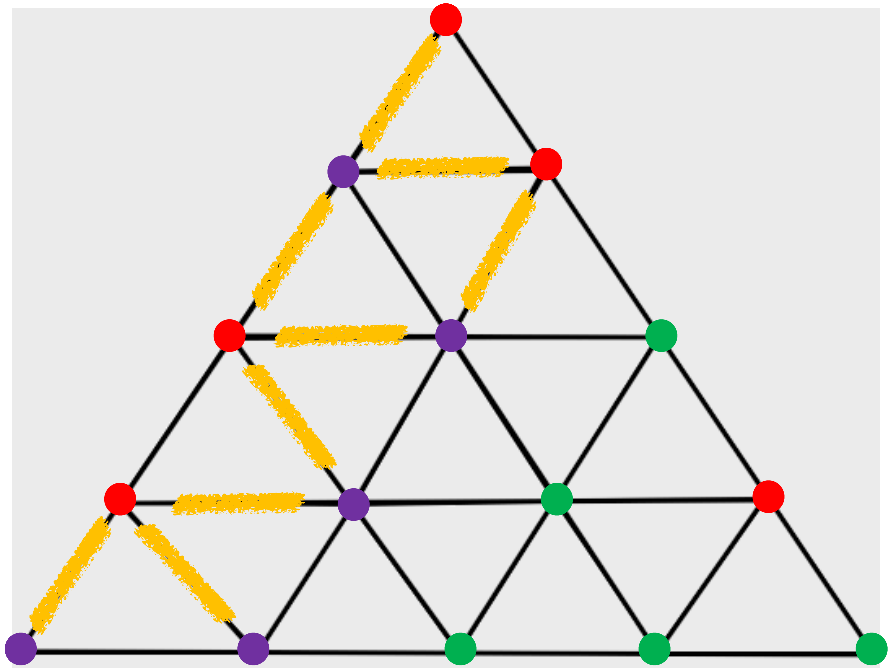
A room can have 0, 1, or 2 doors.

A room with 1 door is a fully colored baby triangle



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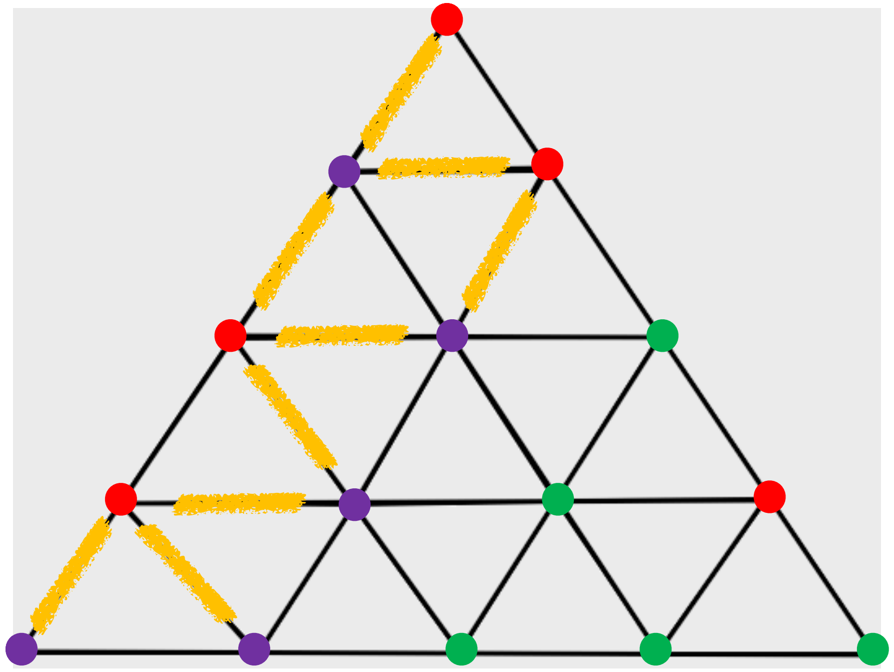
So, we enter the house through a door!



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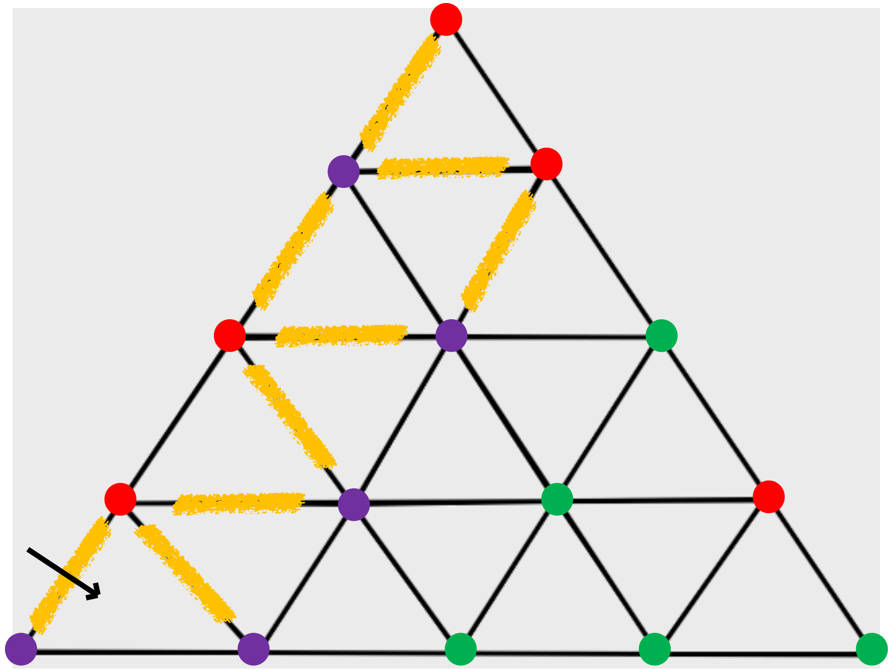
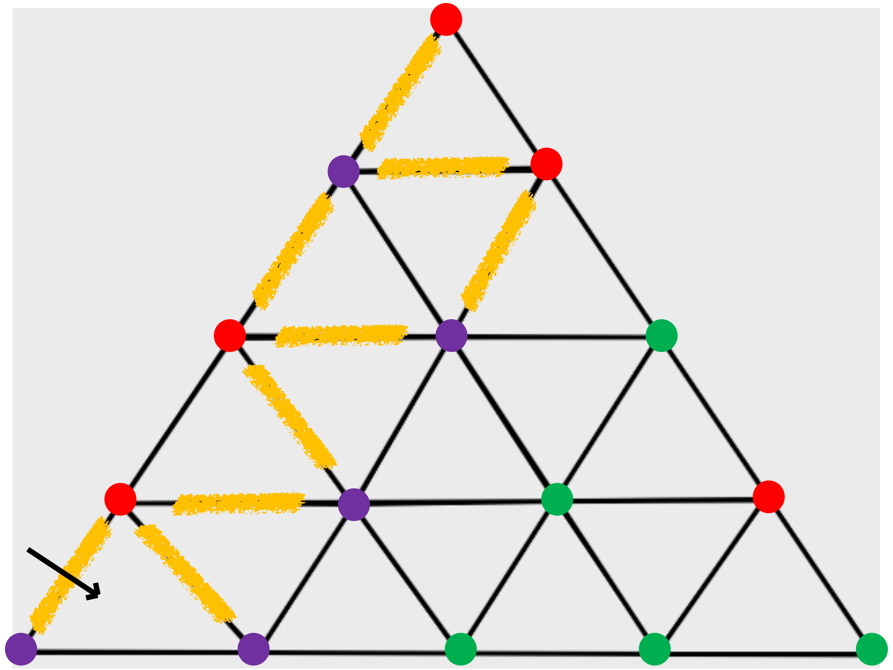


Any Sperner colored triangulation has at least **one fully colored baby triangle**

Sperner's Lemma

Enter the house through a door.

The room we entered can have either 1 or 2 doors



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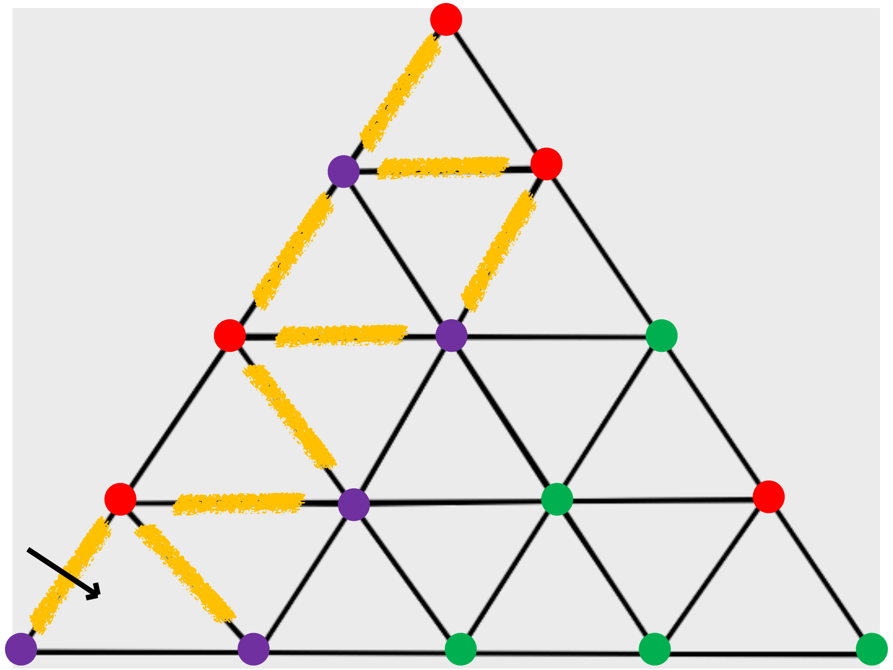

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- 2 doors: leave the room using the other door
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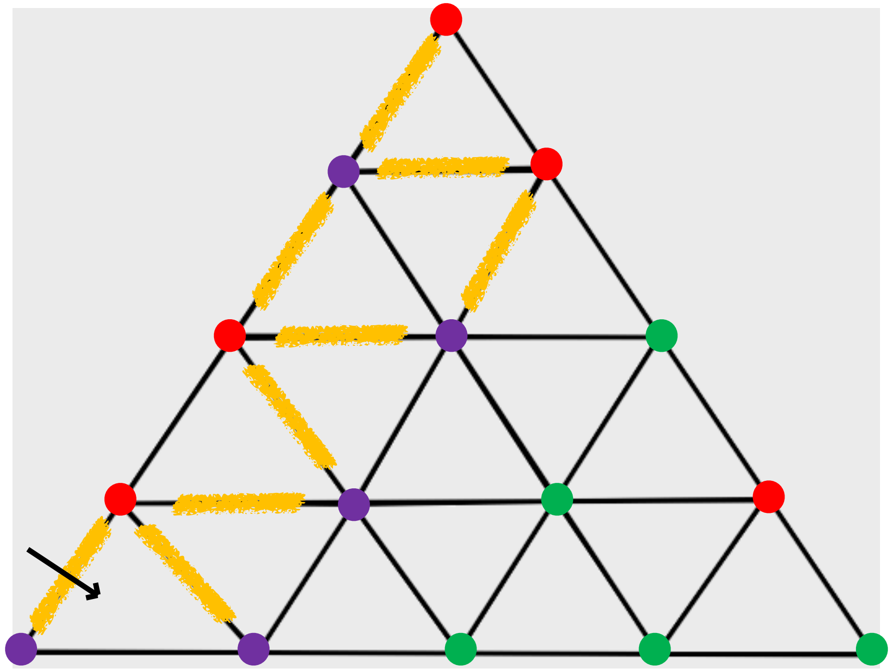

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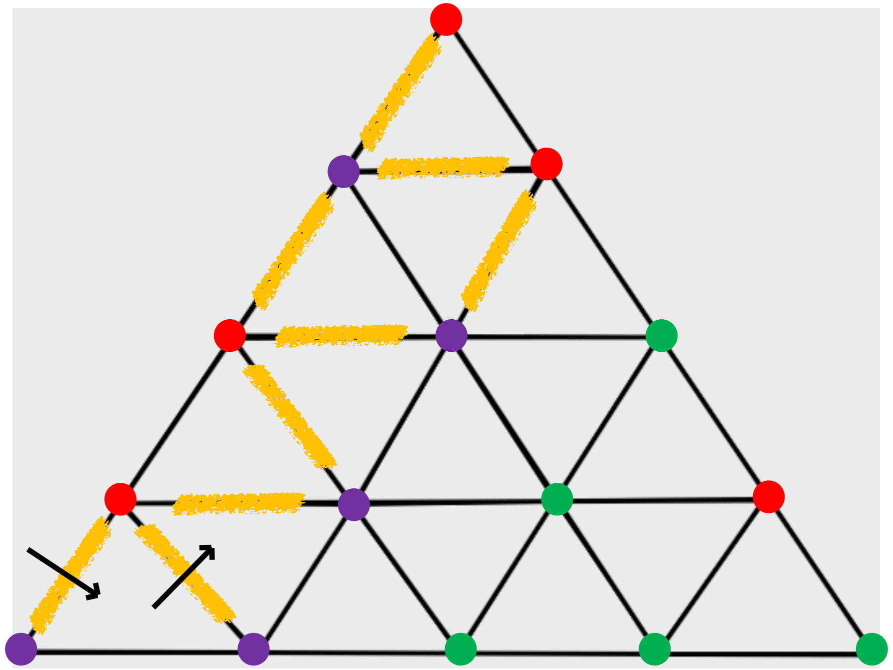

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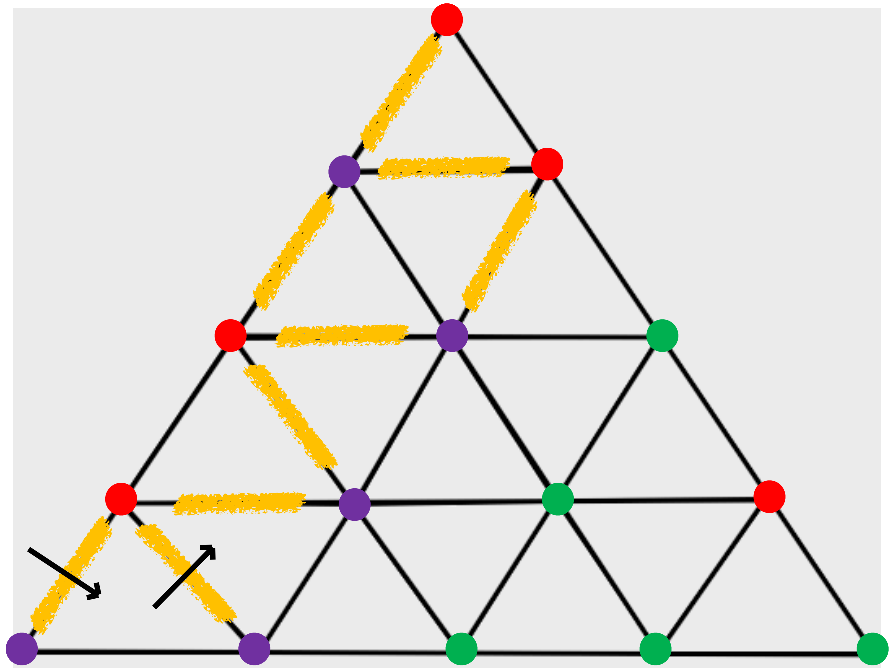

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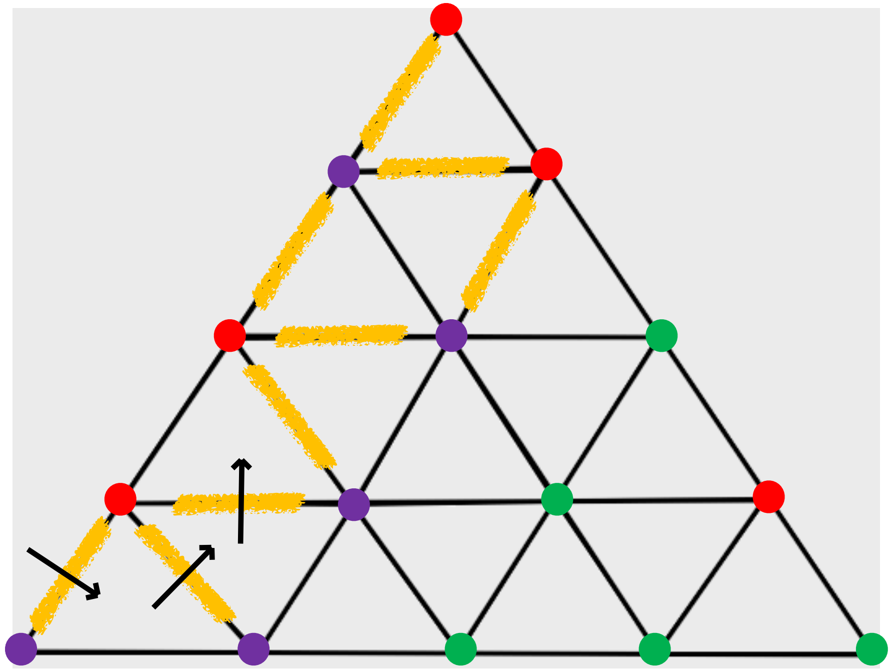

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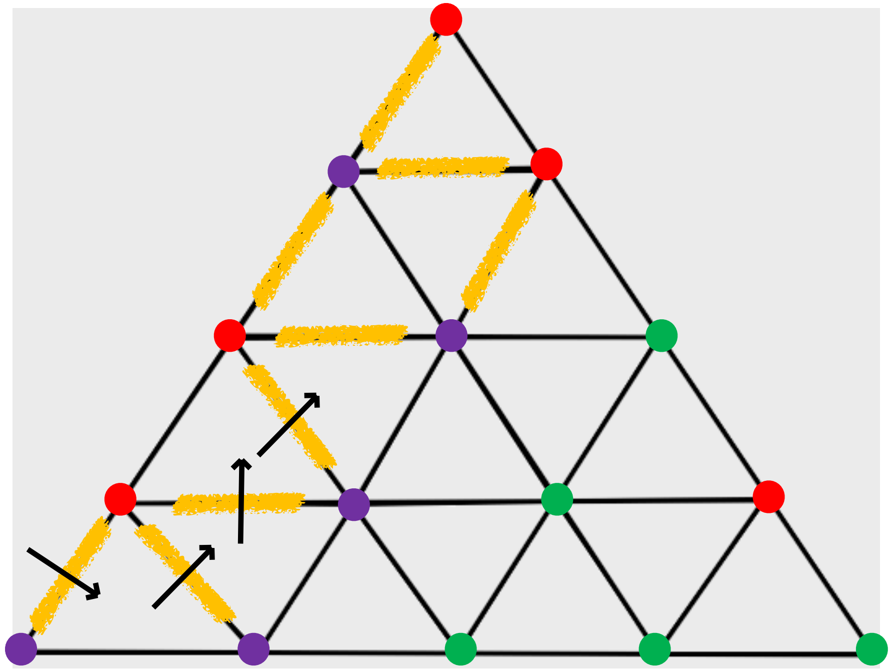


Any Sperner colored triangulation has at least **one fully colored baby triangle**

Sperner's Lemma

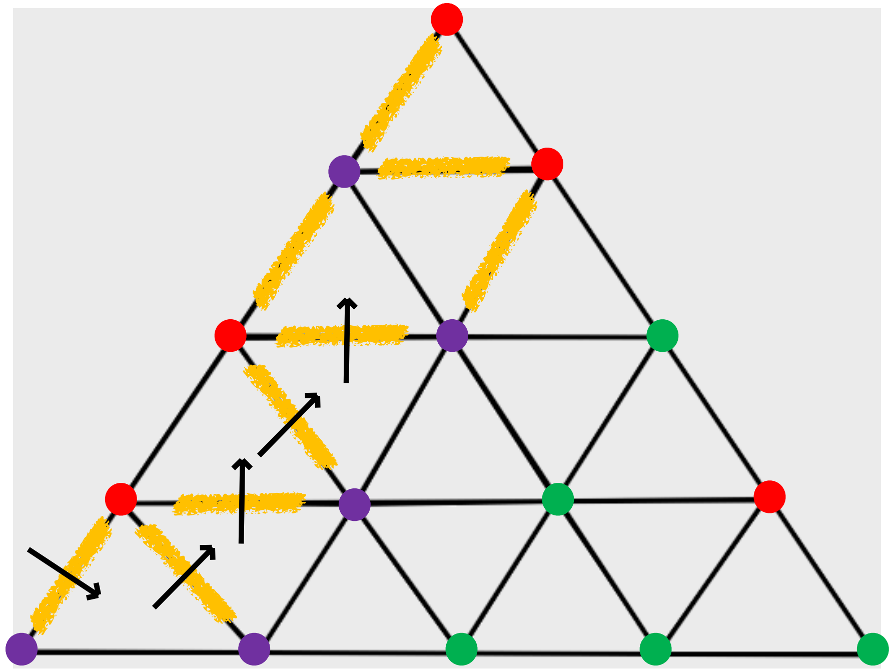
Enter the house through a door.

The room we entered can have either 1 or 2 doors

- 1 door : **sperner solution**
- 2 doors: leave the room using the other door
  and enter a new room

Keep walking!

