“EFX: A Simpler Approach and an (Almost) Optimal Guarantee via Rainbow Cycle Number” [Akrami et al., 2025]

Presented in the Seminar “Mechanism Design Without Money”

Hirota Kinoshita

Max Planck Institute for Informatics

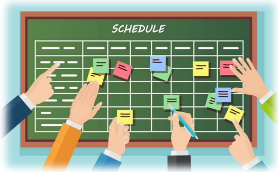
June 24, 2025

Fair Division

- How to divide resources fairly among heterogeneous agents.

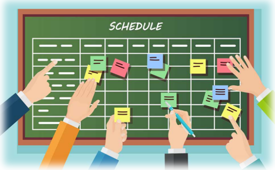
Fair Division

- How to divide resources fairly among heterogeneous agents.
- Divisible resources: land, time, etc.



Fair Division

- How to divide resources fairly among heterogeneous agents.
- Divisible resources: land, time, etc.
- Indivisible resources: people, rooms, tasks, etc.



- How to divide resources fairly among heterogeneous agents.
- Divisible resources: land, time, etc.
- Indivisible resources: people, rooms, tasks, etc. $\leftarrow$ This paper

Fair Division of Indivisible Goods

- An **instance** $(N, M, (v_i)_{i \in N})$ consists of the following:

Fair Division of Indivisible Goods

- An **instance** $(N, M, (v_i)_{i \in N})$ consists of the following:
  - A non-empty set of **agents** N,

Fair Division of Indivisible Goods

- An **instance** $(N, M, (v_i)_{i \in N})$ consists of the following:
  - A non-empty set of **agents** N,
  - A set of indivisible **goods** M,

Fair Division of Indivisible Goods

- An **instance** $(N, M, (v_i)_{i \in N})$ consists of the following:
  - A non-empty set of **agents** N,
  - A set of indivisible **goods** M,
  - The agents' **valuations** $(v_i : 2^M \to \mathbb{R}_{\geq 0})_{i \in N}$.

Fair Division of Indivisible Goods

- An **instance** $(N, M, (v_i)_{i \in N})$ consists of the following:
  - A non-empty set of **agents** N,
  - A set of indivisible **goods** M,
  - The agents' **valuations** $(v_i : 2^M \to \mathbb{R}_{\geq 0})_{i \in N}$.

- A subset of M is called a **bundle**.

Fair Division of Indivisible Goods

- An **instance** $(N, M, (v_i)_{i \in N})$ consists of the following:
  - A non-empty set of **agents** N,
  - A set of indivisible **goods** M,
  - The agents' **valuations** $(v_i : 2^M \to \mathbb{R}_{\geq 0})_{i \in N}$.
- A subset of M is called a **bundle**.
- An **allocation** is a partition $(A_i)_{i \in N}$ of M whose bundles are uniquely labelled by agents.

Fair Division of Indivisible Goods

- An **instance** $(N, M, (v_i)_{i \in N})$ consists of the following:
  - A non-empty set of **agents** N,
  - A set of indivisible **goods** M,
  - The agents' **valuations** $(v_i : 2^M \to \mathbb{R}_{\geq 0})_{i \in N}$.

- A subset of M is called a **bundle**.

- An **allocation** is a partition $(A_i)_{i \in N}$ of M whose bundles are uniquely labelled by agents.

- A **partial allocation** is a labelled partition $(A_i)_{i \in N}$ of a subset of M, where $M \setminus (A_i)_{i \in N}$ is the set of **unallocated** goods (and also called the **charity**).

Envy-Freeness (EF)

EF: one of the most natural concepts of fairness.

Envy-Freeness (EF)

EF: one of the most natural concepts of fairness.

Definition

A partial allocation $(A_i)_{i \in N}$ is said to be **envy-free (EF)** iff

$$v_i(A_i) \geq v_i(A_j) \qquad \forall i, j \in N.$$

In words, every agent i does not envy any other agent j.

Envy-Freeness (EF)

EF: one of the most natural concepts of fairness.

Definition

A partial allocation $(A_i)_{i \in N}$ is said to be **envy-free (EF)** iff

$$v_i(A_i) \geq v_i(A_j) \qquad \forall i, j \in N.$$

In words, every agent i does not envy any other agent j.

Q. Does every instance admit an EF allocation?

Envy-Freeness (EF)

EF: one of the most natural concepts of fairness.

Definition

A partial allocation $(A_i)_{i \in N}$ is said to be **envy-free (EF)** iff

$$v_i(A_i) \geq v_i(A_j) \qquad \forall i, j \in N.$$

In words, every agent i does not envy any other agent j.

Q. Does every instance admit an EF allocation?

A. No, e.g., even when 2 agents divide only 1 good.

Envy-Freeness up to Any Good (EFX)

EFX: “the most compelling” relaxation of EF [Caragiannis et al., 2019].

Envy-Freeness up to Any Good (EFX)

EFX: "the most compelling" relaxation of EF [Caragiannis et al., 2019].

Definition

A partial allocation $(A_i)_{i \in N}$ is said to be **envy-free up to any good (EFX)** iff

$$v_i(A_i) \geq v_i(A_j \setminus \{g\}) \qquad \forall g \in A_j, \forall i, j \in N.$$

Envy-Freeness up to Any Good (EFX)

EFX: "the most compelling" relaxation of EF [Caragiannis et al., 2019].

Definition

A partial allocation $(A_i)_{i \in N}$ is said to be **envy-free up to any good (EFX)** iff

$$v_i(A_i) \geq v_i(A_j \setminus \{g\}) \qquad \forall g \in A_j, \forall i, j \in N.$$

In words, every agent i would not envy any other agent j if any single item in his bundle A_j were removed.

Envy-Freeness up to Any Good (EFX)

EFX: "the most compelling" relaxation of EF [Caragiannis et al., 2019].

Definition

A partial allocation $(A_i)_{i \in N}$ is said to be **envy-free up to any good (EFX)** iff

$$v_i(A_i) \geq v_i(A_j \setminus \{g\}) \qquad \forall g \in A_j, \forall i, j \in N.$$

In words, every agent i would not envy any other agent j if any single item in his bundle A_j were removed.

Q. Does every instance admit an EFX allocation?
— The "most enigmatic" open question [Procaccia, 2020].

EFX for a Few Agents

An EFX allocation has been shown to exist in the following cases:

The other two are arbitrary monotone valuations.

An EFX allocation has been shown to exist in the following cases:

	$\lvert N \rvert$	Valuations
[Plaut and Roughgarden, 2020]	2	Monotone

[1] The other two are arbitrary monotone valuations.

EFX for a Few Agents

An EFX allocation has been shown to exist in the following cases:

	$\|N\|$	Valuations
[Plaut and Roughgarden, 2020]	2	Monotone
[Chaudhury et al., 2020]	3	Additive

[1] The other two are arbitrary monotone valuations.

EFX for a Few Agents

An EFX allocation has been shown to exist in the following cases:

	$\lvert N \rvert$	Valuations
[Plaut and Roughgarden, 2020]	2	Monotone
[Chaudhury et al., 2020]	3	Additive
[Berger et al., 2022]	3	Nice-cancellable

[1] The other two are arbitrary monotone valuations.

EFX for a Few Agents

An EFX allocation has been shown to exist in the following cases:

	$\lvert N \rvert$	Valuations
[Plaut and Roughgarden, 2020]	2	Monotone
[Chaudhury et al., 2020]	3	Additive
[Berger et al., 2022]	3	Nice-cancellable
[Akrami et al., 2025]	3	At least one is MMS-feasible[1]

[1] The other two are arbitrary monotone valuations.

