Resolving the Optimal Metric Distortion Conjecture

Armin Schilling

17.06.2025

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- from voter preferences, choose a candidate

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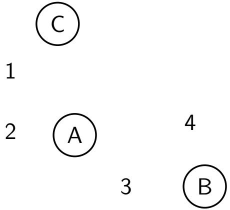

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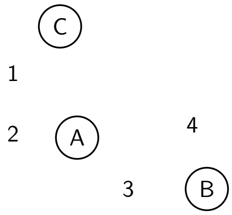

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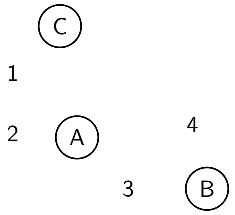

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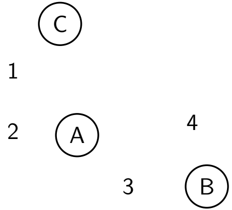

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- object of interest: worst case approximation ratio (distortion)

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- $\mathrm{distortion}(f) = \sup_\sigma \sup_{d \triangleright \sigma} \frac{\mathrm{SC}(f(\sigma), d)}{\min_{c \in C} \mathrm{SC}(c, d)}$

$\Rightarrow$ What is the best possible distortion?

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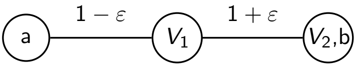

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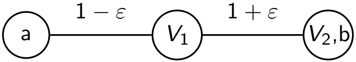

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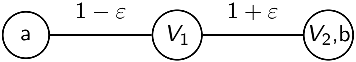

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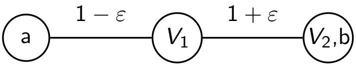

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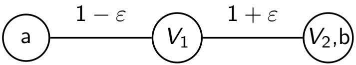

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$\Rightarrow$ distortion$(f) \geq 3$

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Given $\mathcal{E} = (C, V, \sigma)$ and $a \in C$, define the **integral domination graph** as the bipartite graph $G^{\mathcal{E}}(a) = (V, V, E_a)$ with $(i, j) \in E_a$ iff $a \succcurlyeq_i \mathrm{top}(j)$.

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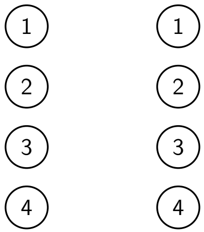

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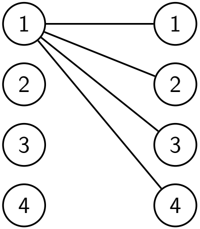

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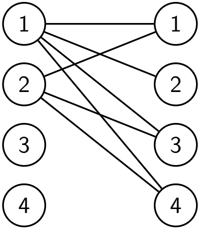

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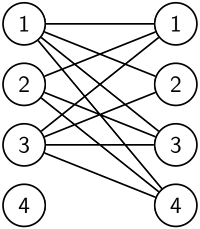

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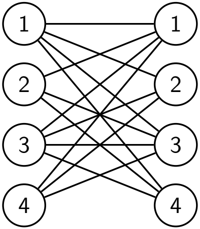

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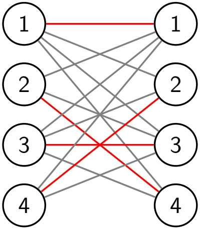

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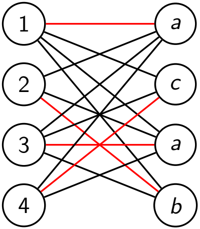

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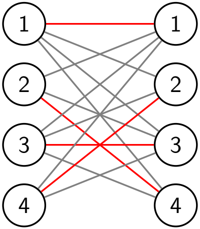

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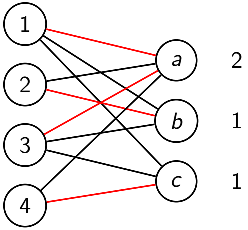