- reach a fully colored baby triangle



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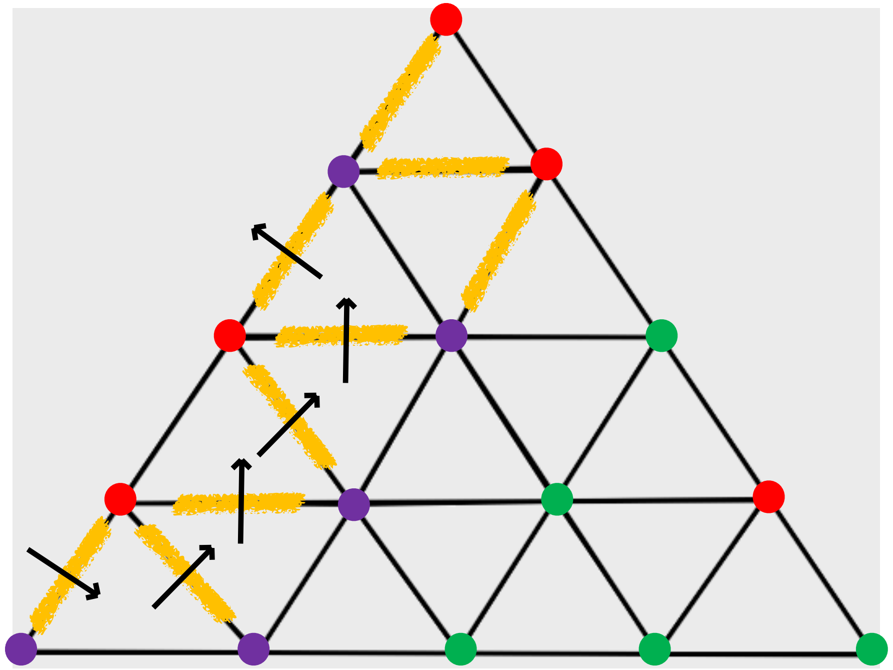
Keep walking!

- reach a fully colored baby triangle
- **thrown out of the house**



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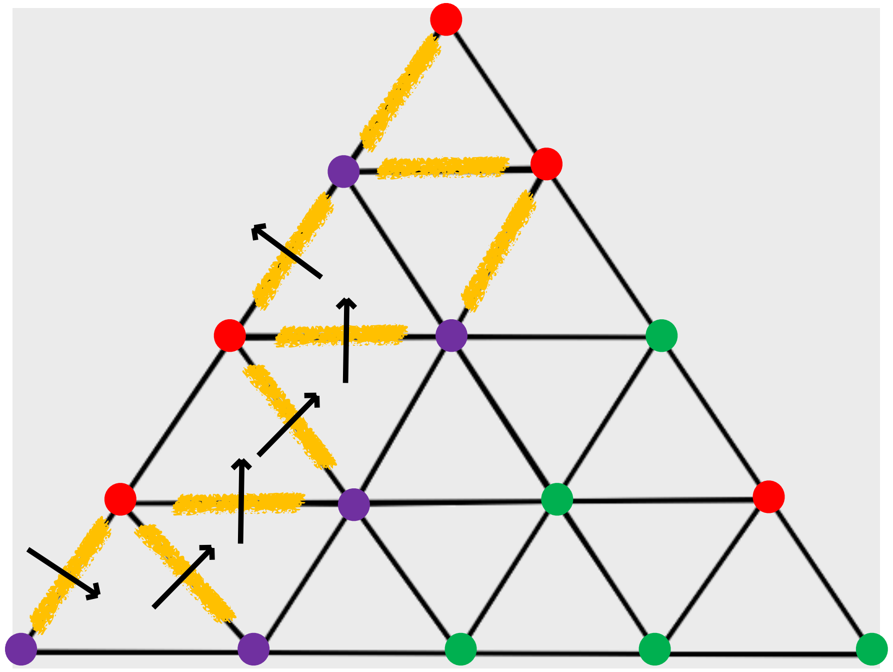
Sperner's Lemma

Thrown out?



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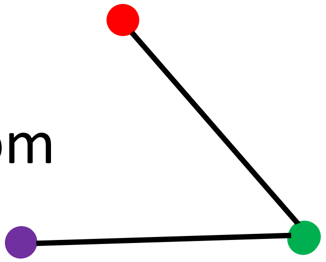
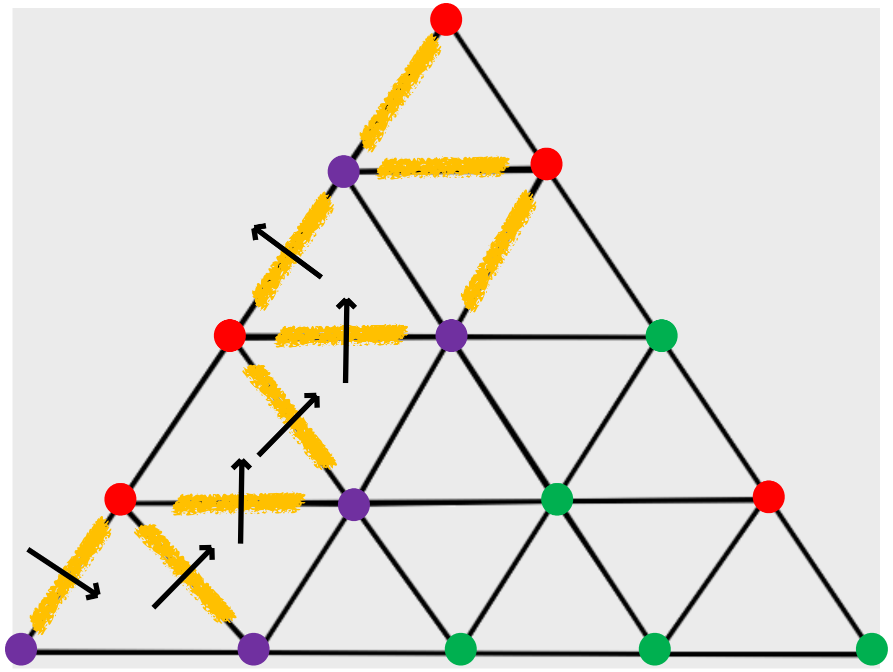
Sperner's Lemma

Thrown out?

- Cannot happen from (since no doors)





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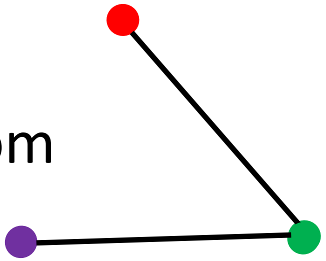
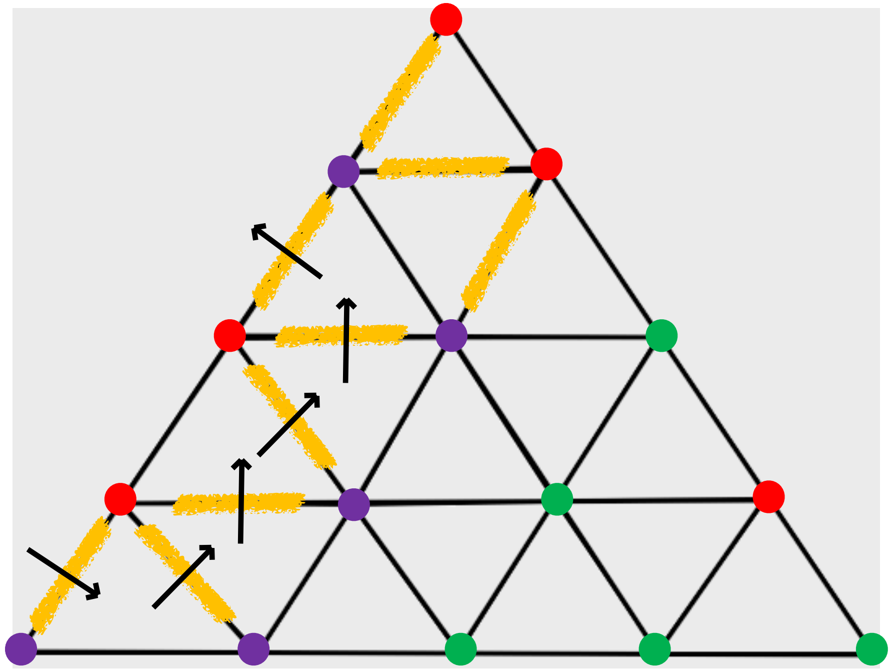
- Cannot happen from  (since no doors)

- Entry and exit doors are paired up



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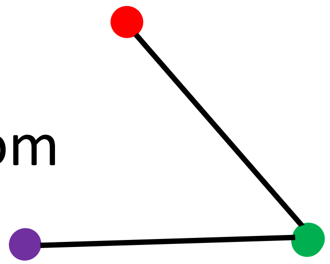
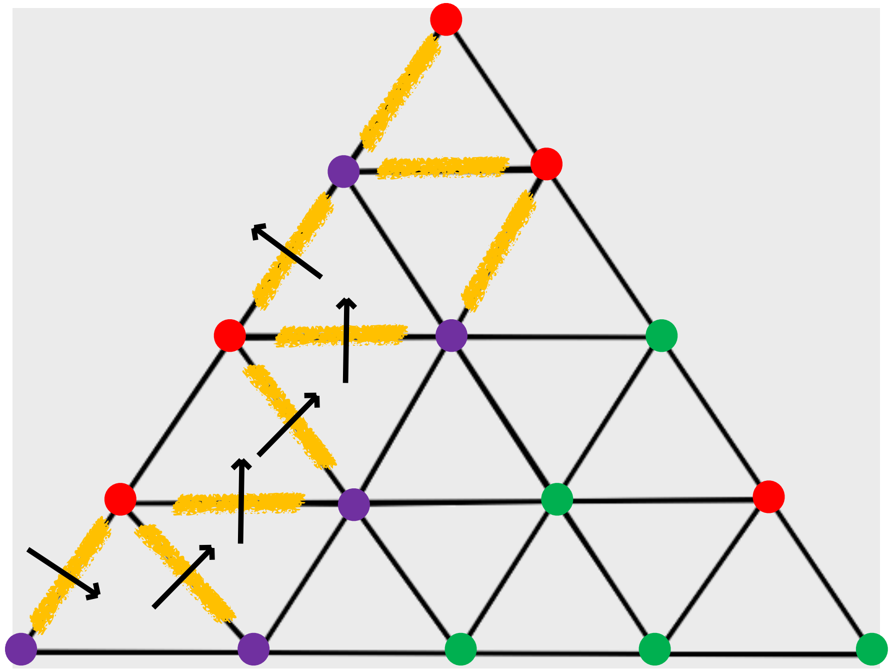


- Entry and exit doors are paired up

- There exists odd number of doors on the boundary.
  $\Longrightarrow$ we can enter again from another door!



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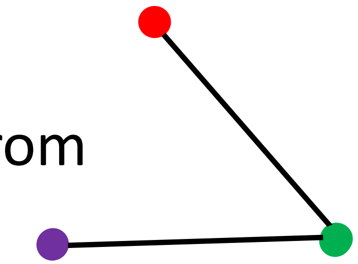
- Cannot happen from 
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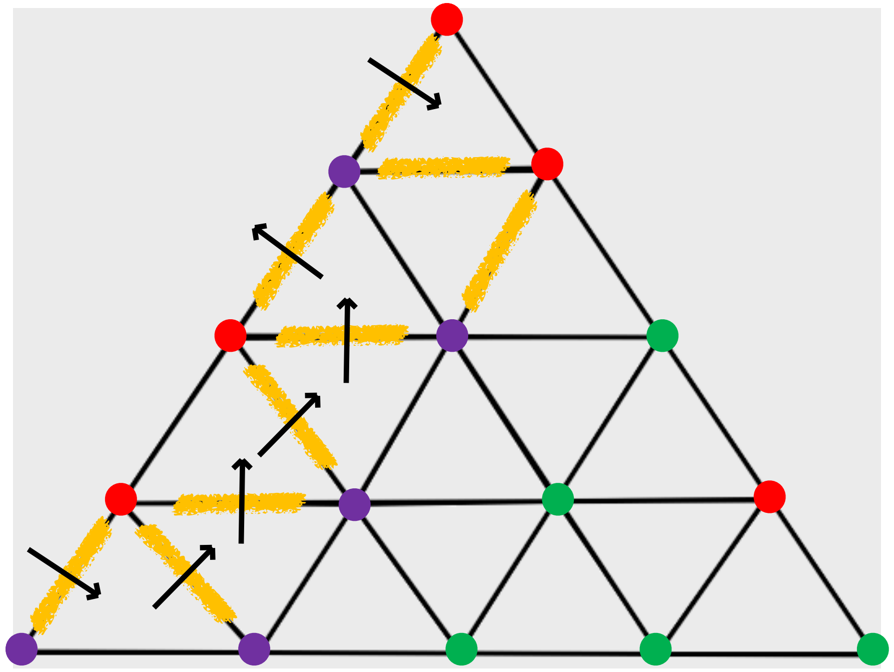

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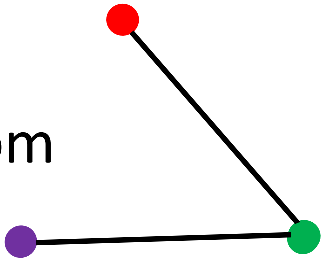

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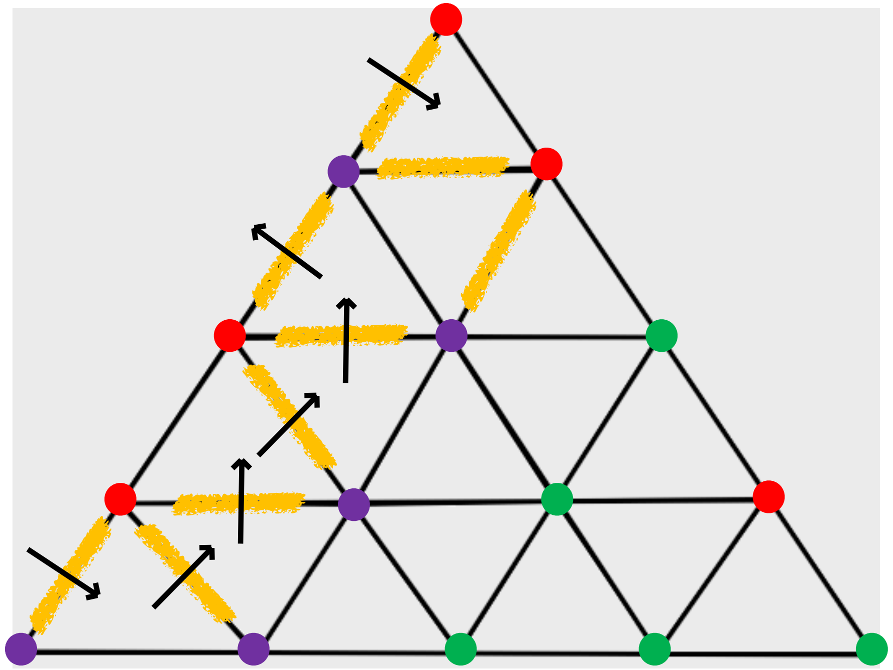

Keep walking!

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Sperner's Lemma

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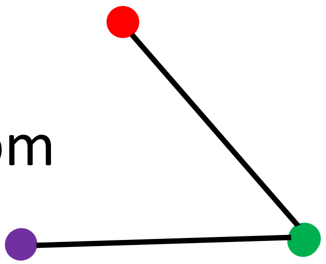

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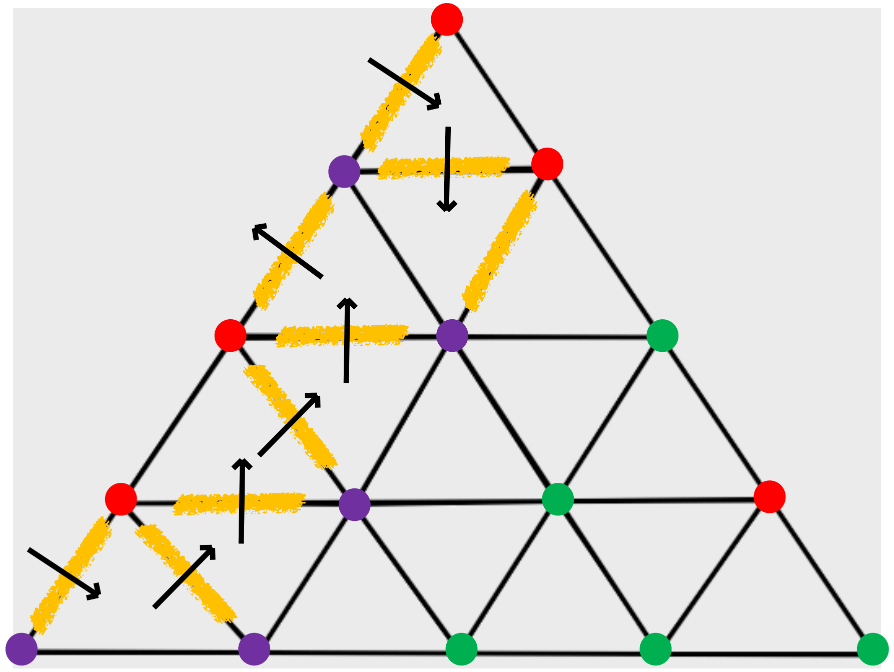

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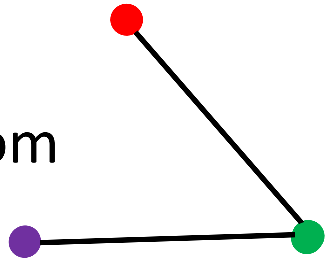

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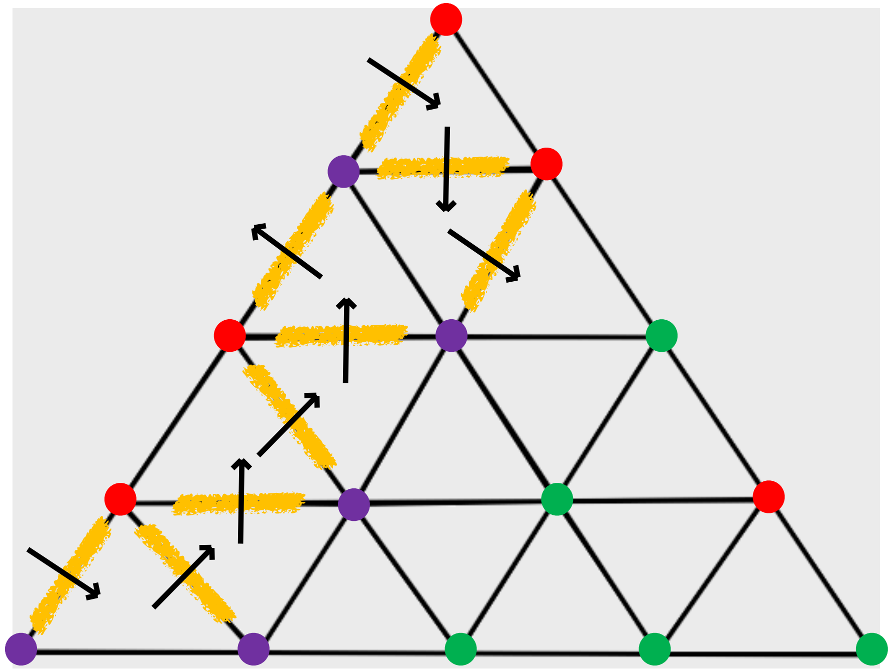

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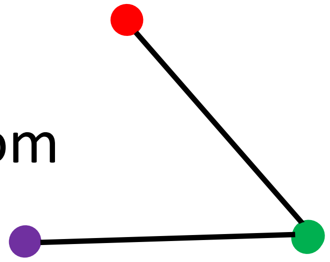

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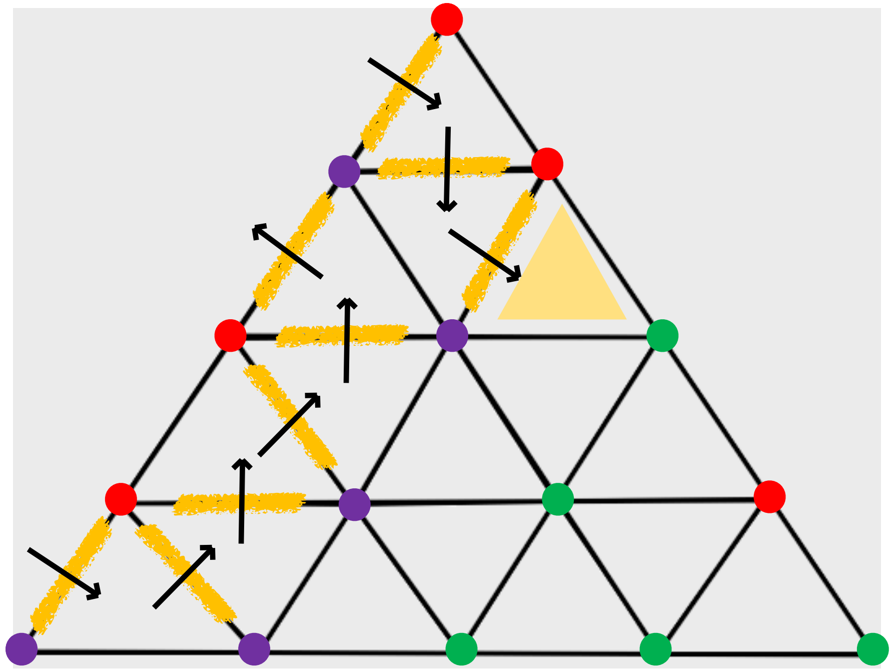

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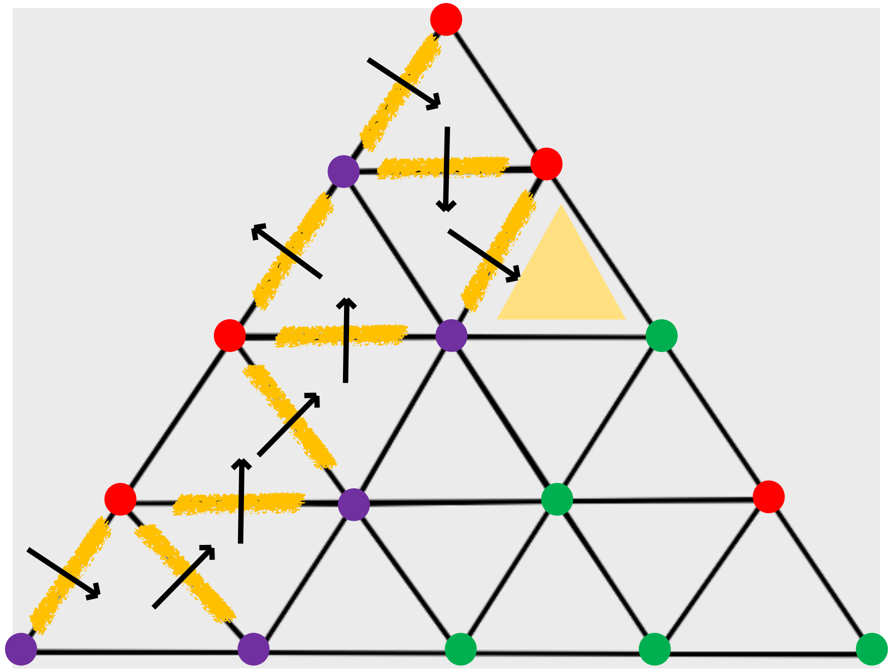
Sperner's Lemma



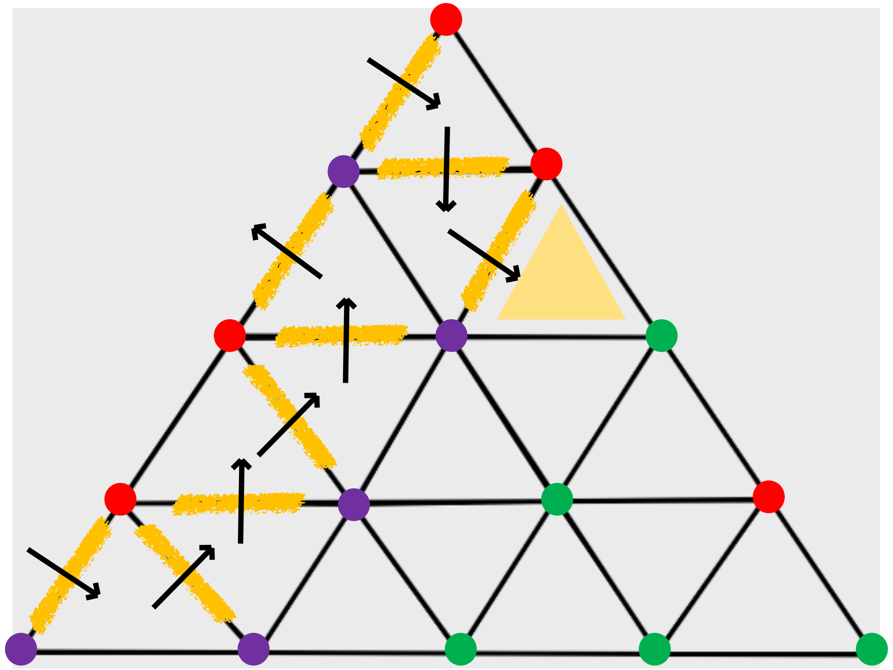
Think:
Why cannot such walks cycle back on themselves?