Valuation Classes

Definition

A valuation $v : 2^M \to \mathbb{R}_{\geq 0}$ is said to be **monotone** iff

$$v(X) \leq v(Y) \qquad\qquad \forall X \subseteq Y \subseteq M.$$

Valuation Classes

Definition

A valuation $v : 2^M \to \mathbb{R}_{\geq 0}$ is said to be **monotone** iff

$$v(X) \leq v(Y) \qquad \forall X \subseteq Y \subseteq M.$$

Theorem (Plaut and Roughgarden [2020])

An EFX allocation exists for 2 agents with monotone valuations.

Valuation Classes

Definition

A valuation $v : 2^M \to \mathbb{R}_{\geq 0}$ is said to be **monotone** iff

$$v(X) \leq v(Y) \qquad \forall X \subseteq Y \subseteq M.$$

Theorem (Plaut and Roughgarden [2020])

An EFX allocation exists for 2 agents with monotone valuations.

Definition

A valuation $v : 2^M \to \mathbb{R}_{\geq 0}$ is said to be **additive** iff

$$v(X) = \sum_{g \in X} v(\{g\}) \qquad \forall X \subseteq M.$$

Valuation Classes

Definition

A valuation $v : 2^M \to \mathbb{R}_{\geq 0}$ is said to be **monotone** iff

$$v(X) \leq v(Y) \qquad \forall X \subseteq Y \subseteq M.$$

Theorem (Plaut and Roughgarden [2020])

An EFX allocation exists for 2 agents with monotone valuations.

Definition

A valuation $v : 2^M \to \mathbb{R}_{\geq 0}$ is said to be **additive** iff

$$v(X) = \sum_{g \in X} v(\{g\}) \qquad \forall X \subseteq M.$$

Theorem (Chaudhury et al. [2020])

An EFX allocation exists for 3 agents with additive valuations.

Valuation Classes

Definition

A monotone valuation $v : 2^M \to \mathbb{R}_{\geq 0}$ is said to be **nice-cancellable** iff there exists an injective valuation $v' : 2^M \to \mathbb{R}_{\geq 0}$ such that

$$v(X) > v(Y) \Rightarrow v'(X) > v'(Y) \qquad \forall X, Y \subseteq M,$$

and that

$$v'(X \cup \{g\}) > v'(Y \cup \{g\}) \Rightarrow v'(X) > v'(Y) \quad \forall g \in M \setminus Y, \forall X, Y \subseteq M.$$

Valuation Classes

Definition

A monotone valuation $v : 2^M \to \mathbb{R}_{\geq 0}$ is said to be **nice-cancellable** iff there exists an injective valuation $v' : 2^M \to \mathbb{R}_{\geq 0}$ such that

$$v(X) > v(Y) \Rightarrow v'(X) > v'(Y) \qquad \forall X, Y \subseteq M,$$

and that

$$v'(X \cup \{g\}) > v'(Y \cup \{g\}) \Rightarrow v'(X) > v'(Y) \quad \forall g \in M \setminus Y, \forall X, Y \subseteq M.$$

Theorem (Berger et al. [2022])

An EFX allocation exists for 3 agents with nice-cancellable valuations.

Valuation Classes

Definition

A monotone valuation $v : 2^M \to \mathbb{R}_{\geq 0}$ is said to be **nice-cancellable** iff there exists an injective valuation $v' : 2^M \to \mathbb{R}_{\geq 0}$ such that

$$v(X) > v(Y) \Rightarrow v'(X) > v'(Y) \qquad \forall X, Y \subseteq M,$$

and that

$$v'(X \cup \{g\}) > v'(Y \cup \{g\}) \Rightarrow v'(X) > v'(Y) \quad \forall g \in M \setminus Y, \forall X, Y \subseteq M.$$

Theorem (Berger et al. [2022])

An EFX allocation exists for 3 agents with nice-cancellable valuations.

Definition

A valuation $v : 2^M \to \mathbb{R}_{\geq 0}$ is said to be **MMS-feasible** iff

$$\max\{v(X), v(S \setminus X)\} \geq \min\{v(Y), v(S \setminus Y)\} \quad \forall X, Y \subseteq S \subseteq M.$$

Valuation Classes

Definition

A monotone valuation $v : 2^M \to \mathbb{R}_{\geq 0}$ is said to be **nice-cancellable** iff there exists an injective valuation $v' : 2^M \to \mathbb{R}_{\geq 0}$ such that

$$v(X) > v(Y) \Rightarrow v'(X) > v'(Y) \qquad \forall X, Y \subseteq M,$$

and that

$$v'(X \cup \{g\}) > v'(Y \cup \{g\}) \Rightarrow v'(X) > v'(Y) \quad \forall g \in M \setminus Y, \forall X, Y \subseteq M.$$

Theorem (Berger et al. [2022])

An EFX allocation exists for 3 agents with nice-cancellable valuations.

Definition

A valuation $v : 2^M \to \mathbb{R}_{\geq 0}$ is said to be **MMS-feasible** iff

$$\max\{v(X), v(S \setminus X)\} \geq \min\{v(Y), v(S \setminus Y)\} \quad \forall X, Y \subseteq S \subseteq M.$$

Additive $\subsetneq$ Nice-cancellable $\subsetneq$ MMS-feasible $\subsetneq$ Monotone.

First Main Result of the Paper

Definition

A valuation $v : 2^M \to \mathbb{R}_{\geq 0}$ is said to be **MMS-feasible** iff

$$\max\{v(X), v(S \setminus X)\} \geq \min\{v(Y), v(S \setminus Y)\} \quad \forall X, Y \subseteq S \subseteq M.$$

Additive $\subsetneq$ Nice-cancellable $\subsetneq$ MMS-feasible $\subsetneq$ Monotone.

First Main Result of the Paper

Definition

A valuation $v : 2^M \to \mathbb{R}_{\geq 0}$ is said to be **MMS-feasible** iff

$$\max\{v(X), v(S \setminus X)\} \geq \min\{v(Y), v(S \setminus Y)\} \quad \forall X, Y \subseteq S \subseteq M.$$

Additive $\subsetneq$ Nice-cancellable $\subsetneq$ MMS-feasible $\subsetneq$ Monotone.

Theorem (Akrami et al. [2025])

An EFX allocation exists when $|N| = 3$ and at least one agent has an MMS-feasible valuation.

First Main Result of the Paper

Definition

A valuation $v : 2^M \to \mathbb{R}_{\geq 0}$ is said to be **MMS-feasible** iff

$$\max\{v(X), v(S \setminus X)\} \geq \min\{v(Y), v(S \setminus Y)\} \quad \forall X, Y \subseteq S \subseteq M.$$

Additive $\subsetneq$ Nice-cancellable $\subsetneq$ MMS-feasible $\subsetneq$ Monotone.

Theorem (Akrami et al. [2025])

An EFX allocation exists when $|N| = 3$ and at least one agent has an MMS-feasible valuation.

- A simple and constructive proof.

First Main Result of the Paper

Definition

A valuation $v : 2^M \to \mathbb{R}_{\geq 0}$ is said to be **MMS-feasible** iff

$$\max\{v(X), v(S \setminus X)\} \geq \min\{v(Y), v(S \setminus Y)\} \quad \forall X, Y \subseteq S \subseteq M.$$

Additive $\subsetneq$ Nice-cancellable $\subsetneq$ MMS-feasible $\subsetneq$ Monotone.

Theorem (Akrami et al. [2025])

An EFX allocation exists when $|N| = 3$ and at least one agent has an MMS-feasible valuation.

- A simple and constructive proof.
- Transition between allocations to improve a certain potential.

First Main Result of the Paper

Definition

A valuation $v : 2^M \to \mathbb{R}_{\geq 0}$ is said to be **MMS-feasible** iff

$$\max\{v(X), v(S \setminus X)\} \geq \min\{v(Y), v(S \setminus Y)\} \quad \forall X, Y \subseteq S \subseteq M.$$

Additive $\subsetneq$ Nice-cancellable $\subsetneq$ MMS-feasible $\subsetneq$ Monotone.