Any Sperner colored triangulation has at least **one fully colored baby triangle**

Sperner's Lemma

- The number of rooms = finite

  $\Longrightarrow$ the walk **terminates**



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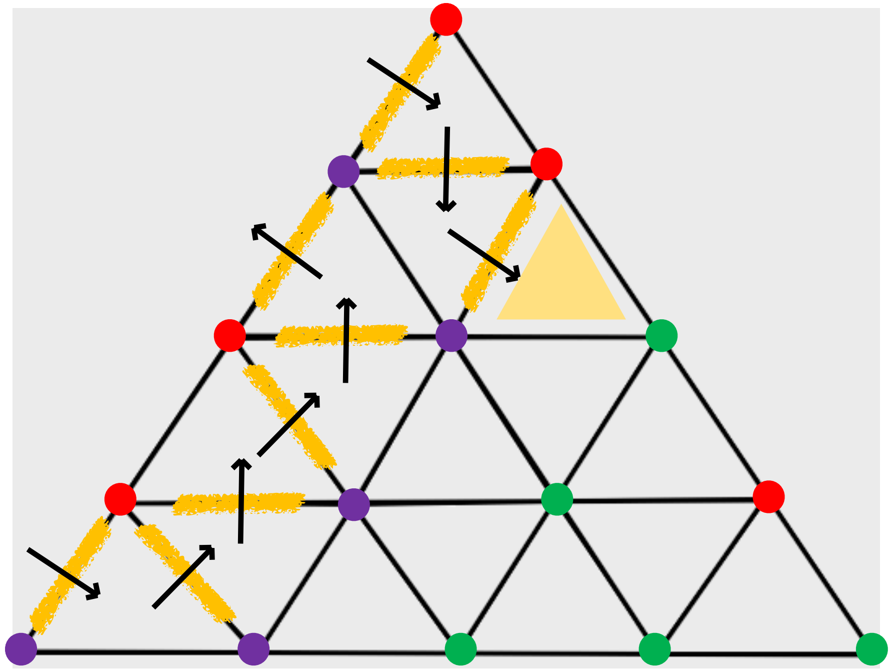
Sperner's Lemma

- The number of rooms = finite
  $\Longrightarrow$ the walk **terminates**

- ∃ at least one walk that will take us to
  a fully colored sperner solution



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(odd number)

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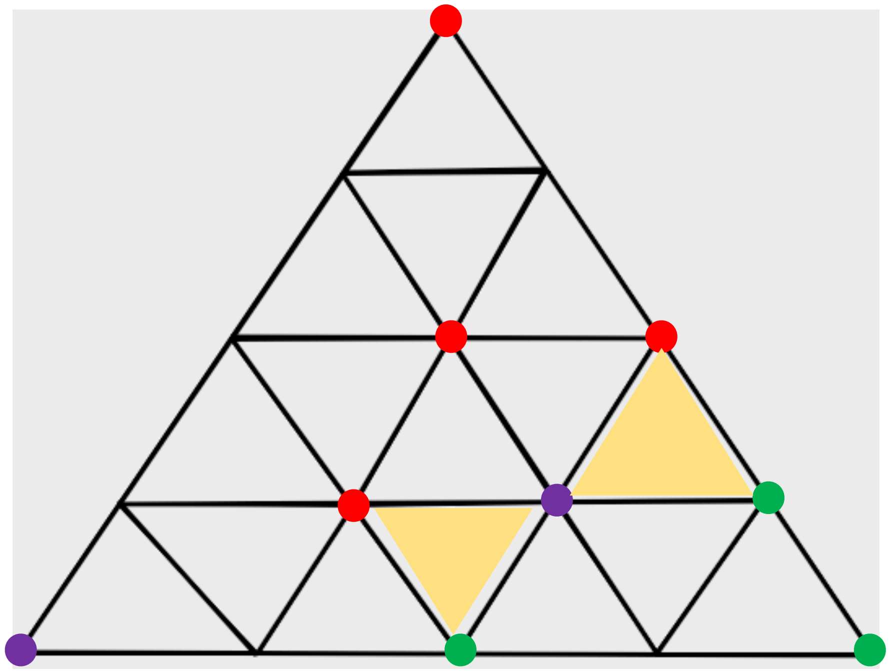

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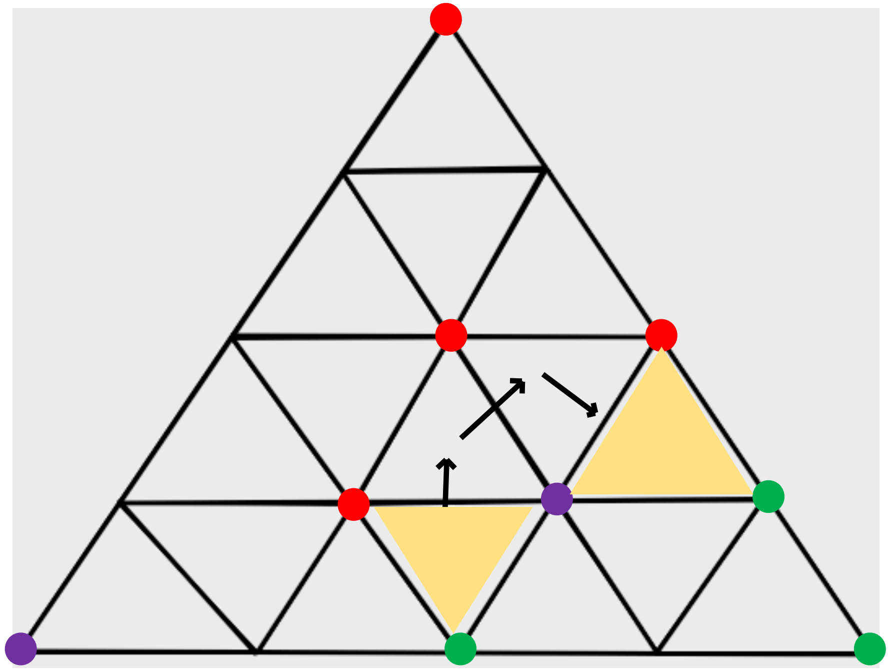

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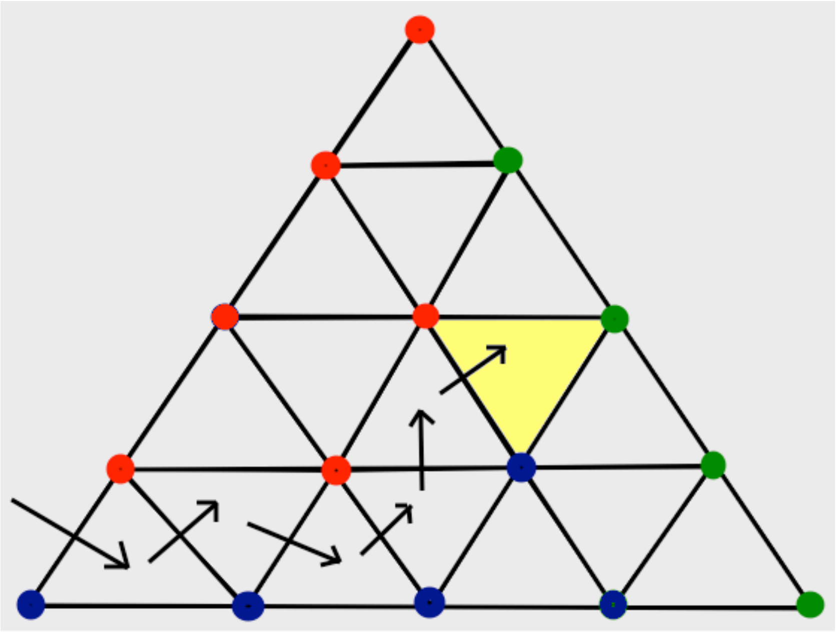

Holds true for any dimension

Any Sperner colored triangulation has at least **one fully colored baby triangle**

Cake division using Sperner's Lemma

Forest Simmons, popularized by Francis Su [1999]

Cake division using Sperner's Lemma

- The resource: **cake [0,1]** and n **agents**
- An allocation $(X_1, \ldots, X_n)$ is *envy-free* if $v_i(X_i) \geq v_i(X_j)$ for all i, j

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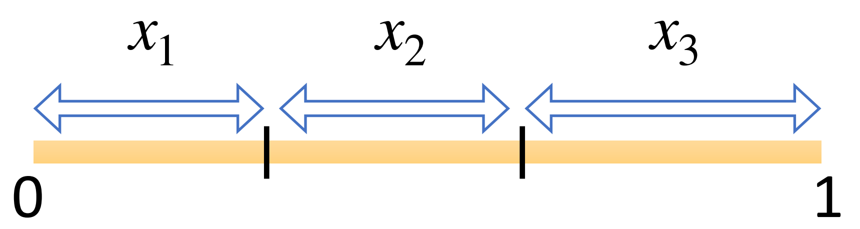

(x_1, x_2, x_3) : a cut

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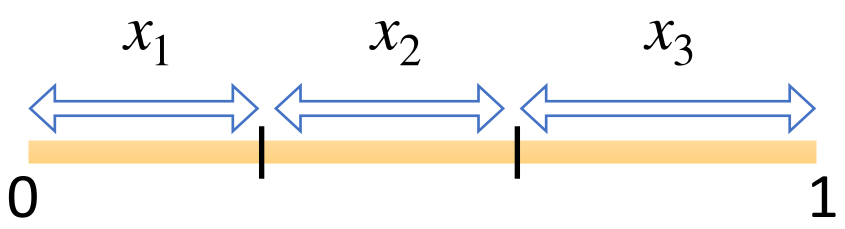

$x_1 + x_2 + x_3 = 1$ and all $x_i \geq 0$

(x_1, x_2, x_3) : a cut

Space of all possible cuts

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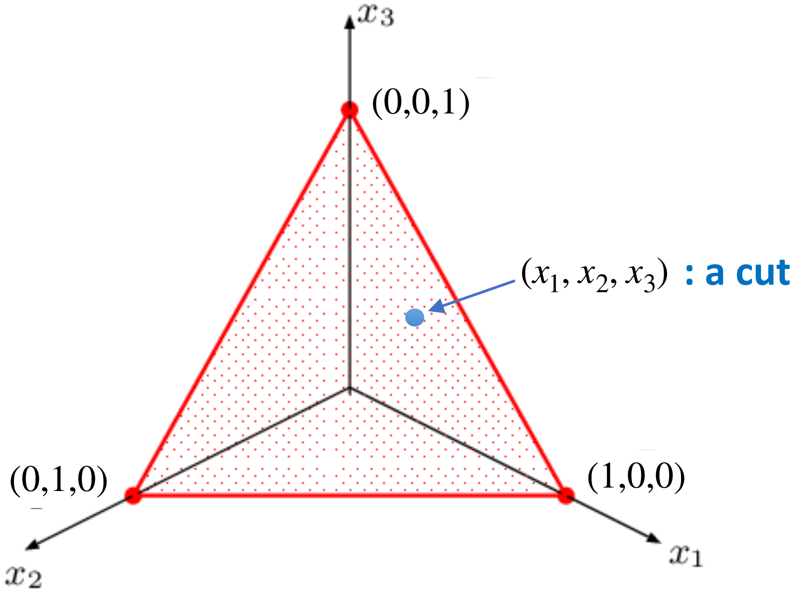
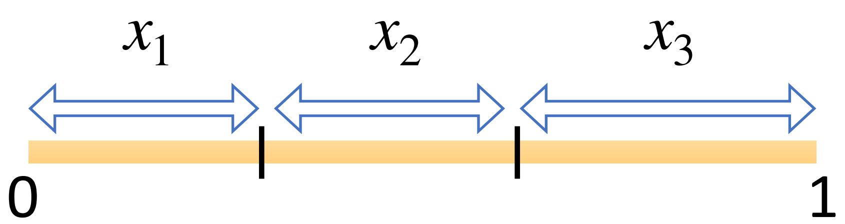


(x_1, x_2, x_3) : a cut



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(2-simplex)

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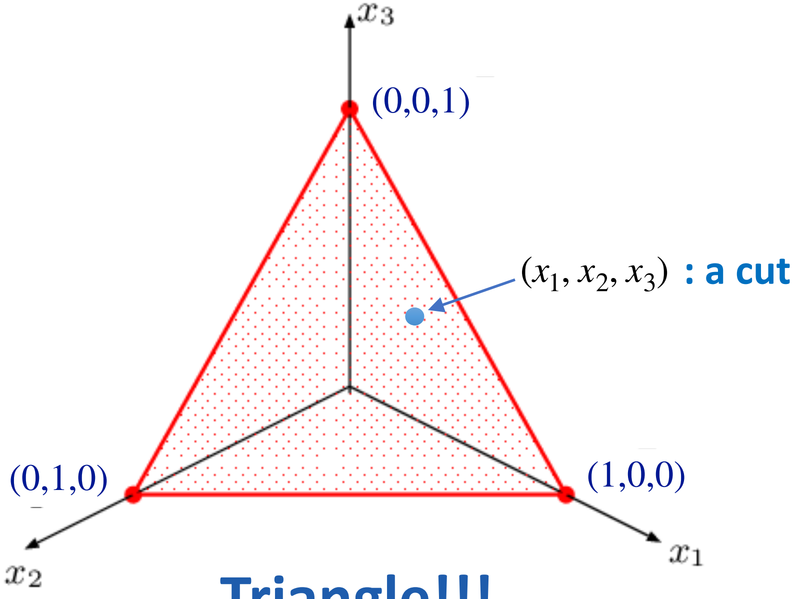
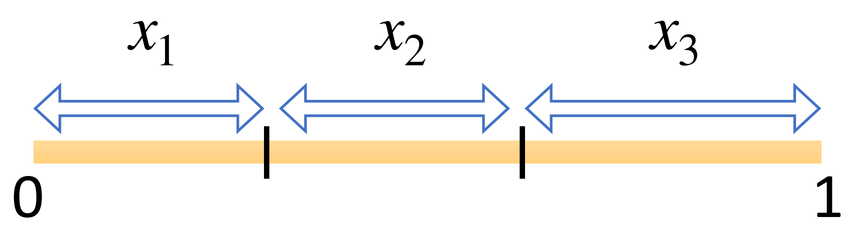


Triangle!!!



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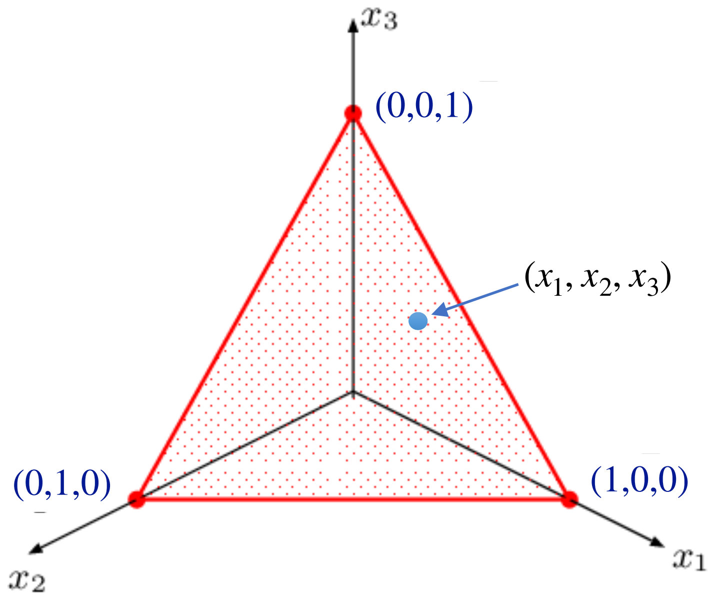
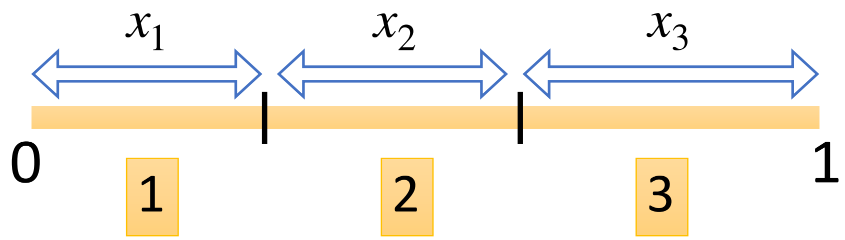
Space of all possible cuts