Theorem (Akrami et al. [2025])

An EFX allocation exists when $|N| = 3$ and at least one agent has an MMS-feasible valuation.

- A simple and constructive proof.
- Transition between allocations to improve a certain potential.
- Doing away with intricate concepts in previous work.

Additional Definitions

Definition

For each relation $\star \in \{\leq, \geq, <, >\}$ over $\mathbb{R}$ and an agent $i \in N$, we let $\star_i$ denote the binary relation over 2^M s.t.

$$X \star_i Y \quad \Leftrightarrow \quad v_i(X) \star v_i(Y) \qquad\qquad \forall X, Y \subseteq M.$$

In inequalities with any $\star_i \in \{\leq_i, \geq_i, <_i, >_i\}$ for any $i \in N$, we let $\max$ and $\min$ denote the maximum and minimum according to $\star_i$, respectively.

Additional Definitions

Definition

For each relation $\star \in \{\leq, \geq, <, >\}$ over $\mathbb{R}$ and an agent $i \in N$, we let $\star_i$ denote the binary relation over 2^M s.t.

$$X \star_i Y \quad \Leftrightarrow \quad v_i(X) \star v_i(Y) \qquad\qquad \forall X, Y \subseteq M.$$

In inequalities with any $\star_i \in \{\leq_i, \geq_i, <_i, >_i\}$ for any $i \in N$, we let $\max$ and $\min$ denote the maximum and minimum according to $\star_i$, respectively.

Definition

In a partition $(X_1, X_2, \ldots, X_n)$ of M, a bundle X_k is said to be **EFX-feasible** for an agent $i \in N$ iff

$$X_k \geq_i \max_{j \in \{1,2,\ldots,n\}} \max_{g \in X_j} (X_j \setminus \{g\}).$$

Reduction to Non-Degenerate Instances

Definition

An instance $(N, M, (v_i)_{i \in N})$ is said to be **non-degenerate** iff each valuation v_i is injective, i.e.,

$$X \neq Y \quad \Rightarrow \quad v_i(X) \neq v_i(Y) \qquad\qquad \forall X, Y \subseteq M, \forall i \in N.$$

Reduction to Non-Degenerate Instances

Definition

An instance $(N, M, (v_i)_{i \in N})$ is said to be **non-degenerate** iff each valuation v_i is injective, i.e.,

$$X \neq Y \quad \Rightarrow \quad v_i(X) \neq v_i(Y) \qquad\qquad \forall X, Y \subseteq M, \forall i \in N.$$

Lemma (Akrami et al. [2025])

For any instance $\mathcal{I} = (N, M, (v_i)_{i \in N})$, one can construct a non-degenerate instance $\tilde{\mathcal{I}} = (N, M, (\tilde{v}_i)_{i \in N})$ such that an allocation X is EFX for $\mathcal{I}$ if it is EFX for $\tilde{\mathcal{I}}$.

Reduction to Non-Degenerate Instances

Definition

An instance $(N, M, (v_i)_{i \in N})$ is said to be **non-degenerate** iff each valuation v_i is injective, i.e.,

$$X \neq Y \quad \Rightarrow \quad v_i(X) \neq v_i(Y) \qquad\qquad \forall X, Y \subseteq M, \forall i \in N.$$

Lemma (Akrami et al. [2025])

For any instance $\mathcal{I} = (N, M, (v_i)_{i \in N})$, one can construct a non-degenerate instance $\tilde{\mathcal{I}} = (N, M, (\tilde{v}_i)_{i \in N})$ such that an allocation X is EFX for $\mathcal{I}$ if it is EFX for $\tilde{\mathcal{I}}$.

In what follows, we consider an arbitrary non-degenerate instance $\mathcal{I} = (N := \{1, 2, 3\}, M, (v_1, v_2, v_3))$ where v_3 is MMS-feasible.

Potential & Invariants

We keep updating a partition $X = (X_1, X_2, X_3)$ of M along with the following potential and invariants.

Potential & Invariants

We keep updating a partition $X = (X_1, X_2, X_3)$ of M along with the following potential and invariants.

Potential

$$\phi(X) \coloneqq \min \{v_1(X_1), v_1(X_2)\}.$$

Potential & Invariants

We keep updating a partition $X = (X_1, X_2, X_3)$ of M along with the following potential and invariants.

Potential

$$\phi(X) := \min \{v_1(X_1), v_1(X_2)\}.$$

It suffices to show that $\phi(X)$ can be strictly increased when X does not induce any EFX allocation.

Potential & Invariants

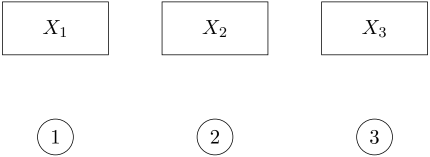
We keep updating a partition $X = (X_1, X_2, X_3)$ of M along with the following potential and invariants.

Potential

$$\phi(X) := \min\{v_1(X_1), v_1(X_2)\}.$$

It suffices to show that $\phi(X)$ can be strictly increased when X does not induce any EFX allocation.

Invariants



Potential & Invariants

We keep updating a partition $X = (X_1, X_2, X_3)$ of M along with the following potential and invariants.

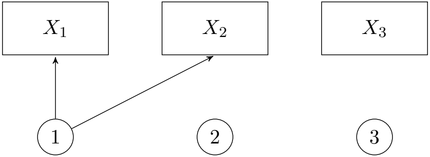
Potential

$$\phi(X) := \min \{v_1(X_1), v_1(X_2)\} .$$

It suffices to show that $\phi(X)$ can be strictly increased when X does not induce any EFX allocation.

Invariants

- Both X_1 and X_2 are EFX-feasible for agent 1.



Potential & Invariants

We keep updating a partition $X = (X_1, X_2, X_3)$ of M along with the following potential and invariants.

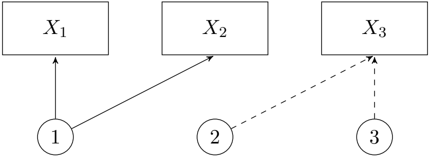
Potential

$$\phi(X) := \min\{v_1(X_1), v_1(X_2)\}.$$

It suffices to show that $\phi(X)$ can be strictly increased when X does not induce any EFX allocation.

Invariants

- Both X_1 and X_2 are EFX-feasible for agent 1.
- X_3 is EFX-feasible for at least one of agents 2 and 3.



Initialization

Theorem (Plaut and Roughgarden [2020])

Given any non-degenerate instance $\mathcal{I} = (N, M, (v)_{i \in N})$ with identical valuations and any non-EFX allocation $(A_i)_{i \in N}$ for $\mathcal{I}$, one can compute an allocation $(B_i)_{i \in N}$ such that $\min_{i \in N} v(A_i) < \min_{i \in N} v(B_i)$.

Their method is referred to as the **PR algorithm** for repeated use.

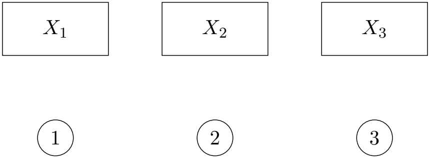
Initialization

Theorem (Plaut and Roughgarden [2020])

Given any non-degenerate instance $\mathcal{I} = (N, M, (v)_{i \in N})$ with identical valuations and any non-EFX allocation $(A_i)_{i \in N}$ for $\mathcal{I}$, one can compute an allocation $(B_i)_{i \in N}$ such that $\min_{i \in N} v(A_i) < \min_{i \in N} v(B_i)$.

Their method is referred to as the **PR algorithm** for repeated use.

Now, we initialize the partition (X_1, X_2, X_3), which enjoys the invariants:



Initialization

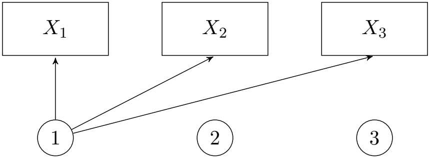
Theorem (Plaut and Roughgarden [2020])

Given any non-degenerate instance $\mathcal{I} = (N, M, (v)_{i \in N})$ with identical valuations and any non-EFX allocation $(A_i)_{i \in N}$ for $\mathcal{I}$, one can compute an allocation $(B_i)_{i \in N}$ such that $\min_{i \in N} v(A_i) < \min_{i \in N} v(B_i)$.

Their method is referred to as the **PR algorithm** for repeated use.

Now, we initialize the partition (X_1, X_2, X_3), which enjoys the invariants:

1. Obtain a partition (X_1, X_2, X_3) s.t. all the bundles are EFX-feasible for agent 1, using the PR algorithm.



Initialization

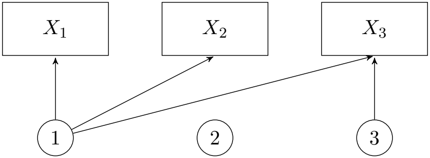
Theorem (Plaut and Roughgarden [2020])

Given any non-degenerate instance $\mathcal{I} = (N, M, (v)_{i \in N})$ with identical valuations and any non-EFX allocation $(A_i)_{i \in N}$ for $\mathcal{I}$, one can compute an allocation $(B_i)_{i \in N}$ such that $\min_{i \in N} v(A_i) < \min_{i \in N} v(B_i)$.

Their method is referred to as the **PR algorithm** for repeated use.

Now, we initialize the partition (X_1, X_2, X_3), which enjoys the invariants:

1. Obtain a partition (X_1, X_2, X_3) s.t. all the bundles are EFX-feasible for agent 1, using the PR algorithm.
2. Assume w.l.o.g. that X_3 is the most valuable for agent 3.



An Immediate Case

Lemma

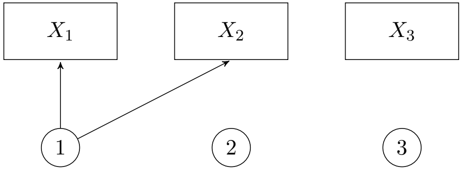
Let a partition X satisfy the invariants. If either X_1 or X_2 is EFX-feasible for either agent 2 or 3 in X, we can obtain an EFX allocation from X.

An Immediate Case

Lemma

Let a partition X satisfy the invariants. If either X_1 or X_2 is EFX-feasible for either agent 2 or 3 in X, we can obtain an EFX allocation from X.

Proof.



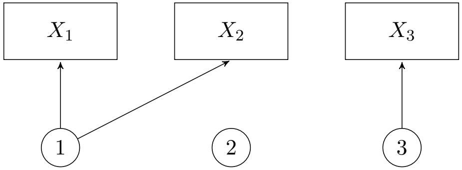
An Immediate Case

Lemma

Let a partition X satisfy the invariants. If either X_1 or X_2 is EFX-feasible for either agent 2 or 3 in X, we can obtain an EFX allocation from X.

Proof.

Suppose w.l.o.g. that X_3 is EFX-feasible for agent 3.



An Immediate Case

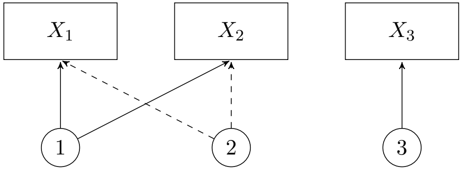
Lemma

Let a partition X satisfy the invariants. If either X_1 or X_2 is EFX-feasible for either agent 2 or 3 in X, we can obtain an EFX allocation from X.

Proof.

Suppose w.l.o.g. that X_3 is EFX-feasible for agent 3.

- If either X_1 or X_2 is EFX-feasible for agent 2, assign bundle X_3 to agent 3, and let agent 2 pick one of X_1 and X_2.
- Otherwise, if either X_1 or X_2 is EFX-feasible for agent 3, let agent 2 pick any bundle and agent 1 then pick one of the rest. □



An Immediate Case

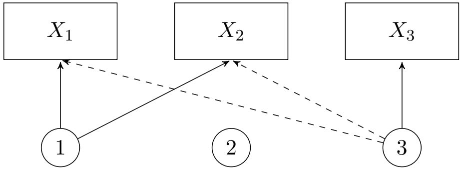
Lemma

Let a partition X satisfy the invariants. If either X_1 or X_2 is EFX-feasible for either agent 2 or 3 in X, we can obtain an EFX allocation from X.

Proof.

Suppose w.l.o.g. that X_3 is EFX-feasible for agent 3.

- If either X_1 or X_2 is EFX-feasible for agent 2, assign bundle X_3 to agent 3, and let agent 2 pick one of X_1 and X_2.
- Otherwise, if either X_1 or X_2 is EFX-feasible for agent 3, let agent 2 pick any bundle and agent 1 then pick one of the rest. □



Further Case Work

Due to the previous lemma, we assume that neither X_1 nor X_2 is EFX-feasible for agent 2 or 3 in X, where the following is observed:

Further Case Work

Due to the previous lemma, we assume that neither X_1 nor X_2 is EFX-feasible for agent 2 or 3 in X, where the following is observed:

Lemma

For each $i \in \{2, 3\}$, it holds under the above assumption that

$$X_3 \setminus \{g_i\} >_i \max\{X_1, X_2\},$$

where g_i denotes the good $g \in X_3$ that maximizes $v_i(X_3 \setminus \{g\})$.

Further Case Work

Due to the previous lemma, we assume that neither X_1 nor X_2 is EFX-feasible for agent 2 or 3 in X, where the following is observed:

Lemma

For each $i \in \{2, 3\}$, it holds under the above assumption that

$$X_3 \setminus \{g_i\} >_i \max\{X_1, X_2\},$$

where g_i denotes the good $g \in X_3$ that maximizes $v_i(X_3 \setminus \{g\})$.

Proof.

Let $i \in \{2, 3\}$ be arbitrary, and suppose w.l.o.g. that $X_1 \geq_i X_2$. As X_1 is not EFX-feasible for agent i in X, it then holds that $X_1 <_i X_3 \setminus \{g_i\}$. $\square$

Further Case Work

Due to the previous lemma, we assume that neither X_1 nor X_2 is EFX-feasible for agent 2 or 3 in X, where the following is observed:

Lemma

For each $i \in \{2, 3\}$, it holds under the above assumption that

$$X_3 \setminus \{g_i\} >_i \max\{X_1, X_2\},$$

where g_i denotes the good $g \in X_3$ that maximizes $v_i(X_3 \setminus \{g\})$.

Proof.

Let $i \in \{2, 3\}$ be arbitrary, and suppose w.l.o.g. that $X_1 \geq_i X_2$. As X_1 is not EFX-feasible for agent i in X, it then holds that $X_1 <_i X_3 \setminus \{g_i\}$. □

Suppose w.l.o.g. that $X_1 \leq_1 X_2$. It remains to discuss the following cases:

Case 1: $X_3 \setminus \{g_i\} >_i X_1 \cup \{g_i\}$ for agent $i = 2$ or $i = 3$.

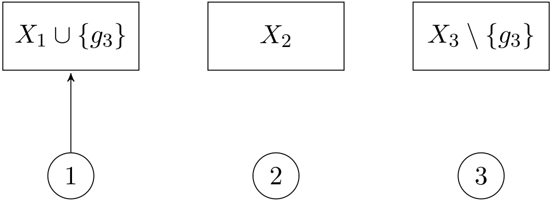
Case 2: $X_3 \setminus \{g_i\} \leq_i X_1 \cup \{g_i\}$ for each agent $i \in \{2, 3\}$.

Case 1: $X_3 \setminus \{g_i\} >_i X_1 \cup \{g_i\}$ for agent $i = 2$ or $i = 3$.