Cake division using Sperner's Lemma

<u>Assumptions on preferences/valuations</u>

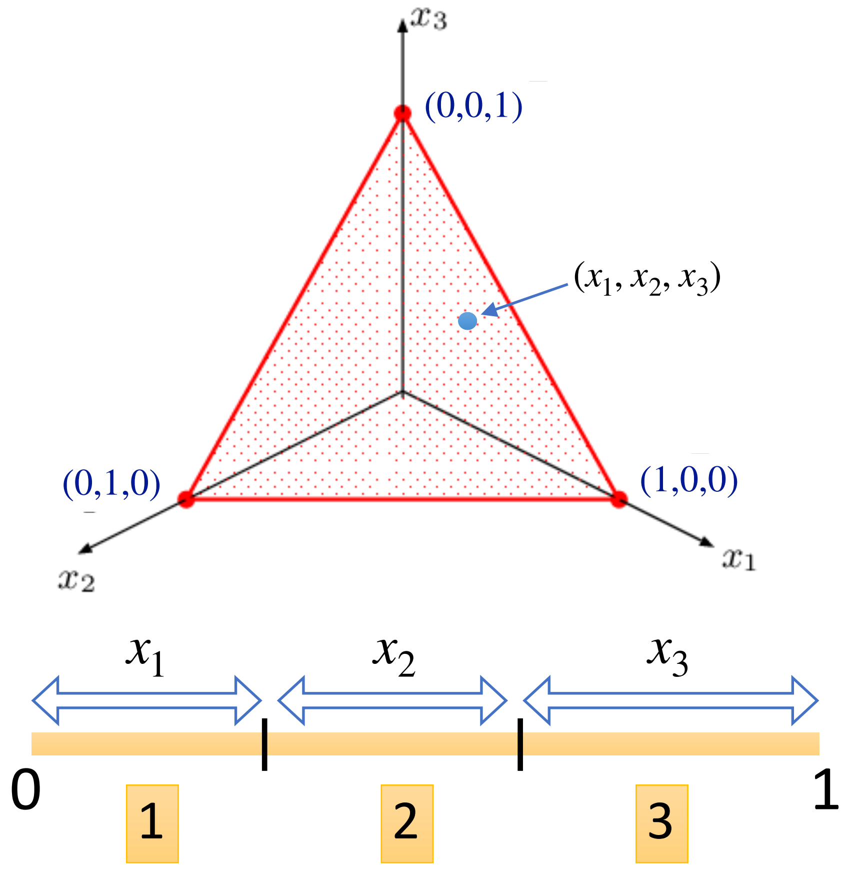




Cake division using Sperner's Lemma

<u>Assumptions on preferences/valuations</u>

- Given any cut (x_1, x_2, x_3), each agent can point to its favorite piece



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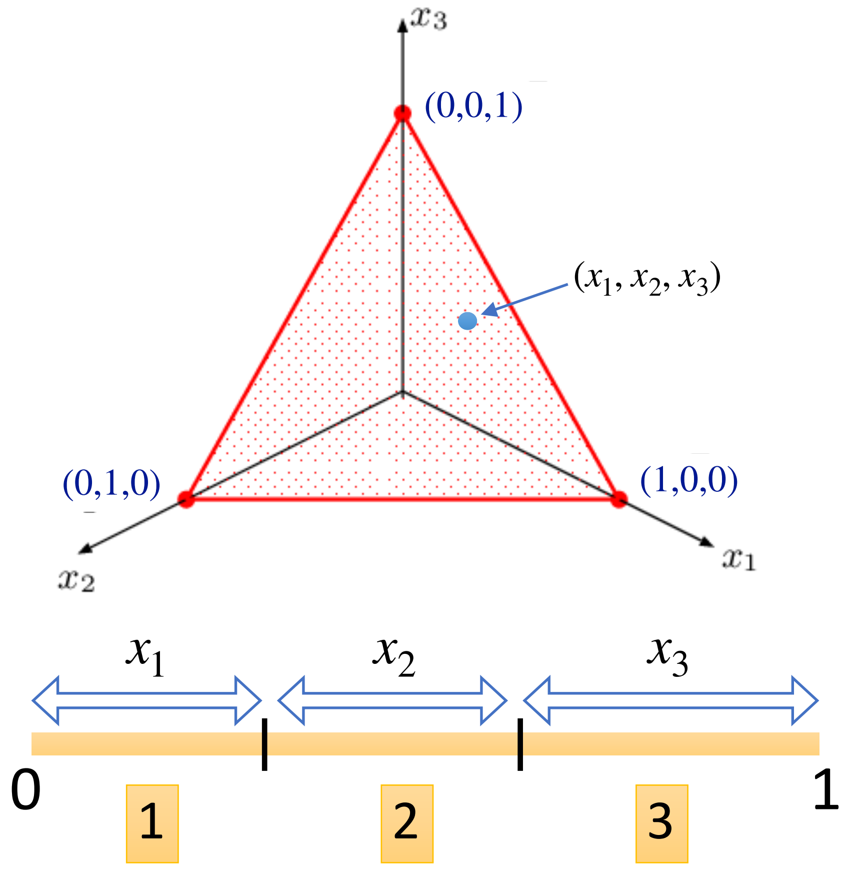

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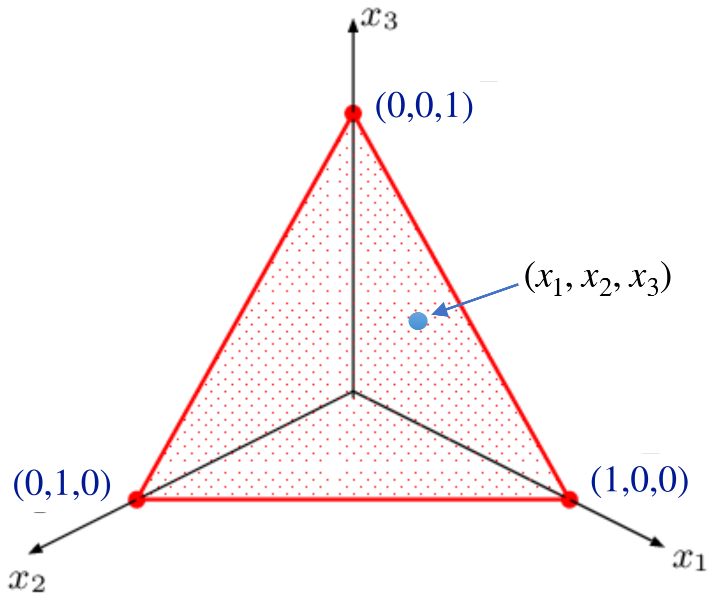
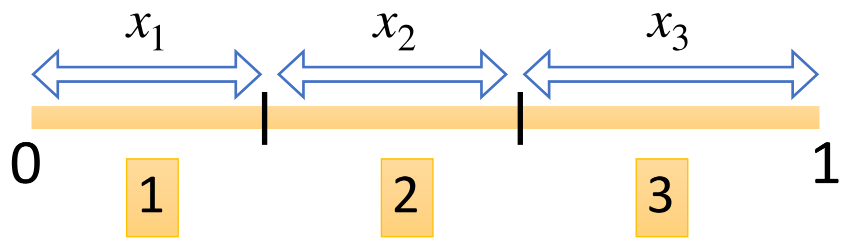
Goal: to invoke Sperner's lemma somehow





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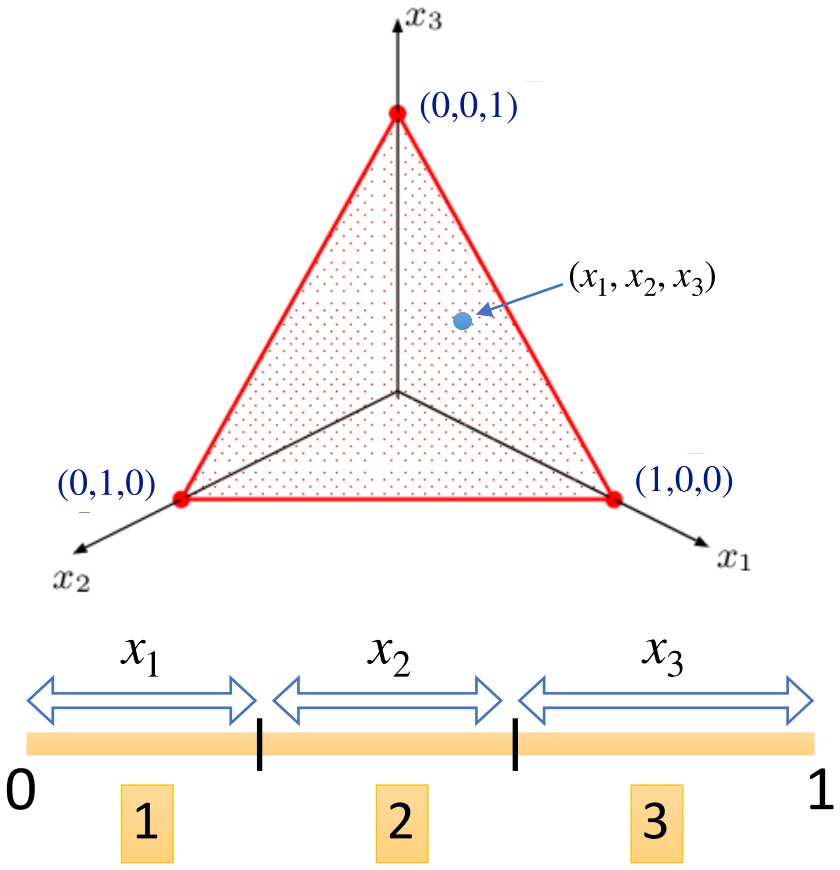
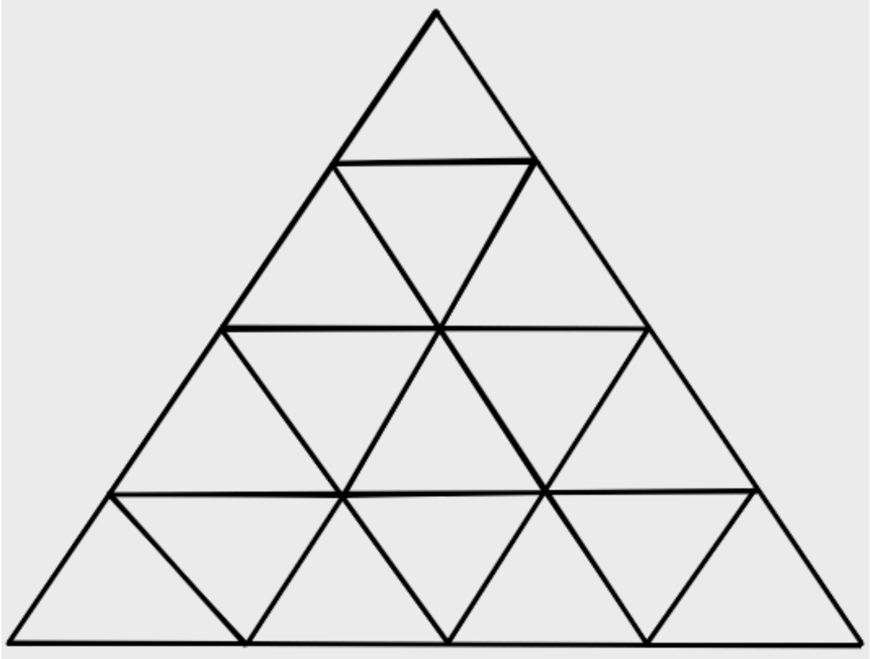
Set of agents: $\{A, B, C\}$



Cake division using Sperner’s Lemma

Cake division using Sperner's Lemma

(0,0,1)

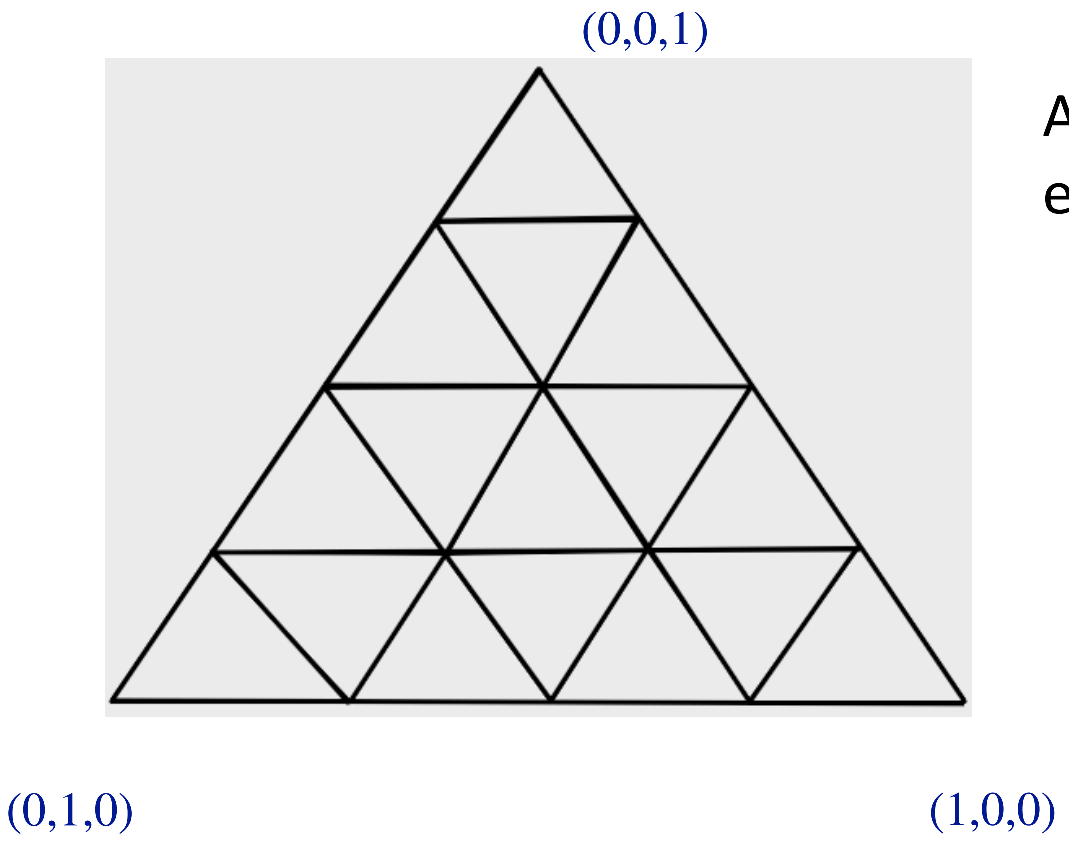


(0,1,0)

(1,0,0)

<u>Ownership labeling</u>

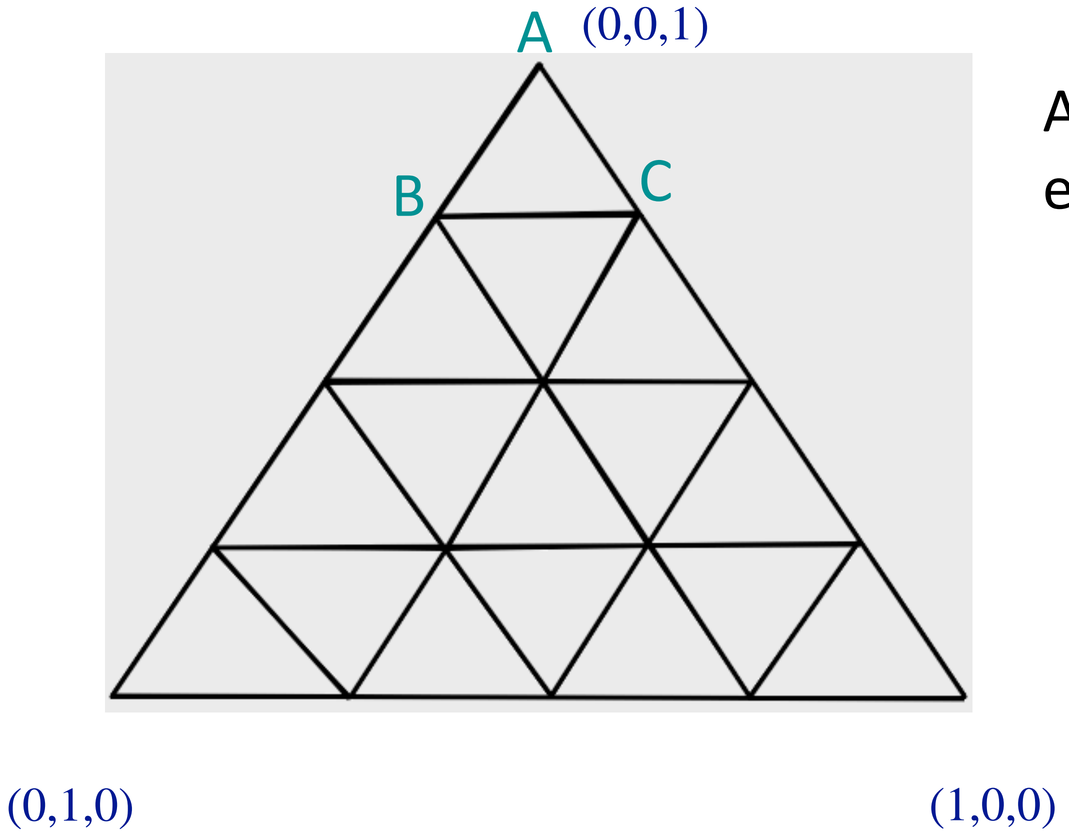
Cake division using Sperner's Lemma



Assign ownerships to each vertex such that
each baby triangle consists of all three owners {A, B, C}.

<u>Ownership labeling</u>

Cake division using Sperner's Lemma



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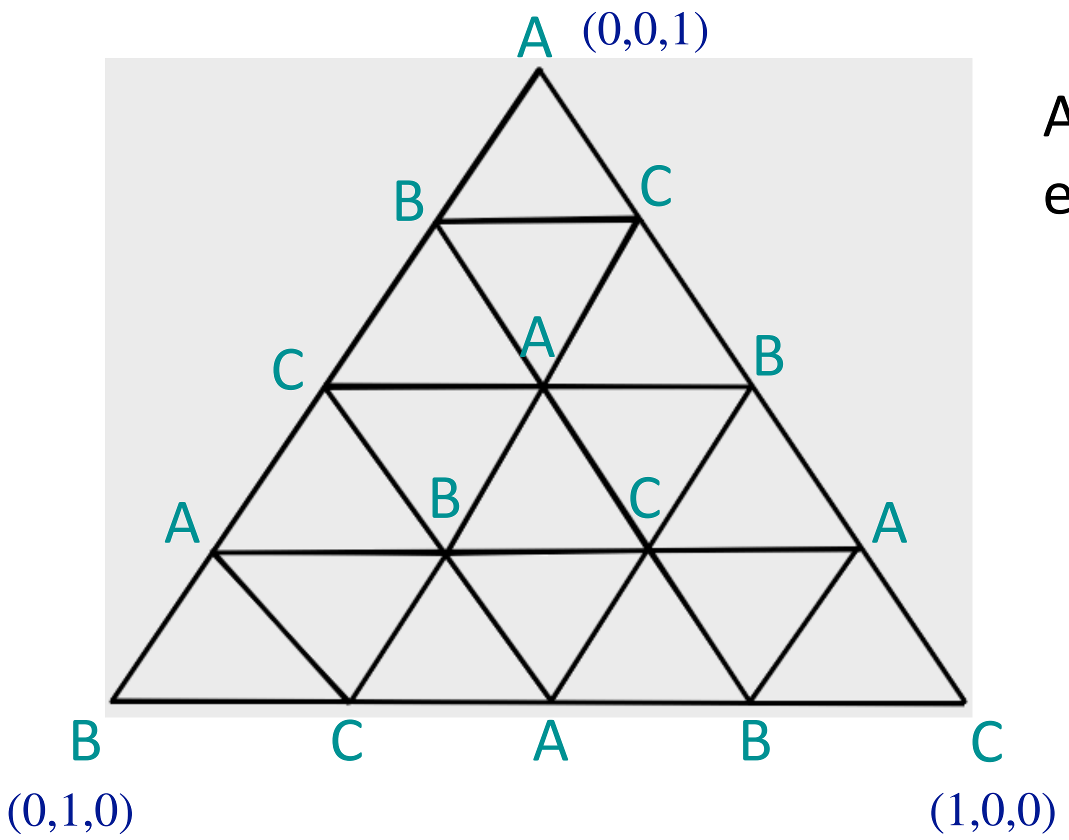

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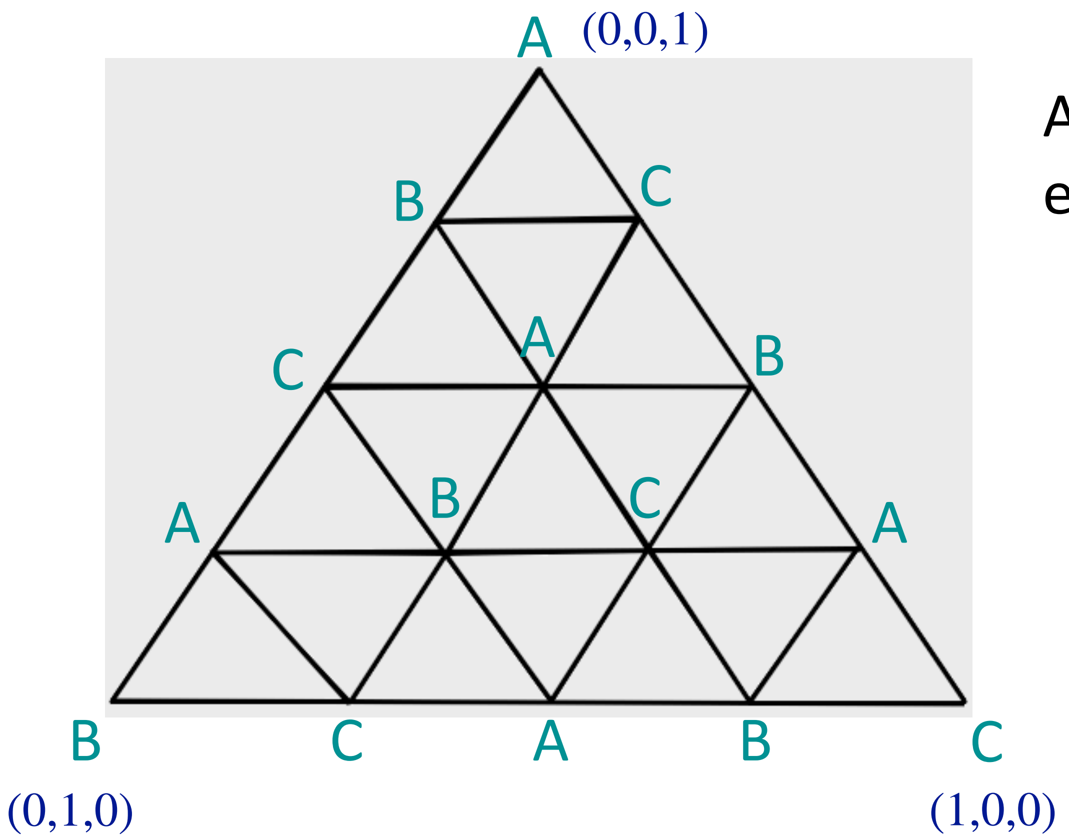