Suppose w.l.o.g. that $X_3 \setminus \{g_3\} >_3 X_1 \cup \{g_3\}$.

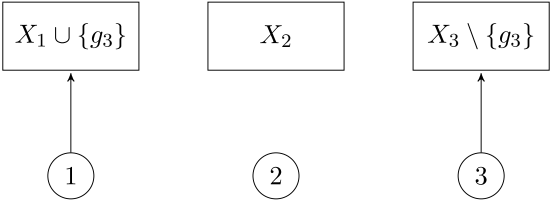
Case 1: $X_3 \setminus \{g_i\} >_i X_1 \cup \{g_i\}$ for agent $i = 2$ or $i = 3$.

Suppose w.l.o.g. that $X_3 \setminus \{g_3\} >_3 X_1 \cup \{g_3\}$.



Case 1: $X_3 \setminus \{g_i\} >_i X_1 \cup \{g_i\}$ for agent $i = 2$ or $i = 3$.

Suppose w.l.o.g. that $X_3 \setminus \{g_3\} >_3 X_1 \cup \{g_3\}$. Together with the previous lemma, we see that $X_3 \setminus \{g_3\}$ is EFX-feasible for agent 3.



Let X_1' be a minimal subset of $X_1 \cup \{g_3\}$ that agent 1 finds more valuable than X_1. Let also $X_2' := X_2$ and $X_3' := M \setminus (X_1' \cup X_2') = (X_1 \cup X_3) \setminus X_1'$.

Case 1: $X_3 \setminus \{g_i\} >_i X_1 \cup \{g_i\}$ for agent $i = 2$ or $i = 3$.

Let X_1' be a minimal subset of $X_1 \cup \{g_3\}$ that agent 1 finds more valuable than X_1. Let also $X_2' := X_2$ and $X_3' := M \setminus (X_1' \cup X_2') = (X_1 \cup X_3) \setminus X_1'$.

Case 1: $X_3 \setminus \{g_i\} >_i X_1 \cup \{g_i\}$ for agent $i = 2$ or $i = 3$.

Let X_1' be a minimal subset of $X_1 \cup \{g_3\}$ that agent 1 finds more valuable than X_1. Let also $X_2' := X_2$ and $X_3' := M \setminus (X_1' \cup X_2') = (X_1 \cup X_3) \setminus X_1'$.

Lemma

$$X_1' >_1 X_2' \setminus \{g\} \qquad \forall g \in X_2',$$
$$X_2' \geq_1 X_1' \setminus \{h\} \qquad \forall h \in X_1'.$$

Case 1: $X_3 \setminus \{g_i\} >_i X_1 \cup \{g_i\}$ for agent $i = 2$ or $i = 3$.

Let X_1' be a minimal subset of $X_1 \cup \{g_3\}$ that agent 1 finds more valuable than X_1. Let also $X_2' := X_2$ and $X_3' := M \setminus (X_1' \cup X_2') = (X_1 \cup X_3) \setminus X_1'$.

Lemma

$$
\begin{aligned}
X_1' &>_1 X_2' \setminus \{g\} && \forall g \in X_2', \\
X_2' &\geq_1 X_1' \setminus \{h\} && \forall h \in X_1'.
\end{aligned}
$$

Proof.

Since both X_1 and X_2 are EFX-feasible for agent 1 in X, it follows that

$$
\begin{aligned}
X_1' &>_1 X_1 \geq_1 X_2 \setminus \{g\} = X_2' \setminus \{g\} && \forall g \in X_2', \\
X_2' &= X_2 \geq_1 X_1 \geq_1 X_1' \setminus \{h\} && \forall h \in X_1'.
\end{aligned}
$$

□

Case 1: $X_3 \setminus \{g_i\} >_i X_1 \cup \{g_i\}$ for agent $i = 2$ or $i = 3$.

Lemma

$$
\begin{aligned}
X_1' &>_1 X_2' \setminus \{g\} && \forall g \in X_2', \\
X_2' &\geq_1 X_1' \setminus \{h\} && \forall h \in X_1'.
\end{aligned}
$$

Case 1: $X_3 \setminus \{g_i\} >_i X_1 \cup \{g_i\}$ for agent $i = 2$ or $i = 3$.

Lemma

$$X'_1 >_1 X'_2 \setminus \{g\} \qquad \forall g \in X'_2,$$
$$X'_2 \geq_1 X'_1 \setminus \{h\} \qquad \forall h \in X'_1.$$

Note that the partition $X' := (X'_1, X'_2, X'_3)$ enjoys that $\phi(X') > \phi(X)$.

Case 1: $X_3 \setminus \{g_i\} >_i X_1 \cup \{g_i\}$ for agent $i = 2$ or $i = 3$.

Lemma

$$\begin{aligned} X_1' &>_1 X_2' \setminus \{g\} && \forall g \in X_2', \\ X_2' &\geq_1 X_1' \setminus \{h\} && \forall h \in X_1'. \end{aligned}$$

Note that the partition $X' \coloneqq (X_1', X_2', X_3')$ enjoys that $\phi(X') > \phi(X)$.

Thus, we are done if both X_1' and X_2' are EFX-feasible for agent 1 in X'.

Case 1: $X_3 \setminus \{g_i\} >_i X_1 \cup \{g_i\}$ for agent $i = 2$ or $i = 3$.

Lemma

$$X_1' >_1 X_2' \setminus \{g\} \qquad \forall g \in X_2',$$
$$X_2' \geq_1 X_1' \setminus \{h\} \qquad \forall h \in X_1'.$$

Note that the partition $X' := (X_1', X_2', X_3')$ enjoys that $\phi(X') > \phi(X)$.

Thus, we are done if both X_1' and X_2' are EFX-feasible for agent 1 in X'.

Otherwise, the lemma implies that there is a good $g \in X_3'$ such that $X_3' \setminus \{g\} >_1 \min\{X_1, X_2\}$.

Case 1: $X_3 \setminus \{g_i\} >_i X_1 \cup \{g_i\}$ for agent $i = 2$ or $i = 3$.

Lemma

$$\begin{aligned} X_1' &>_1 X_2' \setminus \{g\} && \forall g \in X_2', \\ X_2' &\geq_1 X_1' \setminus \{h\} && \forall h \in X_1'. \end{aligned}$$

Note that the partition $X' \coloneqq (X_1', X_2', X_3')$ enjoys that $\phi(X') > \phi(X)$.

Thus, we are done if both X_1' and X_2' are EFX-feasible for agent 1 in X'.

Otherwise, the lemma implies that there is a good $g \in X_3'$ such that $X_3' \setminus \{g\} >_1 \min\{X_1, X_2\}$. The PR algorithm finds a partition $Y = (Y_1, Y_2, Y_3)$ whose bundles are all EFX-feasible for agent 1 in Y and enjoy the second inequality of the following:

Case 1: $X_3 \setminus \{g_i\} >_i X_1 \cup \{g_i\}$ for agent $i = 2$ or $i = 3$.

Lemma

$$X_1' >_1 X_2' \setminus \{g\} \qquad \forall g \in X_2',$$
$$X_2' \geq_1 X_1' \setminus \{h\} \qquad \forall h \in X_1'.$$

Note that the partition $X' := (X_1', X_2', X_3')$ enjoys that $\phi(X') > \phi(X)$.

Thus, we are done if both X_1' and X_2' are EFX-feasible for agent 1 in X'.

Otherwise, the lemma implies that there is a good $g \in X_3'$ such that $X_3' \setminus \{g\} >_1 \min\{X_1, X_2\}$. The PR algorithm finds a partition $Y = (Y_1, Y_2, Y_3)$ whose bundles are all EFX-feasible for agent 1 in Y and enjoy the second inequality of the following:

$$\phi(Y) \geq_1 \min\{Y_1, Y_2, Y_3\} >_1 \min\left\{X_1', X_2', X_3'\right\} = \phi(X') \geq \phi(X).$$

Case 1: $X_3 \setminus \{g_i\} >_i X_1 \cup \{g_i\}$ for agent $i = 2$ or $i = 3$.

Lemma

$$\begin{aligned} X_1' &>_1 X_2' \setminus \{g\} && \forall g \in X_2', \\ X_2' &\geq_1 X_1' \setminus \{h\} && \forall h \in X_1'. \end{aligned}$$

Note that the partition $X' \coloneqq (X_1', X_2', X_3')$ enjoys that $\phi(X') > \phi(X)$.

Thus, we are done if both X_1' and X_2' are EFX-feasible for agent 1 in X'.

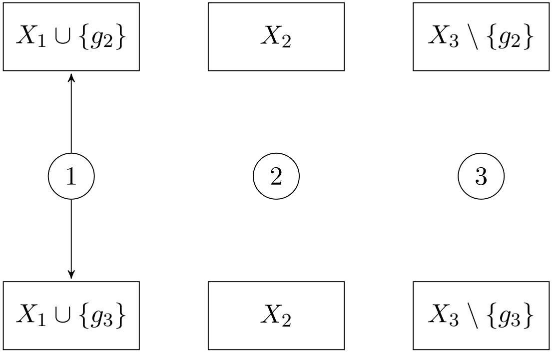
Otherwise, the lemma implies that there is a good $g \in X_3'$ such that $X_3' \setminus \{g\} >_1 \min\{X_1, X_2\}$. The PR algorithm finds a partition $Y = (Y_1, Y_2, Y_3)$ whose bundles are all EFX-feasible for agent 1 in Y and enjoy the second inequality of the following:

$$\phi(Y) \geq_1 \min\{Y_1, Y_2, Y_3\} >_1 \min\left\{X_1', X_2', X_3'\right\} = \phi(X') \geq \phi(X).$$

Suppose w.l.o.g. that agent 3 finds Y_3 the most valuable; then Y satisfies the invariants.

Case 2: $X_3 \setminus \{g_i\} \leq_i X_1 \cup \{g_i\}$ for each agent $i \in \{2, 3\}$.

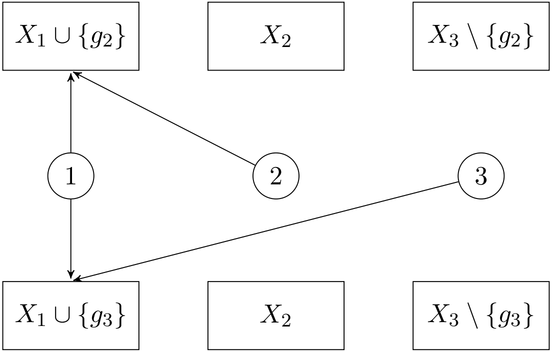
Case 2: $X_3 \setminus \{g_i\} \leq_i X_1 \cup \{g_i\}$ for each agent $i \in \{2, 3\}$.



Case 2: $X_3 \setminus \{g_i\} \leq_i X_1 \cup \{g_i\}$ for each agent $i \in \{2,3\}$.

Since we've shown that $X_3 \setminus \{g_i\} >_i \max\{X_1, X_2\}$ for each $i \in \{2,3\}$, it follows that

$$X_2 \leq_i X_3 \setminus \{g_i\} \leq_i X_1 \cup \{g_i\} \qquad \forall i \in \{2,3\}.$$



Case 2: $X_3 \setminus \{g_i\} \leq_i X_1 \cup \{g_i\}$ for each agent $i \in \{2, 3\}$.

$$X_2 \leq_i X_3 \setminus \{g_i\} \leq_i X_1 \cup \{g_i\} \qquad \forall i \in \{2, 3\}. \tag{1}$$

Case 2: $X_3 \setminus \{g_i\} \leq_i X_1 \cup \{g_i\}$ for each agent $i \in \{2, 3\}$.

$$X_2 \leq_i X_3 \setminus \{g_i\} \leq_i X_1 \cup \{g_i\} \qquad \forall i \in \{2, 3\}. \tag{1}$$

The PR algorithm finds a partition $X' = (X_2, Y_2, Y_3)$ of M such that

$$\min\{X_1 \cup \{g_2\}, X_3 \setminus \{g_2\}\} \leq_2 \min\{Y_2, Y_3\}, \tag{2}$$

$$Y_2 \leq_3 Y_3. \tag{3}$$

Case 2: $X_3 \setminus \{g_i\} \leq_i X_1 \cup \{g_i\}$ for each agent $i \in \{2,3\}$.

$$X_2 \leq_i X_3 \setminus \{g_i\} \leq_i X_1 \cup \{g_i\} \qquad \forall i \in \{2,3\}. \tag{1}$$

The PR algorithm finds a partition $X' = (X_2, Y_2, Y_3)$ of M such that

$$\min\{X_1 \cup \{g_2\}, X_3 \setminus \{g_2\}\} \leq_2 \min\{Y_2, Y_3\}, \tag{2}$$

$$Y_2 \leq_3 Y_3. \tag{3}$$

Lemma

For each $i \in \{2,3\}$, Y_i is EFX-feasible for agent i in X'.

Case 2: $X_3 \setminus \{g_i\} \leq_i X_1 \cup \{g_i\}$ for each agent $i \in \{2,3\}$.

$$X_2 \leq_i X_3 \setminus \{g_i\} \leq_i X_1 \cup \{g_i\} \qquad \forall i \in \{2,3\}. \tag{1}$$

The PR algorithm finds a partition $X' = (X_2, Y_2, Y_3)$ of M such that

$$\min\{X_1 \cup \{g_2\}, X_3 \setminus \{g_2\}\} \leq_2 \min\{Y_2, Y_3\}, \tag{2}$$

$$Y_2 \leq_3 Y_3. \tag{3}$$

Lemma

For each $i \in \{2,3\}$, Y_i is EFX-feasible for agent i in X'.

Proof.

Eqs. (1) and (2) yield that $Y_2 \geq_2 X_3 \setminus \{g\}$ for any $g \in X_3$, and that

$$Y_2 \geq_2 \min\{X_1 \cup \{g_2\}, X_3 \setminus \{g_2\}\} = X_3 \setminus \{g_2\} \geq_2 X_2,$$

implying the claim for $i = 2$.

□

Case 2: $X_3 \setminus \{g_i\} \leq_i X_1 \cup \{g_i\}$ for each agent $i \in \{2, 3\}$.

$$X_2 \leq_i X_3 \setminus \{g_i\} \leq_i X_1 \cup \{g_i\} \qquad \forall i \in \{2, 3\}. \tag{1}$$

The PR algorithm finds a partition $X' = (X_2, Y_2, Y_3)$ of M such that

$$\min\{X_1 \cup \{g_2\}, X_3 \setminus \{g_2\}\} \leq_2 \min\{Y_2, Y_3\}, \tag{2}$$

$$Y_2 \leq_3 Y_3. \tag{3}$$

Lemma

For each $i \in \{2, 3\}$, Y_i is EFX-feasible for agent i in X'.

Proof.