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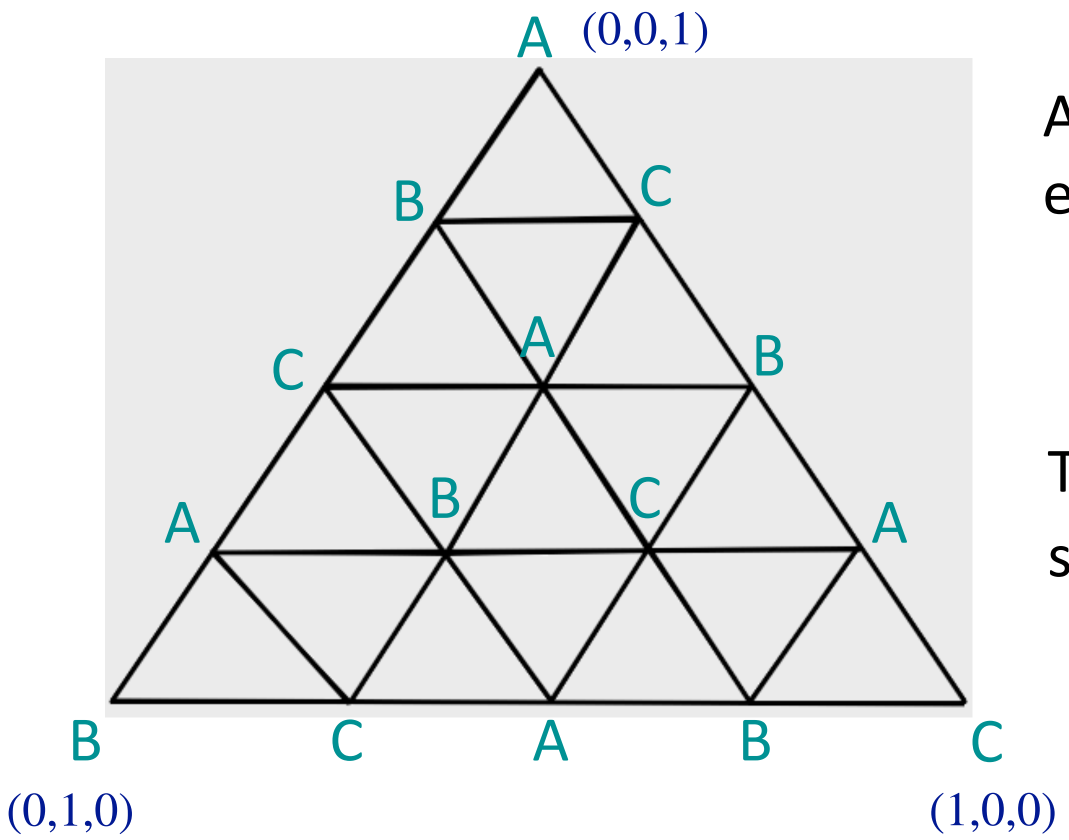

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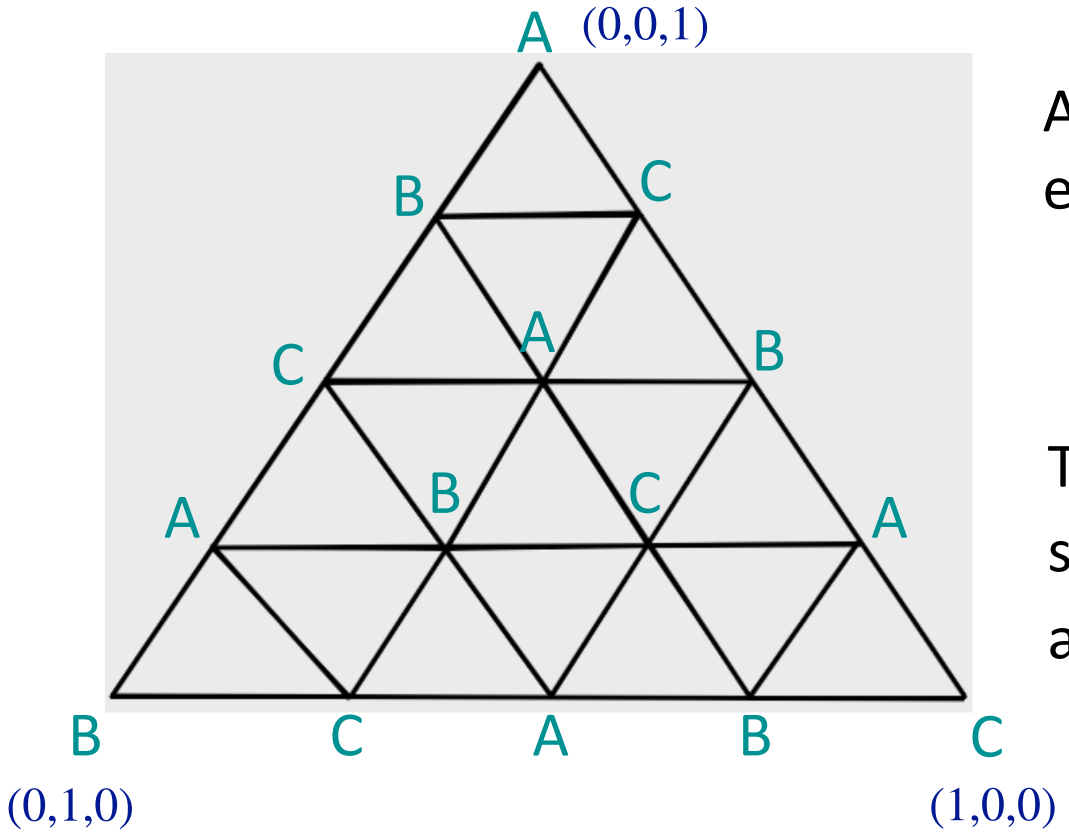
(There exists an efficient way to do this)

To generate a **Sperner coloring**, we go to a vertex, say some (x_1, x_2, x_3), and

Cake division using Sperner's Lemma



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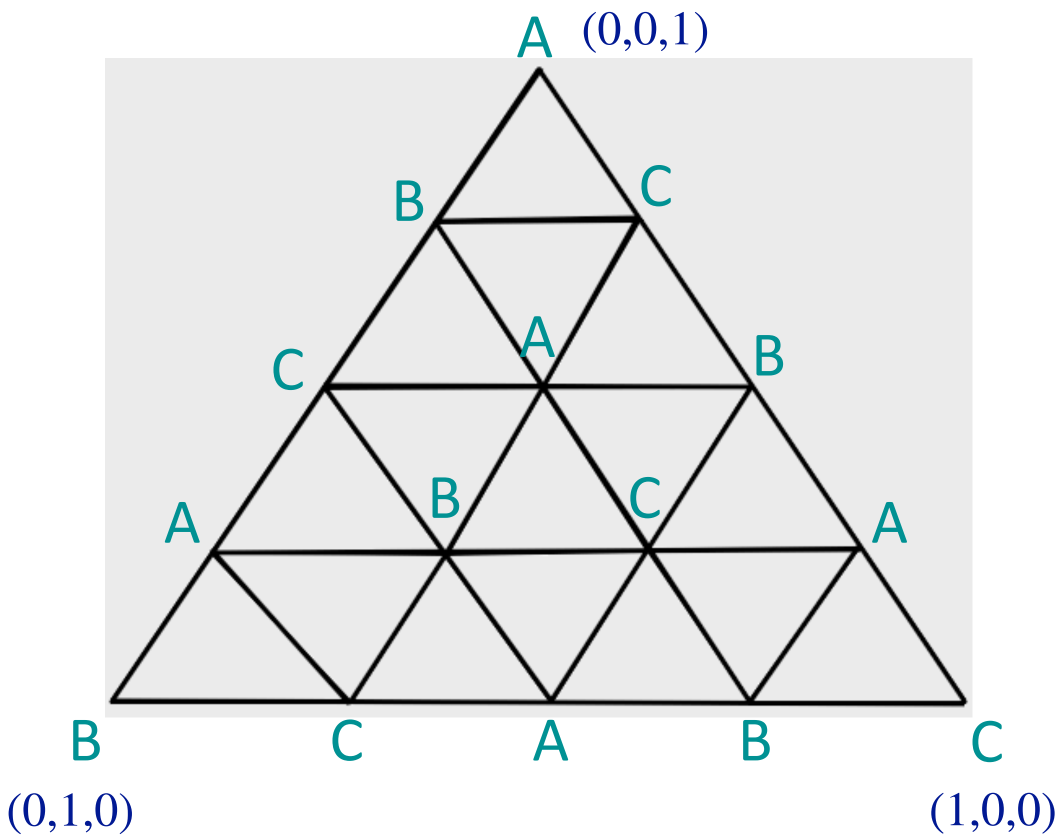
To generate a **Sperner coloring**, we go to a vertex, say some (x_1, x_2, x_3), and ask its owner agent her most **favorite** piece in this cut

Cake division using Sperner's Lemma



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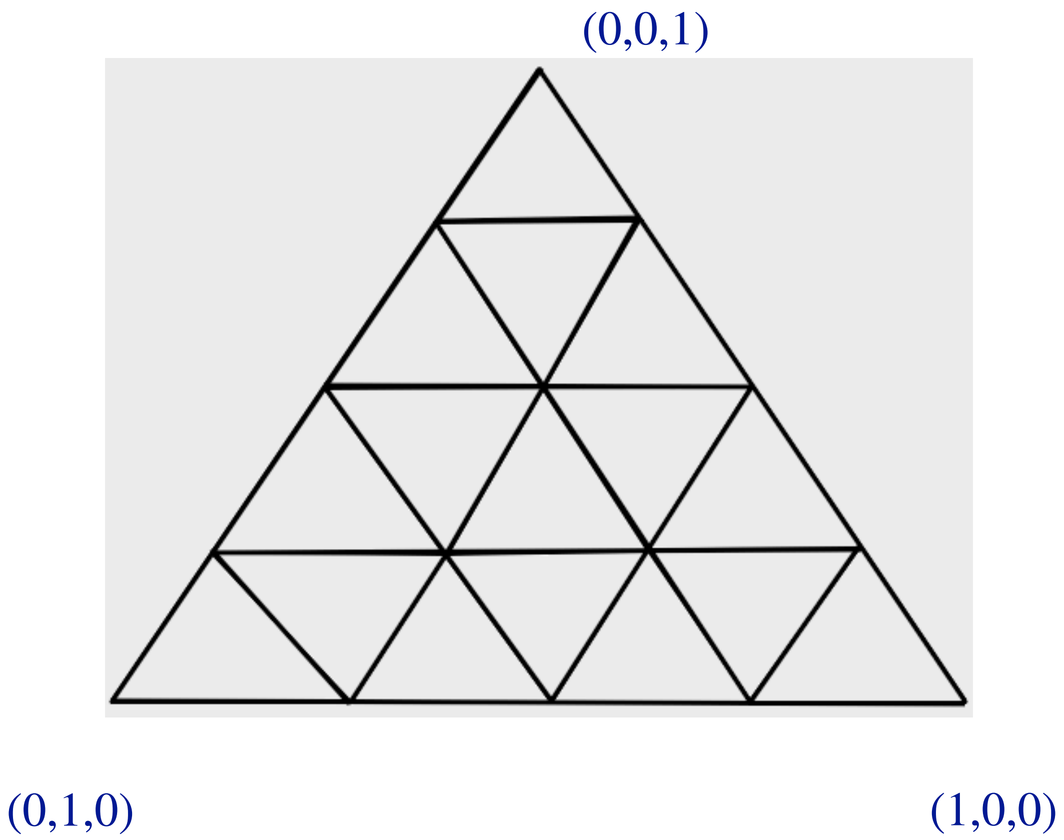

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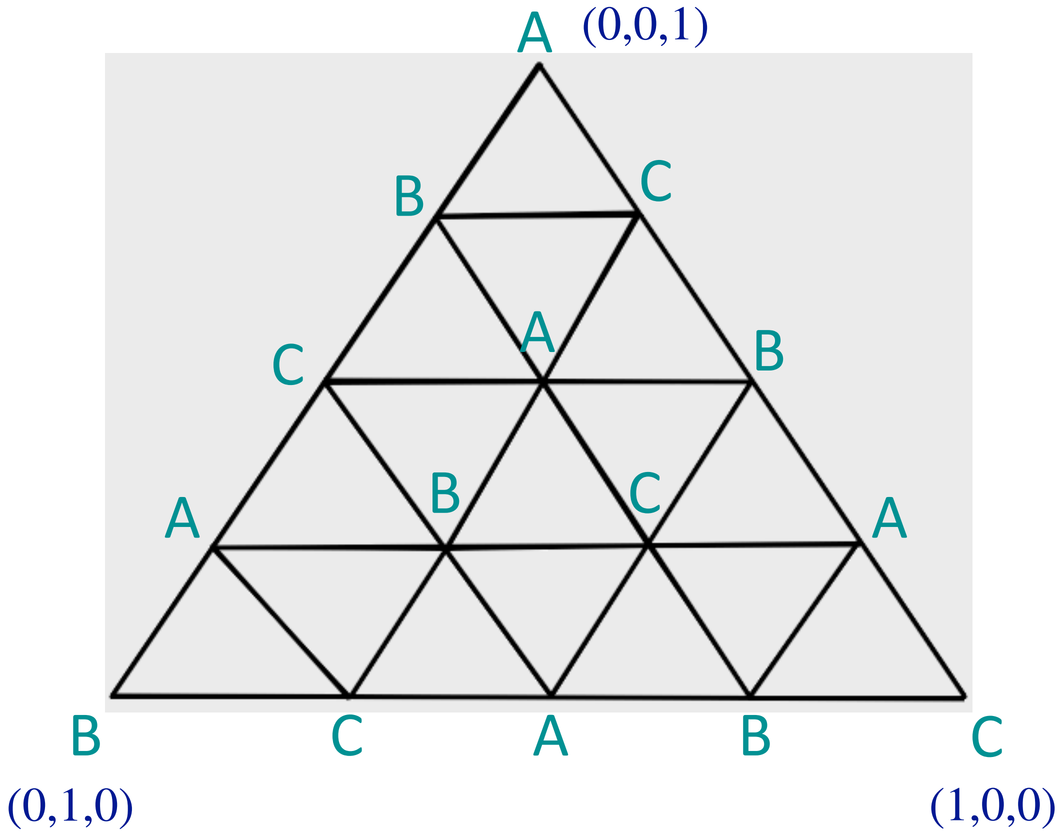

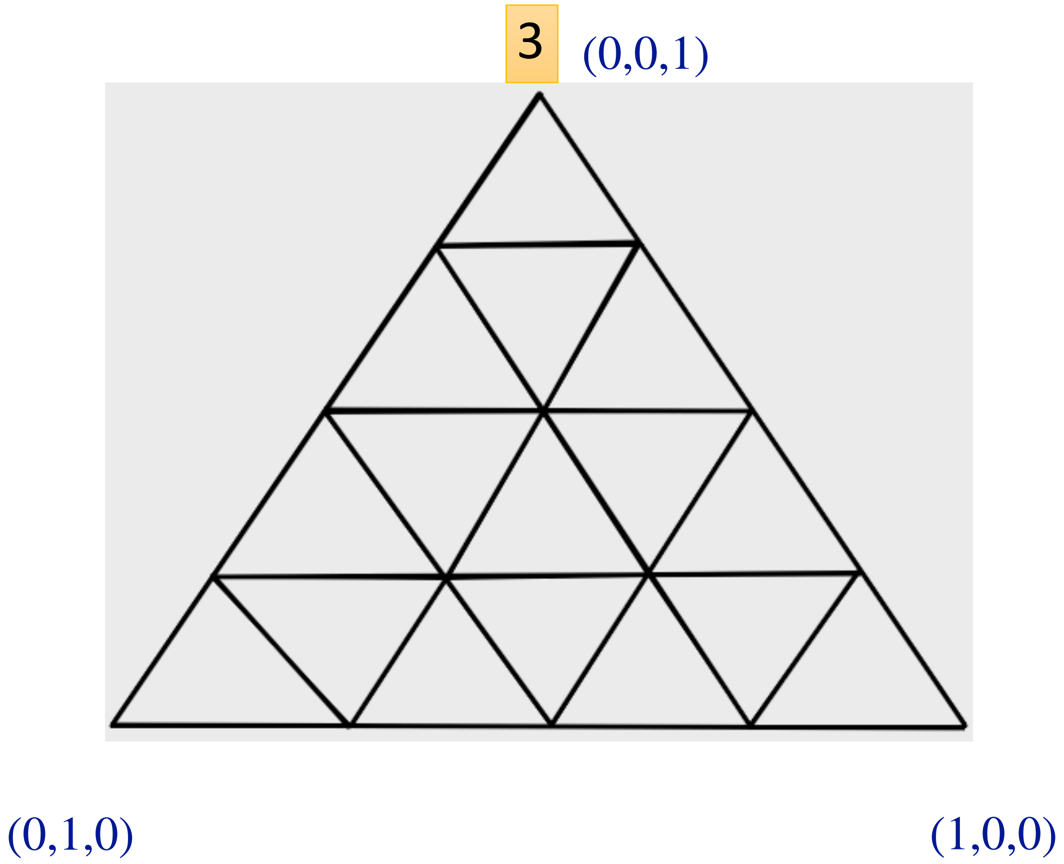

Ownership labeling

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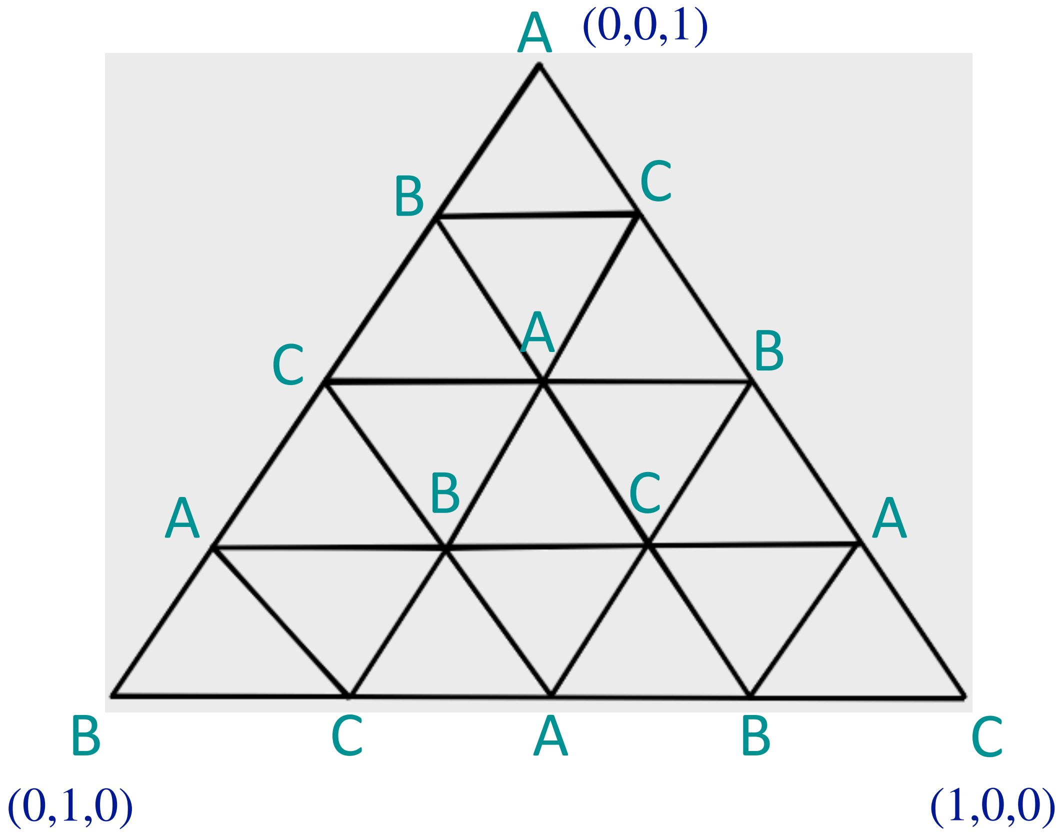