Eqs. (1) and (2) yield that $Y_2 \geq_2 X_3 \setminus \{g\}$ for any $g \in X_3$, and that

$$Y_2 \geq_2 \min\{X_1 \cup \{g_2\}, X_3 \setminus \{g_2\}\} = X_3 \setminus \{g_2\} \geq_2 X_2,$$

implying the claim for $i = 2$. Eq. (3) and MMS-feasibility of v_3 give that

$$Y_3 \geq_3 \min\{Y_2, Y_3\} \geq_3 \min\{X_1 \cup \{g_3\}, X_3 \setminus \{g_3\}\} \geq_3 X_2.$$

Combining this with Eq. (3) leads to EFX-feasibility of Y_3 for agent 3. □

Case 2: $X_3 \setminus \{g_i\} \leq_i X_1 \cup \{g_i\}$ for each agent $i \in \{2, 3\}$.

$$X_2 \leq_i X_3 \setminus \{g_i\} \leq_i X_1 \cup \{g_i\} \qquad \forall i \in \{2, 3\}. \tag{1}$$

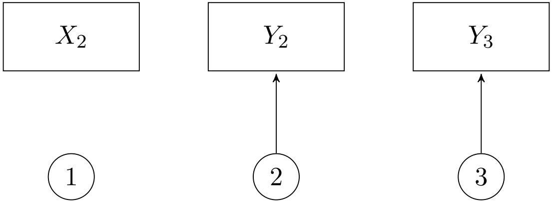
The PR algorithm finds a partition $X' = (X_2, Y_2, Y_3)$ of M such that

$$\min \{X_1 \cup \{g_2\}, X_3 \setminus \{g_2\}\} \leq_2 \min \{Y_2, Y_3\}, \tag{2}$$

$$Y_2 \leq_3 Y_3. \tag{3}$$

Lemma

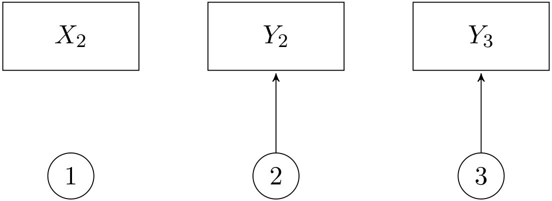
For each $i \in \{2, 3\}$, Y_i is EFX-feasible for agent i in X'.



Case 2: $X_3 \setminus \{g_i\} \leq_i X_1 \cup \{g_i\}$ for each agent $i \in \{2, 3\}$.

Lemma

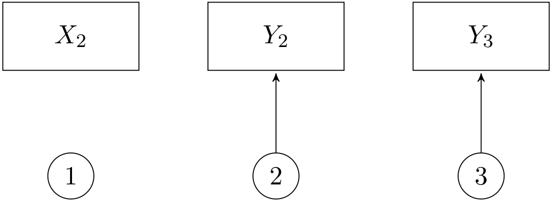
For each $i \in \{2, 3\}$, Y_i is EFX-feasible for agent i in X'.



Case 2: $X_3 \setminus \{g_i\} \leq_i X_1 \cup \{g_i\}$ for each agent $i \in \{2, 3\}$.

Lemma

For each $i \in \{2, 3\}$, Y_i is EFX-feasible for agent i in X'.

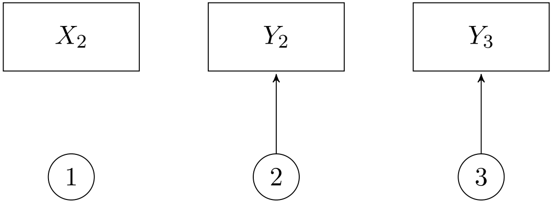


- If X_2 is EFX-feasible for agent 1 in X', we are done.

Case 2: $X_3 \setminus \{g_i\} \leq_i X_1 \cup \{g_i\}$ for each agent $i \in \{2, 3\}$.

Lemma

For each $i \in \{2, 3\}$, Y_i is EFX-feasible for agent i in X'.

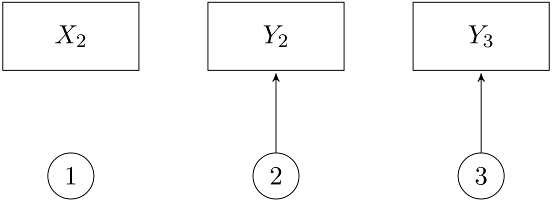


- If X_2 is EFX-feasible for agent 1 in X', we are done.
- Otherwise, it remains to do the following (informally stated):

Case 2: $X_3 \setminus \{g_i\} \leq_i X_1 \cup \{g_i\}$ for each agent $i \in \{2, 3\}$.

Lemma

For each $i \in \{2, 3\}$, Y_i is EFX-feasible for agent i in X'.

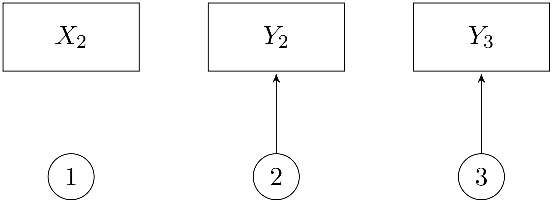


- If X_2 is EFX-feasible for agent 1 in X', we are done.
- Otherwise, it remains to do the following (informally stated):
  1. If $Y_3 \leq_1 X_2$, move some goods from Y_2 to Y_3.

Case 2: $X_3 \setminus \{g_i\} \leq_i X_1 \cup \{g_i\}$ for each agent $i \in \{2, 3\}$.

Lemma

For each $i \in \{2, 3\}$, Y_i is EFX-feasible for agent i in X'.

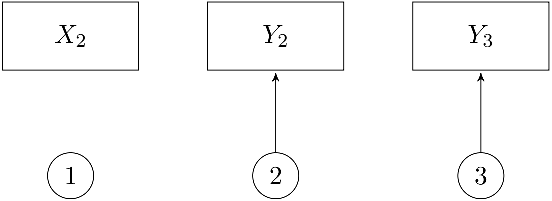


- If X_2 is EFX-feasible for agent 1 in X', we are done.
- Otherwise, it remains to do the following (informally stated):
  1. If $Y_3 \leq_1 X_2$, move some goods from Y_2 to Y_3.
  2. Apply the PR algorithm in terms of v_1.

Case 2: $X_3 \setminus \{g_i\} \leq_i X_1 \cup \{g_i\}$ for each agent $i \in \{2,3\}$.

Lemma

For each $i \in \{2,3\}$, Y_i is EFX-feasible for agent i in X'.



- If X_2 is EFX-feasible for agent 1 in X', we are done.
- Otherwise, it remains to do the following (informally stated):
  1. If $Y_3 \leq_1 X_2$, move some goods from Y_2 to Y_3.
  2. Apply the PR algorithm in terms of v_1.
  3. Let agent 2 pick their favorite bundle.

EFX with Charity

It has been shown that a partial EFX allocation exists with the following number of unallocated goods:

EFX with Charity

It has been shown that a partial EFX allocation exists with the following number of unallocated goods:

	Charity
Chaudhury et al. [2021b]	$n-1$
Berger et al. [2022]	$n-2$

EFX with Charity

It has been shown that a partial EFX allocation exists with the following number of unallocated goods:

	Charity
Chaudhury et al. [2021b]	$n-1$
Berger et al. [2022]	$n-2$

Q. Does an EFX allocation with a sublinear charity exist?
— An open question.
— But yes, if an approximation is allowed.

Rainbow Cycle Number

Definition

For each integer $d > 0$, the **rainbow cycle number** $R(d)$ denotes the largest integer k such that there exists a k-partite directed graph $G = (V_1 \cup V_2 \cup \cdots \cup V_k, E)$ that satisfies the following:

- For every $i \in \{1, 2, \ldots, k\}$, $1 \leq |V_i| \leq d$.
- For every $i, j \in \{1, 2, \ldots, k\}$ with $i \neq j$, each vertex in V_i has an incoming edge from some vertex in V_j.
- There is no cycle that visits each part at most once.

Rainbow Cycle Number

Definition

For each integer $d > 0$, the **rainbow cycle number** $R(d)$ denotes the largest integer k such that there exists a k-partite directed graph $G = (V_1 \cup V_2 \cup \cdots \cup V_k, E)$ that satisfies the following:

- For every $i \in \{1, 2, \ldots, k\}$, $1 \leq |V_i| \leq d$.
- For every $i, j \in \{1, 2, \ldots, k\}$ with $i \neq j$, each vertex in V_i has an incoming edge from some vertex in V_j.
- There is no cycle that visits each part at most once.