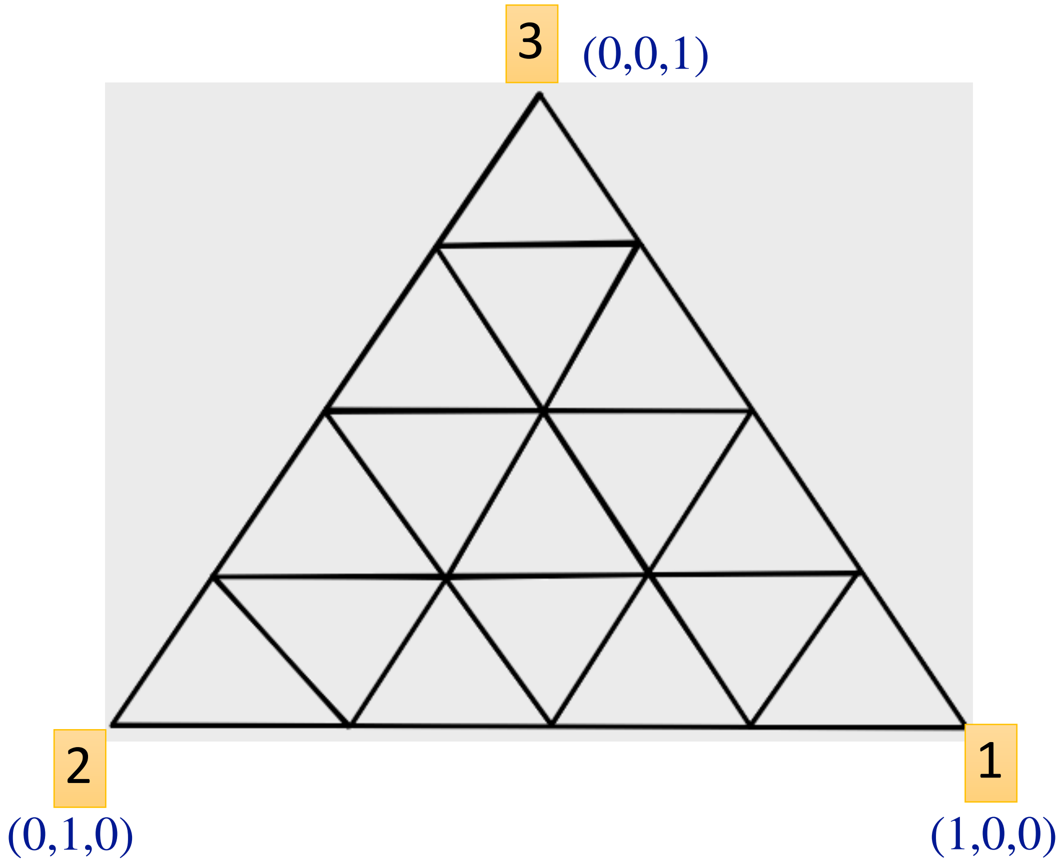

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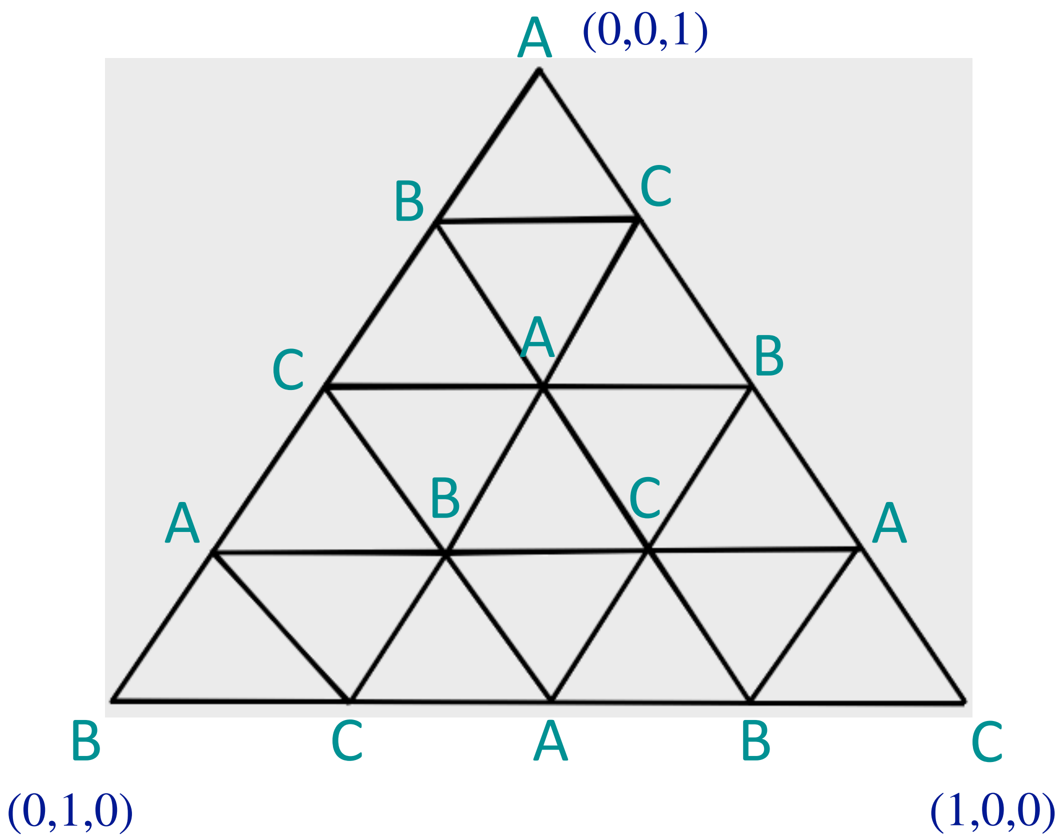
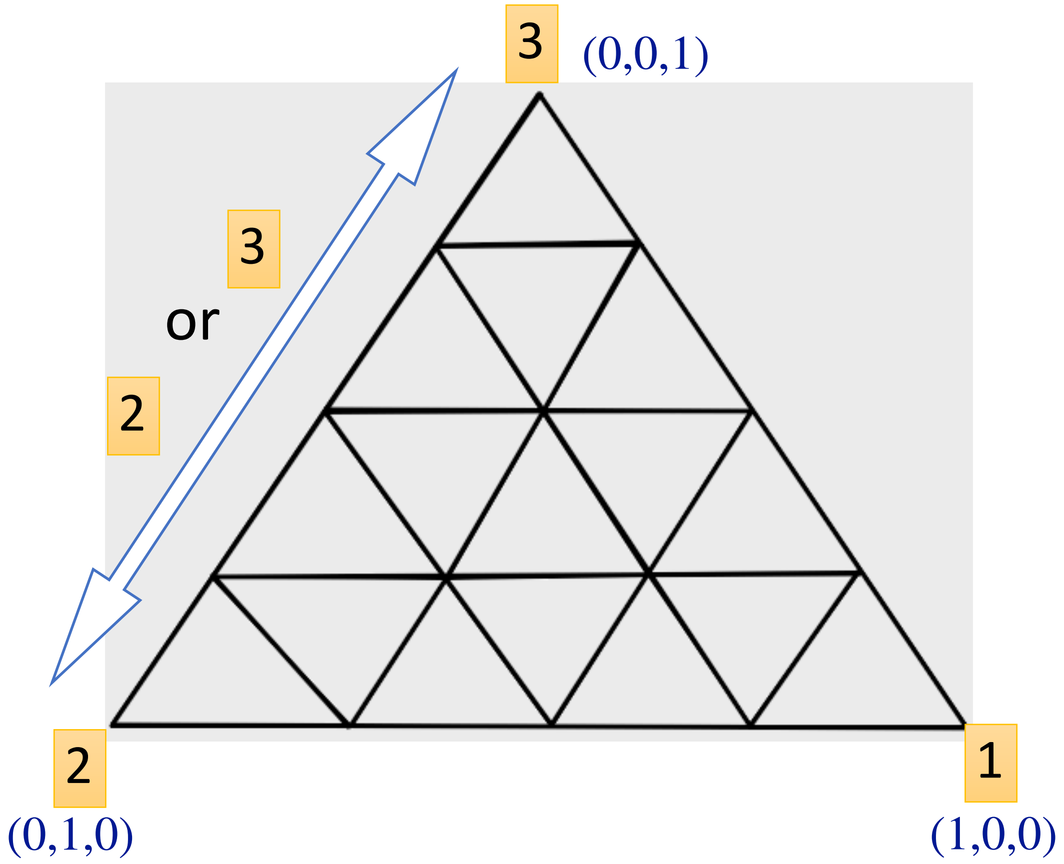
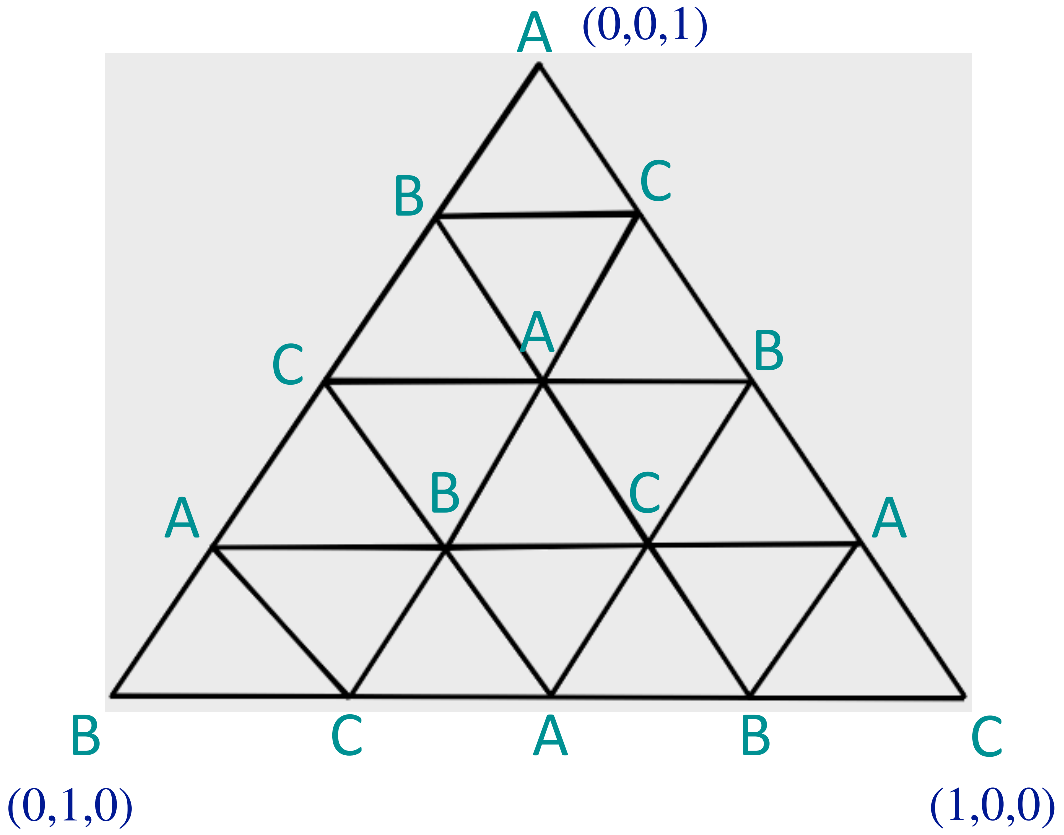
Cake division using Sperner's Lemma





Ownership labeling

Cake division using Sperner's Lemma



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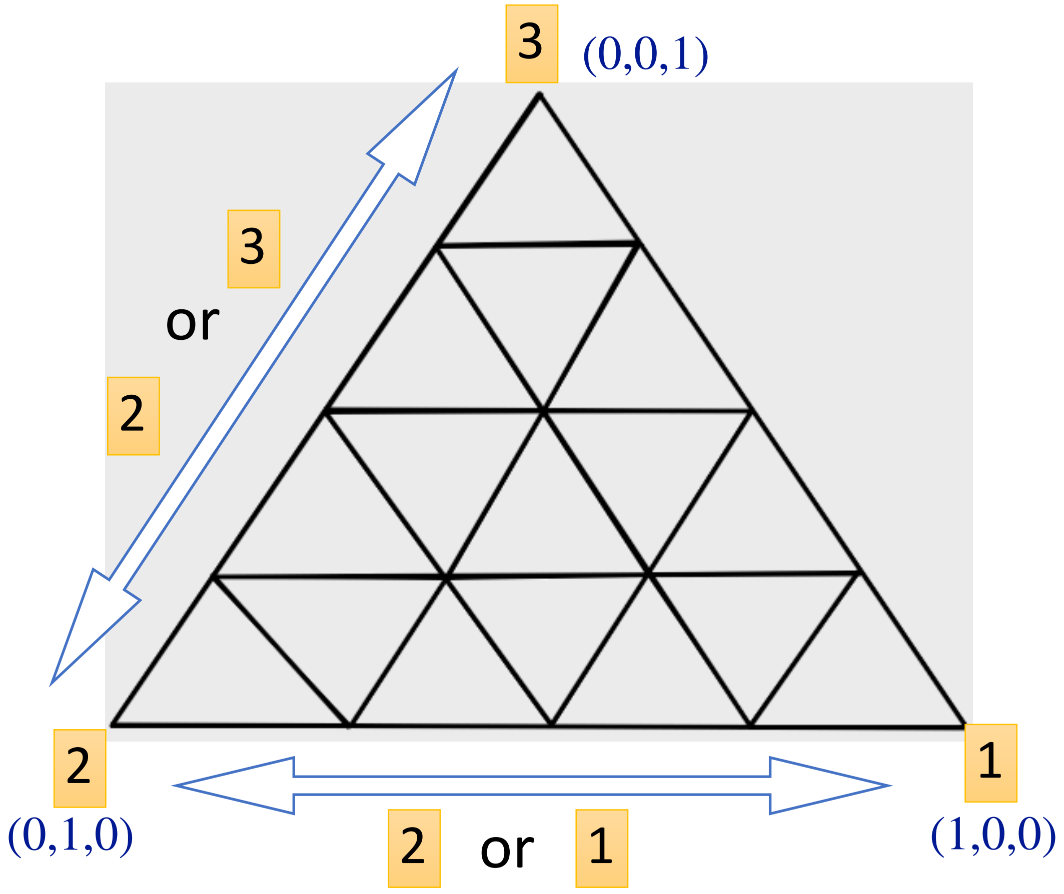
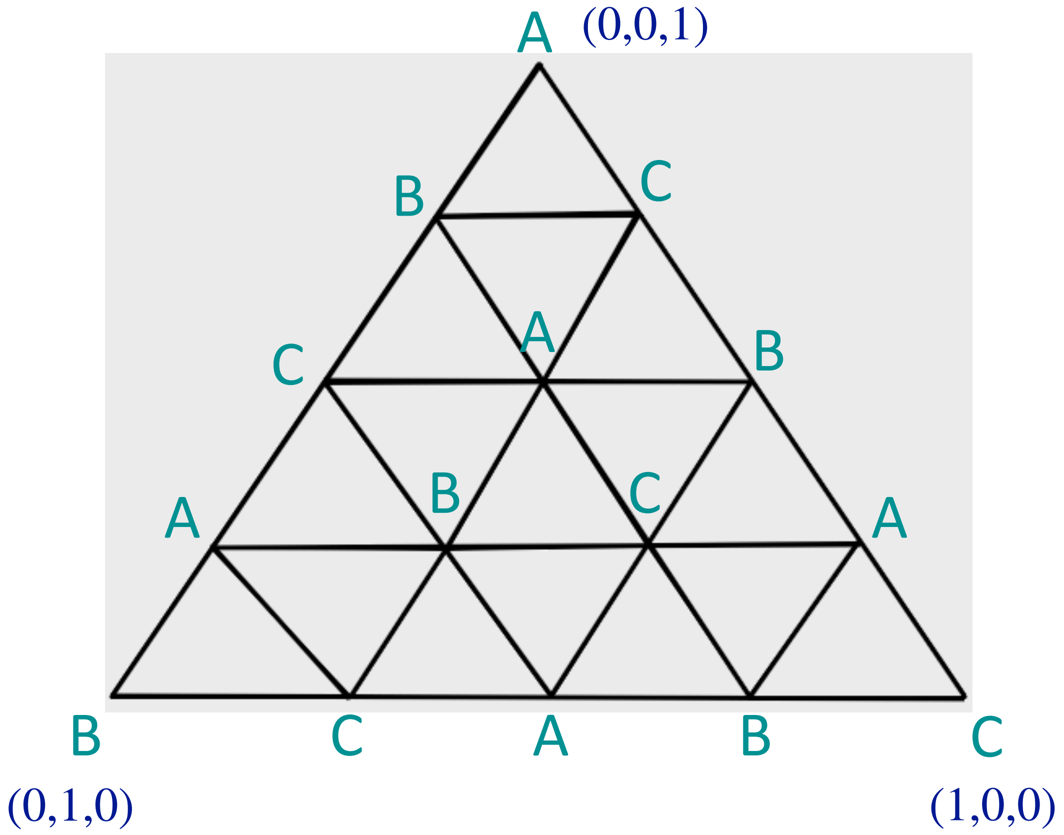


Cake division using Sperner's Lemma



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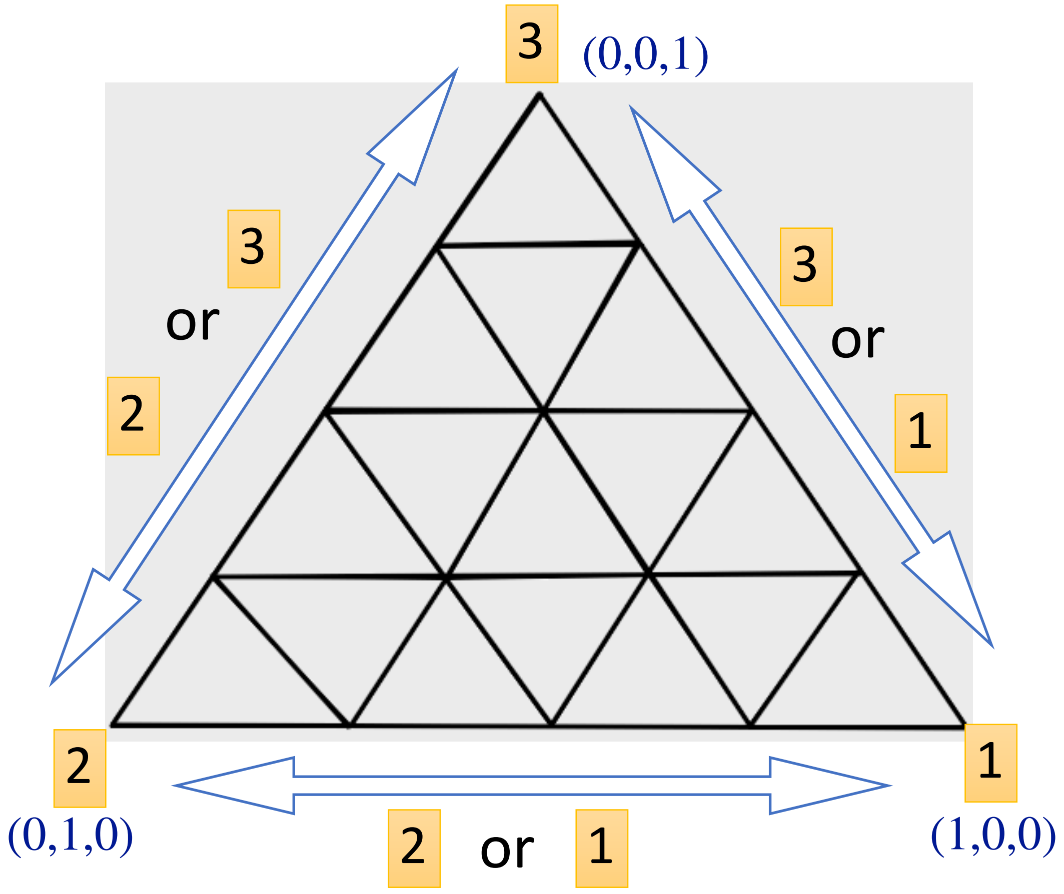

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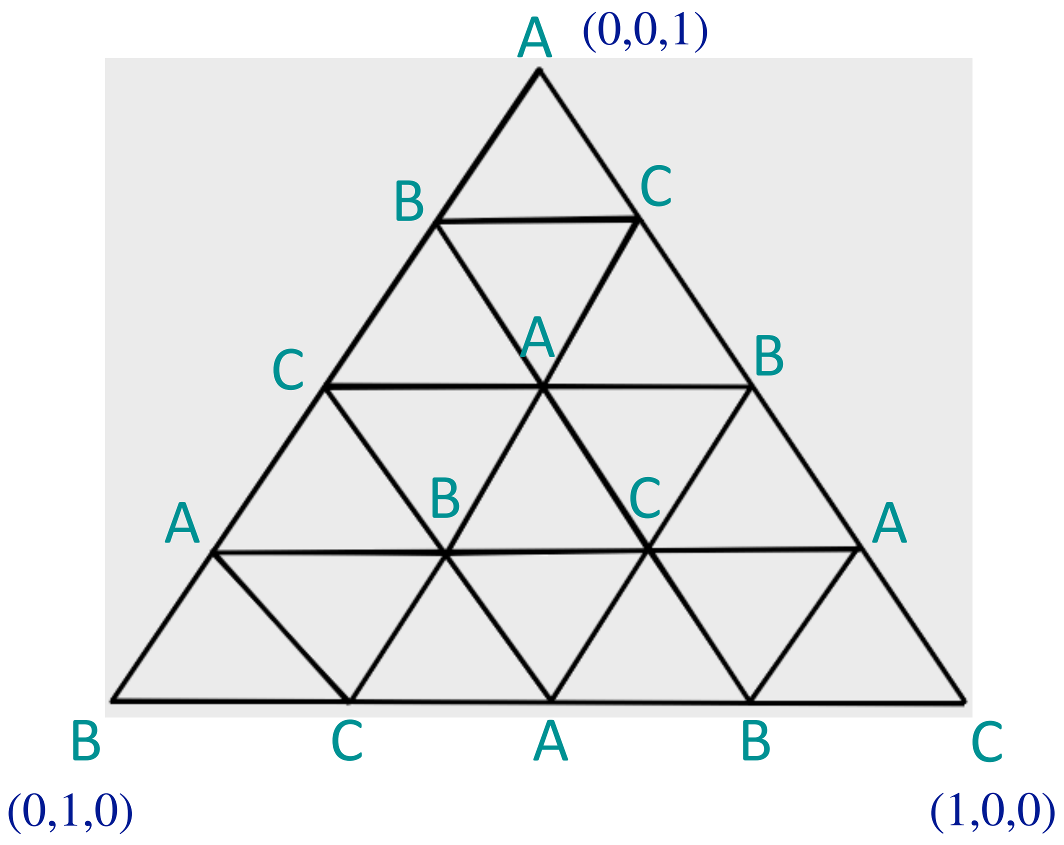

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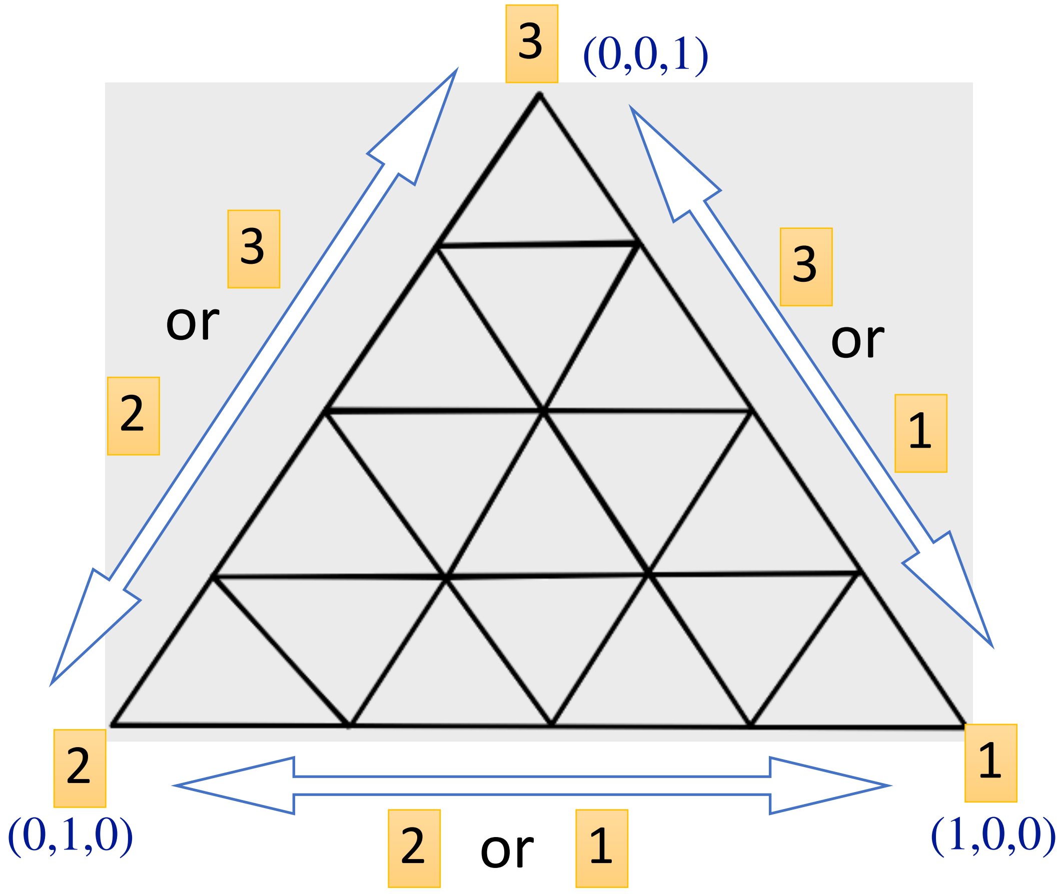
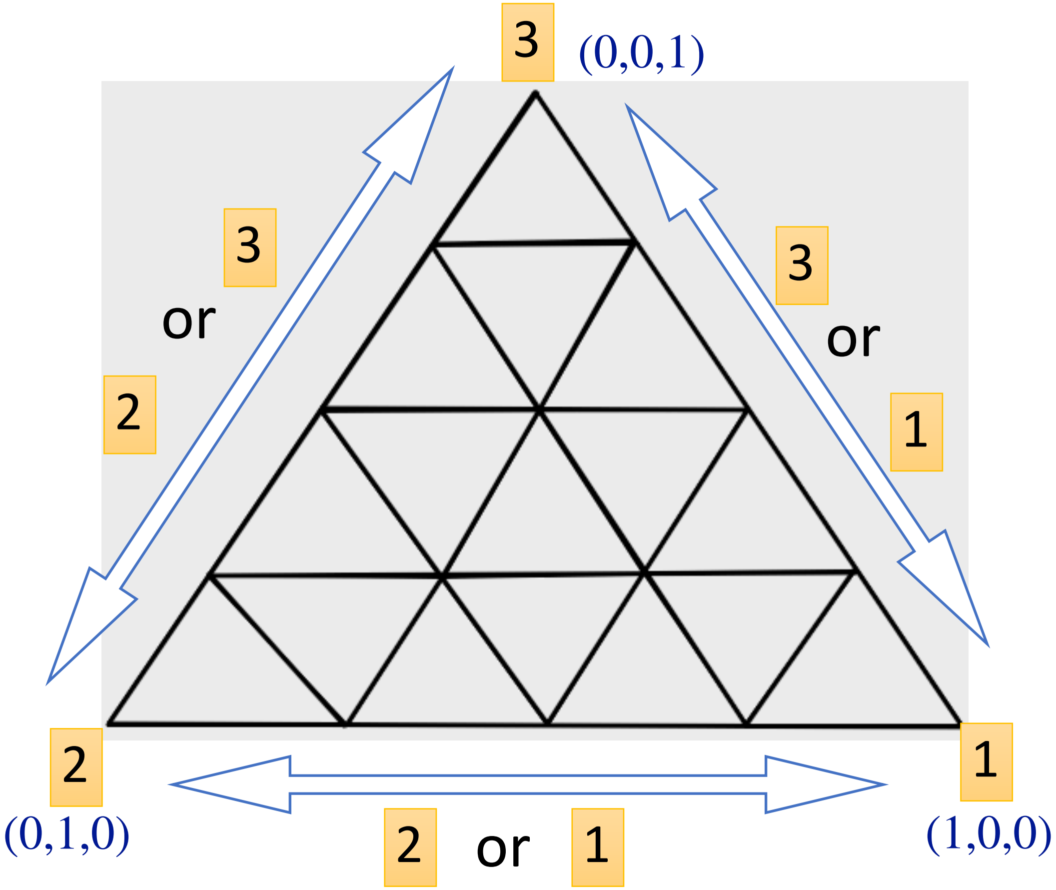


Sperner coloring

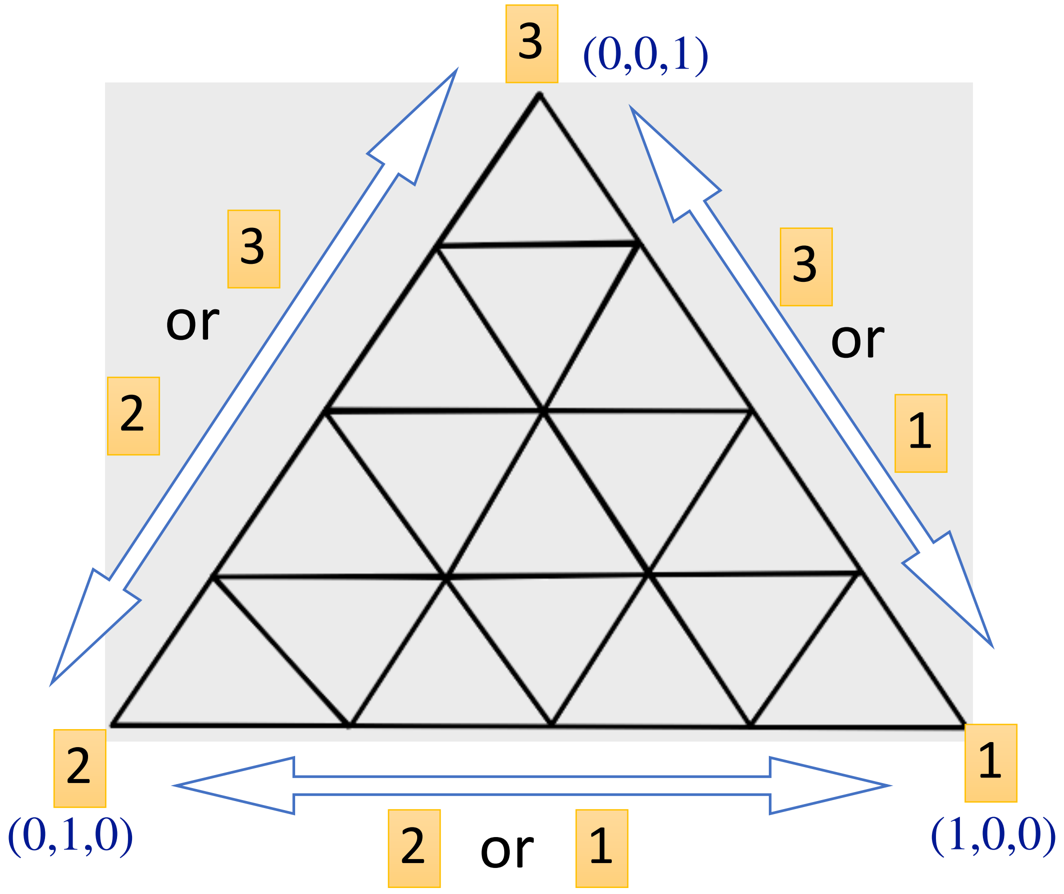
Cake division using Sperner's Lemma

Sperner's lemma $\Longrightarrow$



Cake division using Sperner's Lemma

Sperner's lemma $\Longrightarrow$ Existence of a baby triangle that has all the labels 1, 2 & 3



Cake division using Sperner's Lemma

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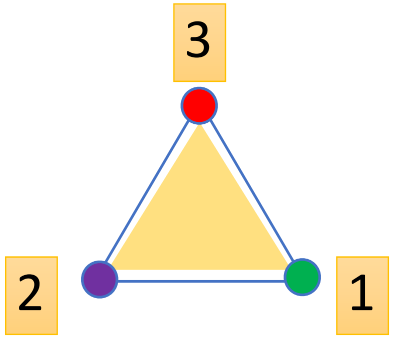
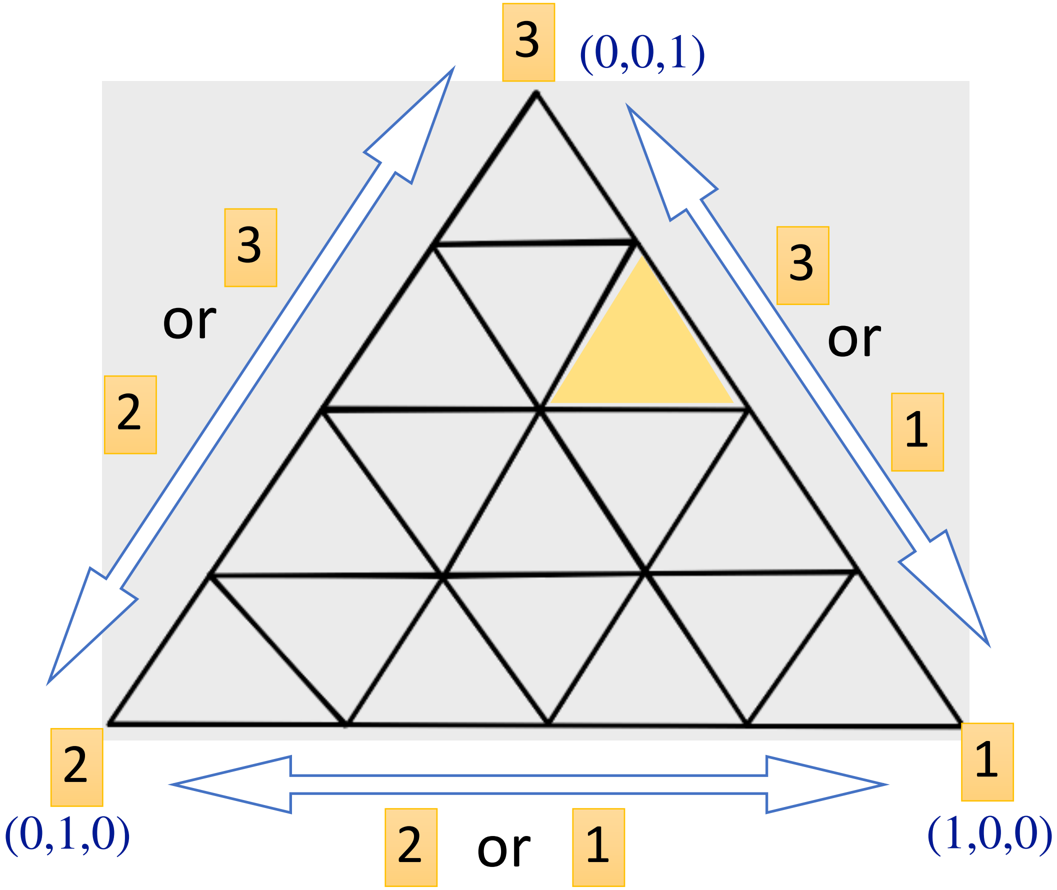


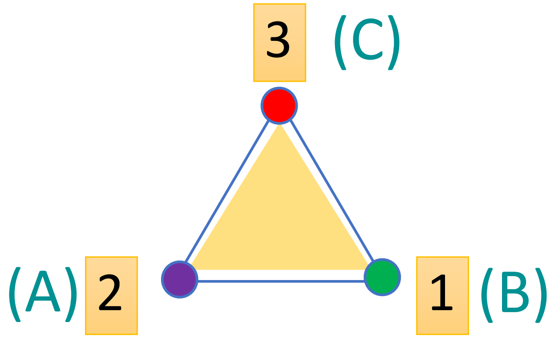


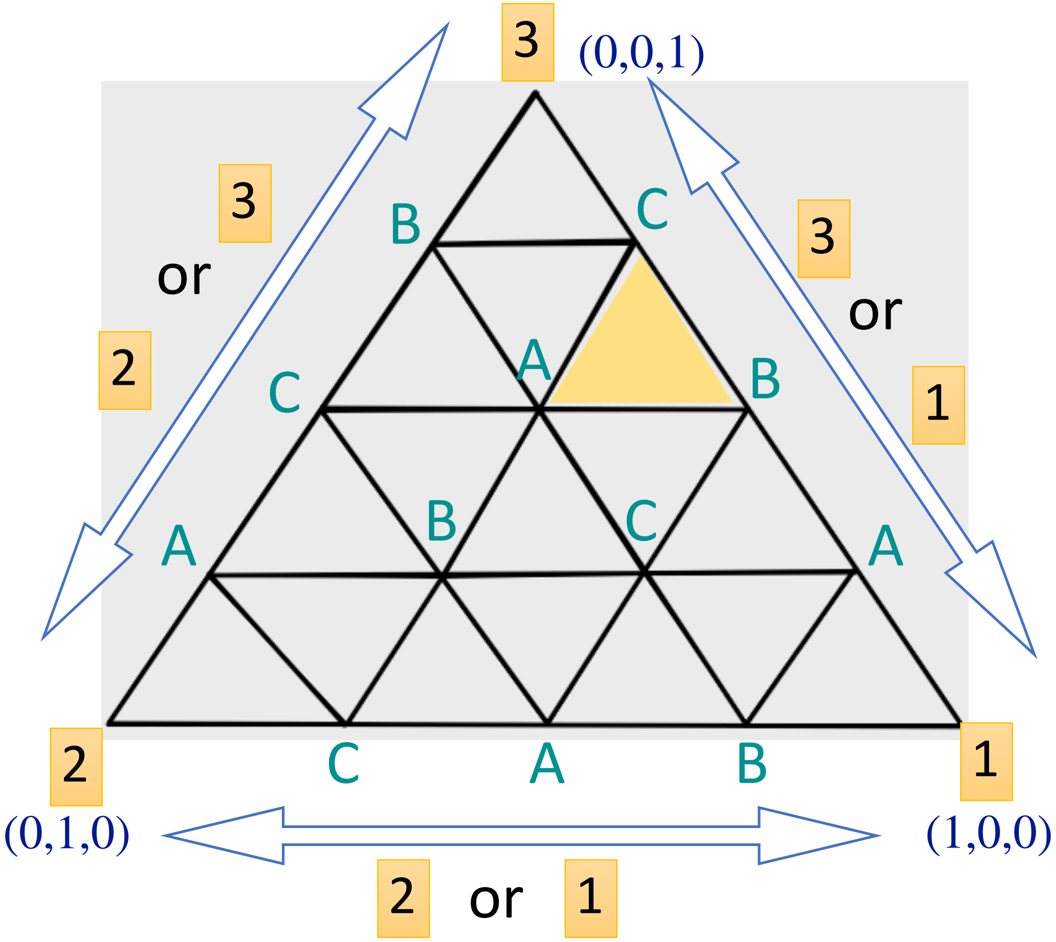
Cake division using Sperner's Lemma

Sperner's lemma ⟹ Existence of a baby triangle that has all the labels 1 , 2 & 3

+

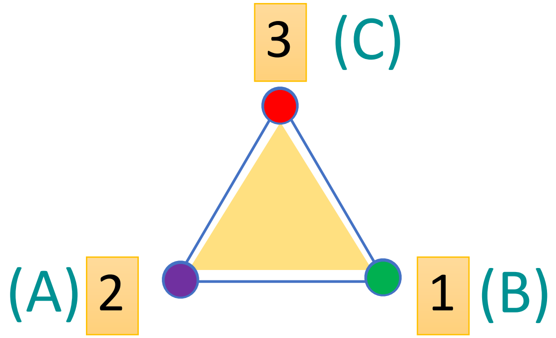
Ownership labeling ⟹





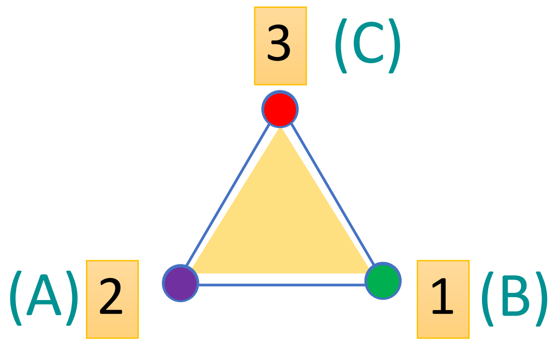
Sperner coloring

Cake division using Sperner's Lemma



At this cut,
agent A prefers piece 2

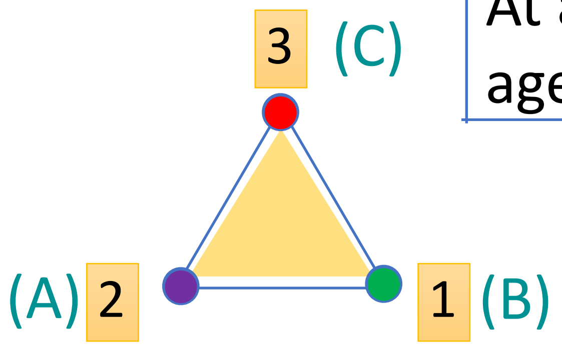
Cake division using Sperner's Lemma



At this cut,
agent A prefers piece 2

At a nearby cut,
agent B prefers piece 1

Cake division using Sperner's Lemma

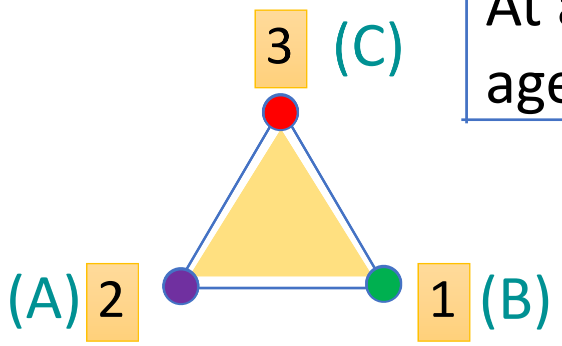


At a nearby cut,
agent C prefers piece 3

At this cut,
agent A prefers piece 2

At a nearby cut,
agent B prefers piece 1

Cake division using Sperner's Lemma



At a nearby cut,
agent C prefers piece 3

At this cut,
agent A prefers piece 2

At a nearby cut,
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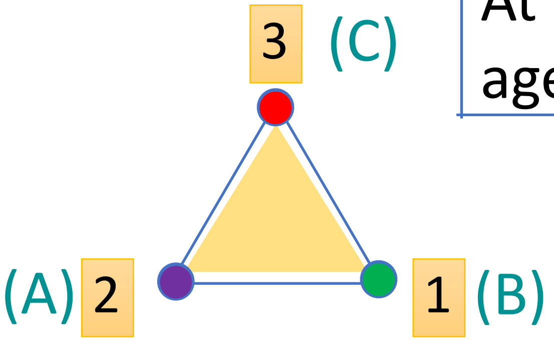
What we have is ***not*** a single cut (and hence not a single allocation), but three *nearby cuts*, where *envy-free-type* of thing is going on.