Previous work shows the following reduction to a problem in graph theory.

Theorem (Chaudhury et al. [2021a])

Let $\epsilon \in \left(0, \frac{1}{2}\right]$ be arbitrary. For any instance with n agents, there is a $(1-\epsilon)$-EFX allocation with $O\left(\frac{n}{\epsilon\, d_{n,\epsilon}}\right)$ unallocated goods, where $d_{n,\epsilon}$ denotes the smallest integer $d > 0$ that enjoys $d\, R(d) \geq \frac{n}{\epsilon}$.

Upper Bounds in Previous Work

Theorem (Chaudhury et al. [2021a])

Let $\epsilon \in \left(0, \frac{1}{2}\right]$ be arbitrary. For any instance with n agents, there is a $(1-\epsilon)$-EFX partial allocation with $O\left(\frac{n}{\epsilon\, d_{n,\epsilon}}\right)$ many unallocated goods, where $d_{n,\epsilon}$ denotes the smallest integer $d > 0$ that enjoys $d\, R(d) \geq \frac{n}{\epsilon}$.

Upper bounds on $R(d)$ imply those on the number of unallocated goods.

Upper Bounds in Previous Work

Theorem (Chaudhury et al. [2021a])

Let $\epsilon \in \left(0, \frac{1}{2}\right]$ be arbitrary. For any instance with n agents, there is a $(1-\epsilon)$-EFX partial allocation with $O\left(\frac{n}{\epsilon\, d_{n,\epsilon}}\right)$ many unallocated goods, where $d_{n,\epsilon}$ denotes the smallest integer $d > 0$ that enjoys $d\,R(d) \geq \frac{n}{\epsilon}$.

Upper bounds on $R(d)$ imply those on the number of unallocated goods.

Previous work gives the following upper bounds:

	$R(d)$	Charity
Chaudhury et al. [2021a]	$O\left(d^4\right)$	$O\left(\left(\frac{n}{\epsilon}\right)^{0.8}\right)$
Berendsohn et al. [2022]	$O\left(d^{2+o(1)}\right)$	$O\left(\left(\frac{n}{\epsilon}\right)^{0.67}\right)$

Second Main Result of the Paper

This paper establishes an improved and almost tight upper bound.[2]

[2] It is shown by Chaudhury et al. [2021a] that $R(d) \geq d$.

Second Main Result of the Paper

This paper establishes an improved and almost tight upper bound.[2]

Theorem (Akrami et al. [2025], Jahan et al. [2023])

It holds that $R(d) = O\left(d \log d\right)$.

[2] It is shown by Chaudhury et al. [2021a] that $R(d) \geq d$.

Second Main Result of the Paper

This paper establishes an improved and almost tight upper bound.[2]

Theorem (Akrami et al. [2025], Jahan et al. [2023])

It holds that $R(d) = O\left(d \log d\right)$.

Corollary

Let $\epsilon \in \left(0, \frac{1}{2}\right]$ be arbitrary. For any instance with n agents, there is a $(1-\epsilon)$-EFX partial allocation with $\tilde{O}\left(\left(\frac{n}{\epsilon}\right)^{0.5}\right)$ many unallocated goods.

[2] It is shown by Chaudhury et al. [2021a] that $R(d) \geq d$.

Second Main Result of the Paper

This paper establishes an improved and almost tight upper bound.[2]

Theorem (Akrami et al. [2025], Jahan et al. [2023])

It holds that $R(d) = O\left(d \log d\right)$.

Corollary

Let $\epsilon \in \left(0, \frac{1}{2}\right]$ be arbitrary. For any instance with n agents, there is a $(1-\epsilon)$-EFX partial allocation with $\tilde{O}\left(\left(\frac{n}{\epsilon}\right)^{0.5}\right)$ many unallocated goods.

	$R(d)$	Charity
[Chaudhury et al., 2021a]	$O\left(d^4\right)$	$O\left(\left(\frac{n}{\epsilon}\right)^{0.8}\right)$
[Berendsohn et al., 2022]	$O\left(d^{2+o(1)}\right)$	$O\left(\left(\frac{n}{\epsilon}\right)^{0.67}\right)$
[Akrami et al., 2025, Jahan et al., 2023]	$\tilde{O}\left(d\right)$	$\tilde{O}\left(\left(\frac{n}{\epsilon}\right)^{0.5}\right)$

[2] It is shown by Chaudhury et al. [2021a] that $R(d) \geq d$.

Summary of the Paper

The paper improves the following two fronts towards the fundamental open problem about the existence of EFX allocations:

Summary of the Paper

The paper improves the following two fronts towards the fundamental open problem about the existence of EFX allocations:

A small number of agents:

3 agents and at least one MMS-feasible valuation.

Summary of the Paper

The paper improves the following two fronts towards the fundamental open problem about the existence of EFX allocations:

A small number of agents:

3 agents and at least one MMS-feasible valuation.

Charity (and approximation):

$(1-\epsilon)$-EFX with $O\left(\left(\frac{n}{\epsilon}\right)^{1/2}\right)$ many unallocated goods.

Summary of the Paper

The paper improves the following two fronts towards the fundamental open problem about the existence of EFX allocations:

A small number of agents:

3 agents and at least one MMS-feasible valuation.

Charity (and approximation):

$(1-\epsilon)$-EFX with $O\left(\left(\frac{n}{\epsilon}\right)^{1/2}\right)$ many unallocated goods.

Relevant open problems include the following:

- Existence of EFX allocations for 3 agents with general valuations.
- Existence of EFX allocations for 4 agents with additive valuations.

References

H. Akrami, N. Alon, B. R. Chaudhury, J. Garg, K. Mehlhorn, and R. Mehta. EFX: A simpler approach and an (almost) optimal guarantee via rainbow cycle number. Operations Research, 73(2):738–751, 2025.

B. A. Berendsohn, S. Boyadzhiyska, and L. Kozma. Fixed-point cycles and approximate EFX allocations. In Proceedings of the 47th International Symposium on Mathematical Foundations of Computer Science, page 17, 2022.

B. Berger, A. Cohen, M. Feldman, and A. Fiat. Almost full EFX exists for four agents. In Proceedings of the 36th AAAI Conference on Artificial Intelligence, pages 4826–4833, 2022.

I. Caragiannis, D. Kurokawa, H. Moulin, A. D. Procaccia, N. Shah, and J. Wang. The unreasonable fairness of maximum Nash welfare. ACM Transactions on Economics and Computation, 7(3):1–32, 2019.

B. R. Chaudhury, J. Garg, and K. Mehlhorn. EFX exists for three agents. In Proceedings of the 21st ACM Conference on Economics and Computation, pages 1–19, 2020.

B. R. Chaudhury, J. Garg, K. Mehlhorn, R. Mehta, and P. Misra. Improving EFX guarantees through rainbow cycle number. In Proceedings of the 22nd ACM Conference on Economics and Computation, pages 310–311, 2021a.

B. R. Chaudhury, T. Kavitha, K. Mehlhorn, and A. Sgouritsa. A little charity guarantees almost envy-freeness. SIAM Journal on Computing, 50(4):1336–1358, 2021b.

S. C. Jahan, M. Seddighin, S.-M. Seyed-Javadi, and M. Sharifi. Rainbow cycle number and EFX allocations: (almost) closing the gap. In Proceedings of the 32nd International Joint Conference on Artificial Intelligence, pages 2572–2580, 2023.

B. Plaut and T. Roughgarden. Almost envy-freeness with general valuations. SIAM Journal on Discrete Mathematics, 34(2):1039–1068, 2020.

A. D. Procaccia. Technical perspective: An answer to fair division's most enigmatic question. Communications of the ACM, 63(4):118–118, 2020.