Cake division using Sperner's Lemma



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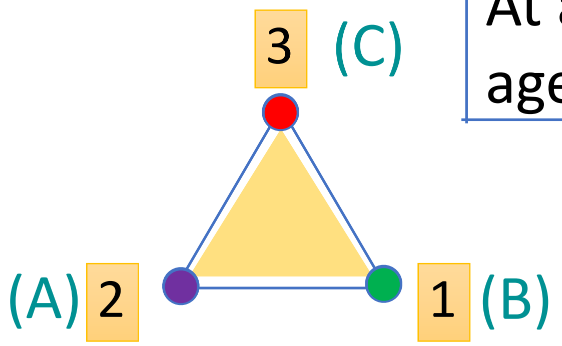
A single cut where all three agents prefer different pieces $\Longrightarrow$

Cake division using Sperner's Lemma



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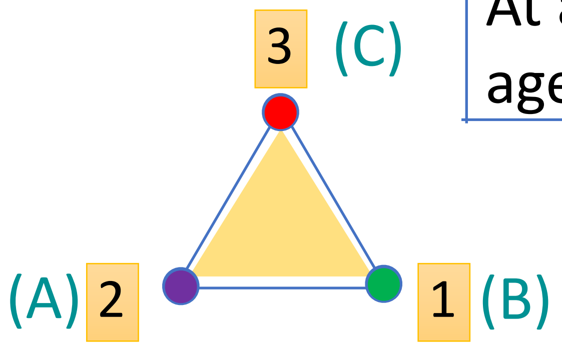
A single cut where all three agents prefer different pieces $\Longrightarrow$ **EF cake division**

Cake division using Sperner's Lemma



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Sperner's Lemma ⟹ A set of three *'nearby'* cuts where different agents prefer different pieces

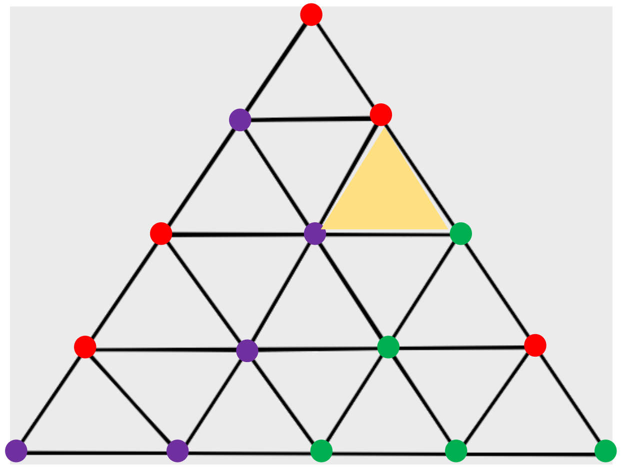
A single cut where all three agents prefer different pieces ⟹ **EF cake division**

Cake division using Sperner's Lemma

Sperner's Lemma $\Longrightarrow$ A set of three *'nearby'* cuts where different agents prefer different pieces

'*Approximate*' envy-free connected division

Cake division using Sperner's Lemma



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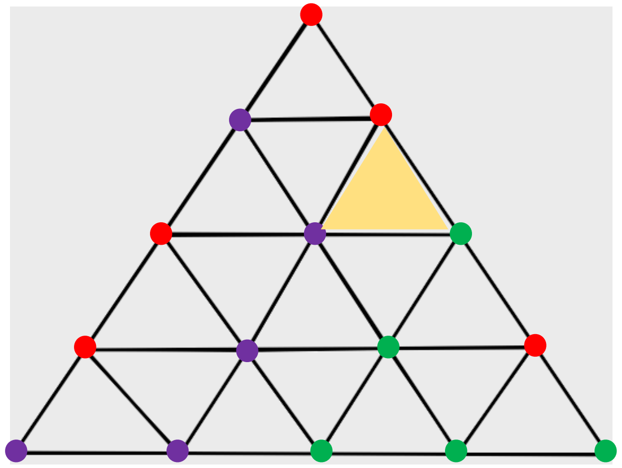
Cake division using Sperner's Lemma

Imagine making this triangulation finer and finer



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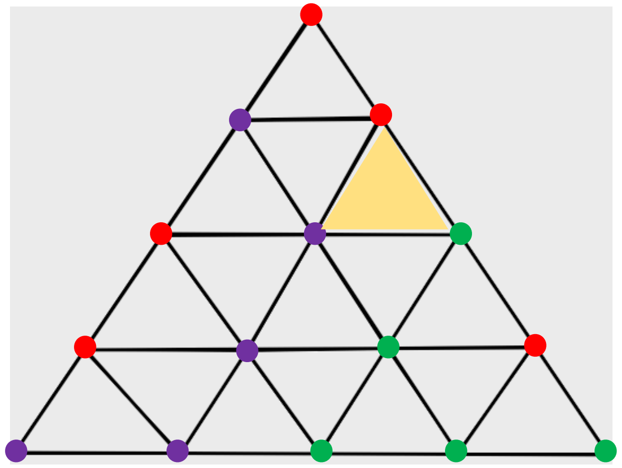
Cake division using Sperner's Lemma

Imagine making this triangulation finer and finer
- we will have increasingly *'nearby'* cuts
- where we have *envy-free like things* happening



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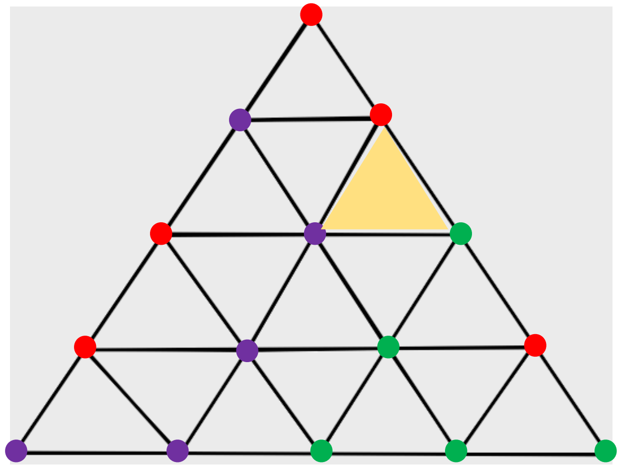

Imagine making this triangulation finer and finer
- we will have increasingly *'nearby'* cuts
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We can do something more: use convergence properties

Sperner's Lemma $\Longrightarrow$ A set of three *'nearby'* cuts where different agents prefer different pieces

'*Approximate*' envy-free connected division

Cake division using Sperner's Lemma

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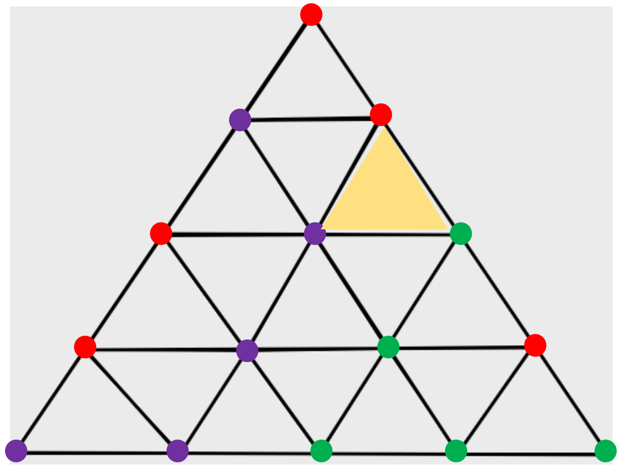
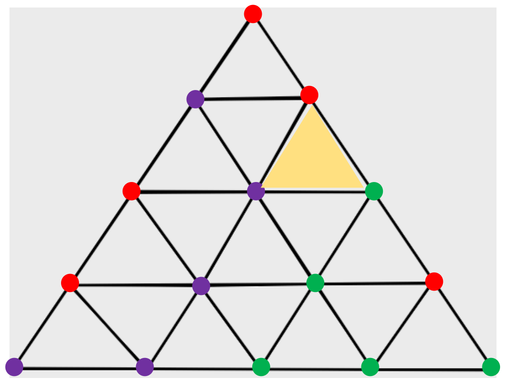


Valuations are (*topological*) ***closed*** $\Longrightarrow$ the ***limiting cut*** has to be **envy-free**

Sperner's Lemma $\Longrightarrow$ A set of three *'nearby'* cuts where different agents prefer different pieces

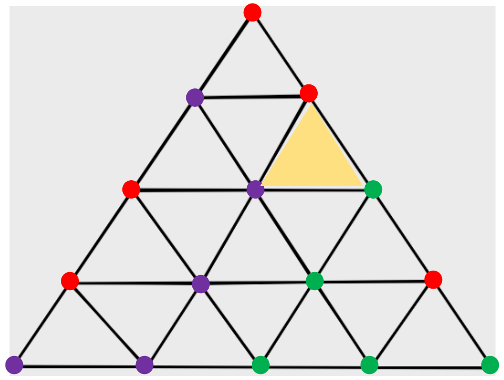
'*Approximate*' envy-free connected division

Cake division using Sperner's Lemma



Cake division using Sperner's Lemma

<u>Third Assumption: valuations are closed</u>



Cake division using Sperner's Lemma

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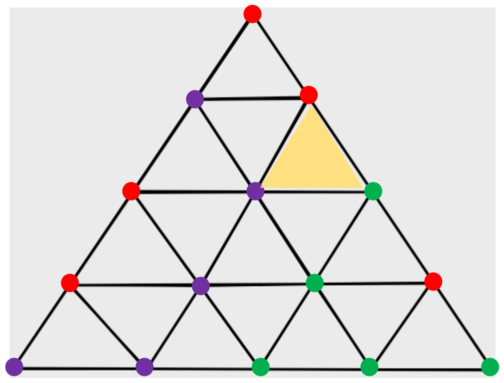
Denote a cut $X = (x_1, x_2, x_3)$. Consider a sequence of cuts $X^{(1)}, X^{(2)}, X^{(3)}, \ldots$



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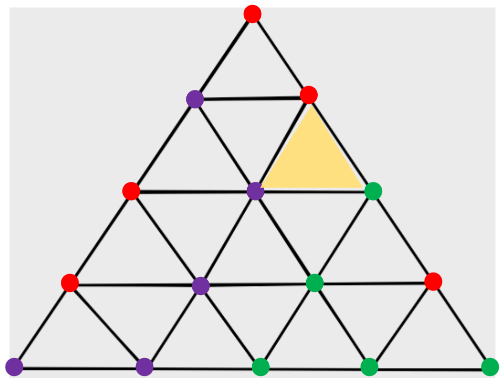
Triangle is bounded



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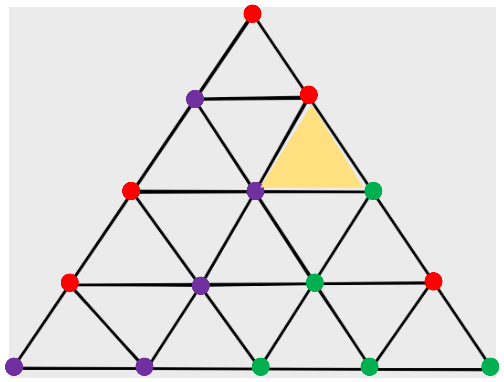
Denote a cut $X = (x_1, x_2, x_3)$. Consider a sequence of cuts $X^{(1)}, X^{(2)}, X^{(3)}, \ldots$

Triangle is bounded $\Longrightarrow$ the above sequence has a ***convergent subsequence***



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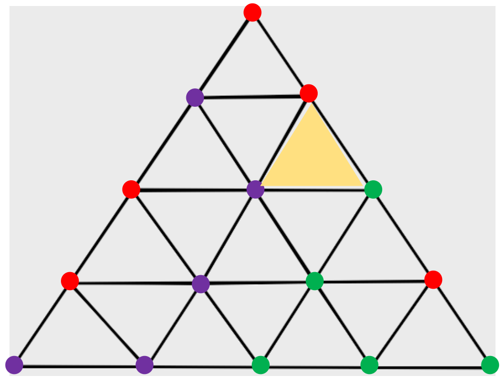
(Using Bolzano-Weistrass convergence theorem)



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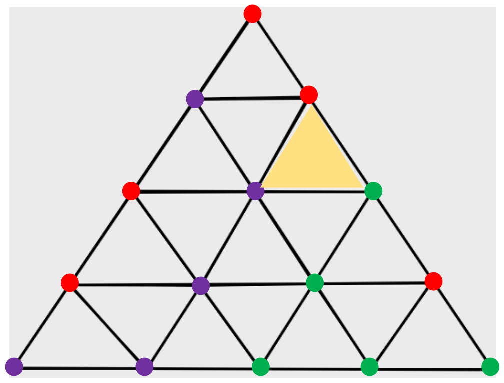

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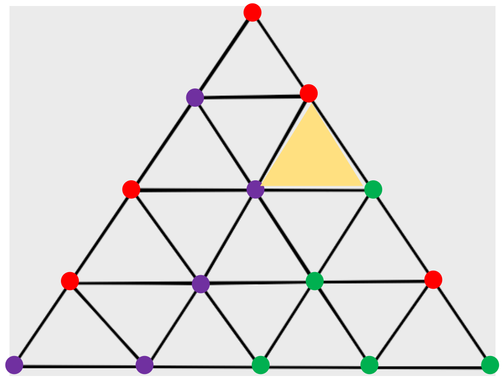

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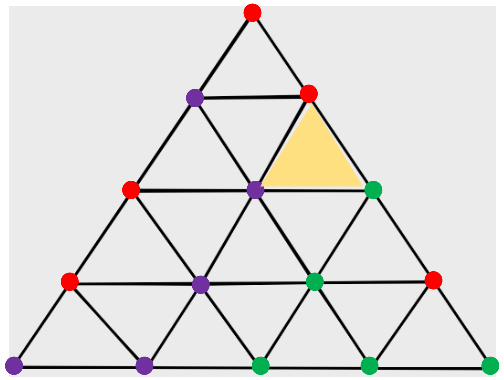

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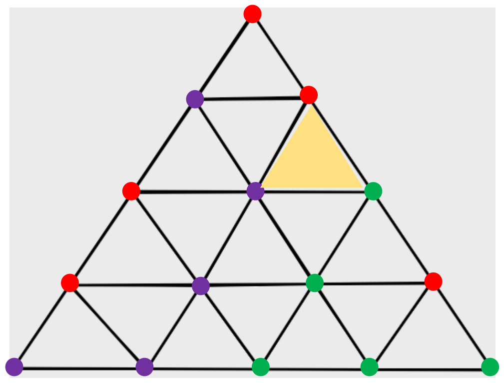

Cake division using Sperner's Lemma

Third Assumption: valuations are closed **Closed under limit!**

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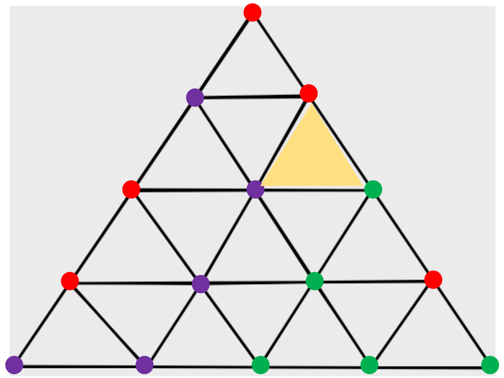

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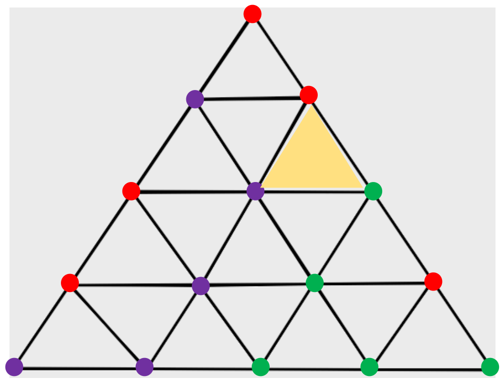

Cake division using Sperner's Lemma

Third Assumption: valuations are closed **Closed under limit!**

Idea: There is a limiting cut where we can turn *approximate EF into exact EF*

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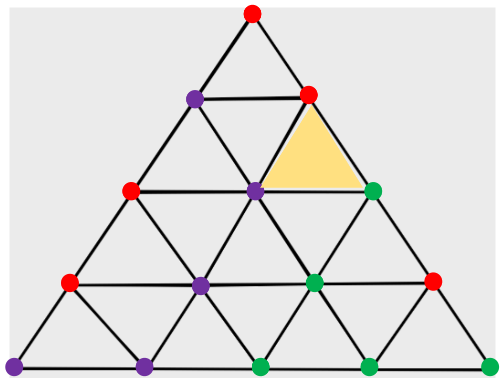

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They will all converge to a single cut-point,
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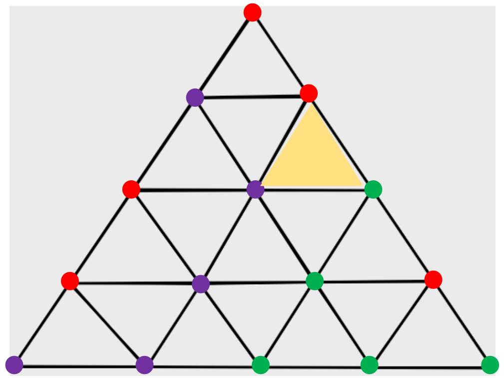

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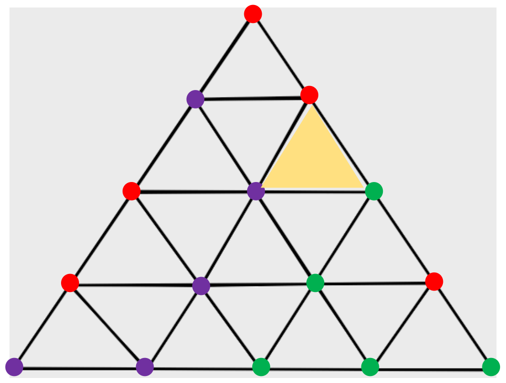
Exact envy-free connected division



Cake division using Sperner's Lemma

Sperner's Lemma

Convergence-based existential proof of envy-free cake divisions with connected pieces

Cake division using Sperner's Lemma

Sperner's Lemma

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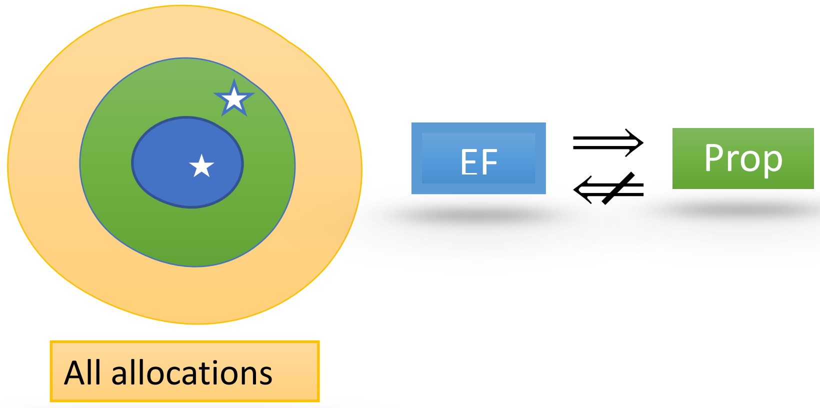
(and hence, it does not lead to an efficient algorithm)

Cake division using Sperner's Lemma

Sperner's Lemma

Convergence-based existential proof of envy-free cake divisions with connected pieces (and hence, it does not lead to an efficient algorithm)



Stromquist [1980], Su [1999]

Envy-free cake division with connected pieces exist for any number of agents

Envy-free cake divisions

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Stromquist, *J. of Combinatorics 2008*

No finite-query protocols exists for connected EF cake division

even for *three* agents!

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[ABK**R**] *WINE'19*

(Fair and Efficient Cake Division with Connected Pieces)(28 May)

An efficient algorithm: **1/2-EF + 1/3-NSW** allocation for *connected* EF cake division

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[ABK**R**] *WINE'19* (Fair and Efficient Cake Division with Connected Pieces)(28 May)

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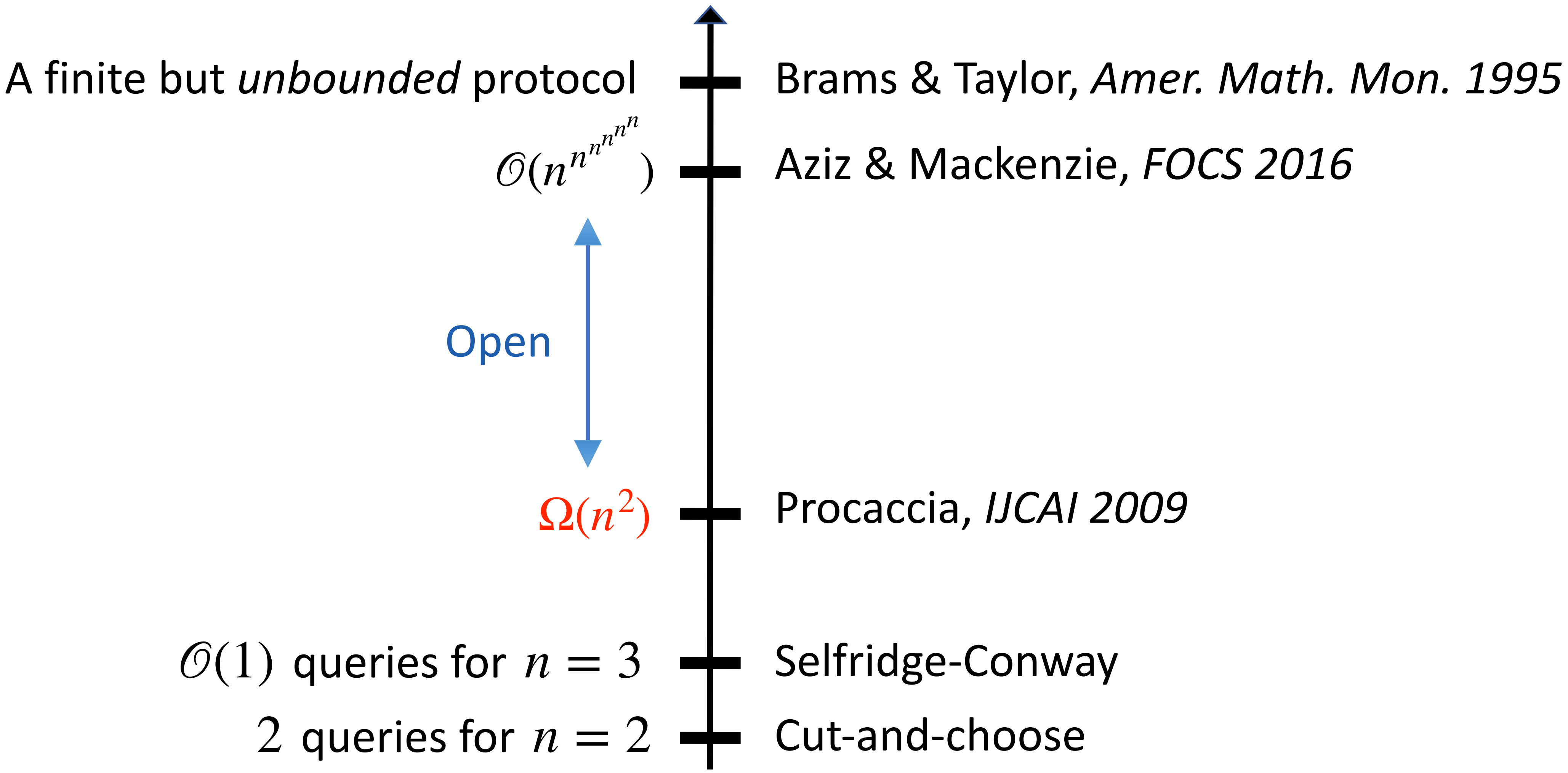
[ABK**R**] *EC'20* (Fair Cake Division under Monotone Likelihood Ratios)(25 June)

Efficient algorithms for connected EF cake division for a *broad class of instances*

Query Complexity of Envy-freeness



Source: 'Computational Social Choice' Lecture Slides by Rohit Vaish, IITD