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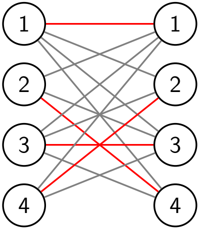

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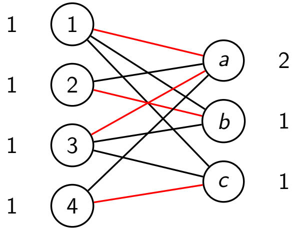

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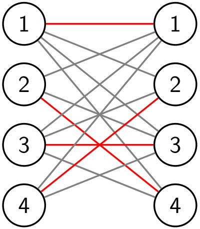

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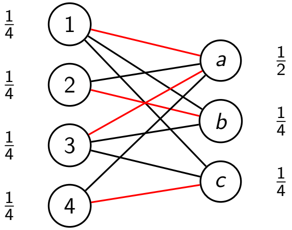

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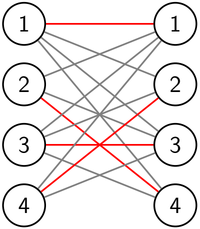

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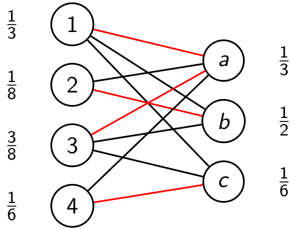

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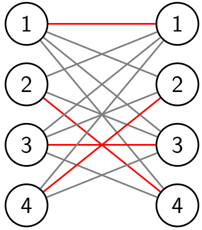

(p,q)-Domination Graph and Fractional Perfect Matching

Definition: (p,q)-Domination Graph

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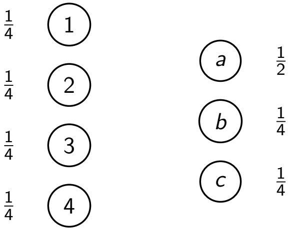

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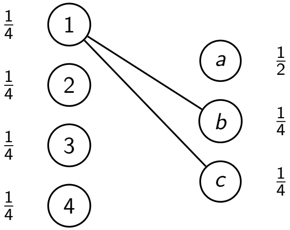

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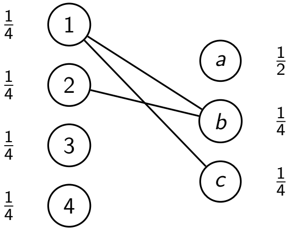

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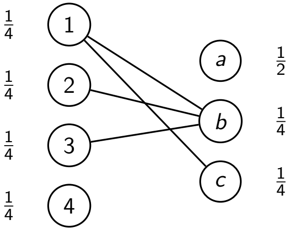

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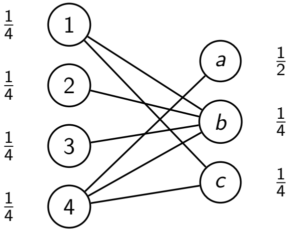

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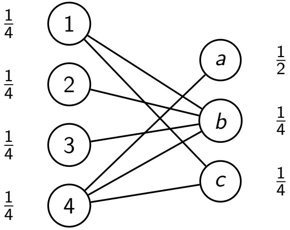

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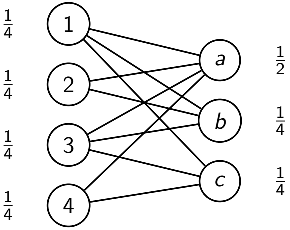

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Define $D^{\mathcal{E}}_a(S) := \{c \in C : \exists i \in S : a \succcurlyeq_i c\}$,

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Given $\mathcal{E} = (V, C, \sigma)$, normalized weight vectors p, q and $a \in C$, then $G^{\mathcal{E}}_{p,q}(a)$ admits a **fractional perfect matching** iff $\forall S \subseteq V : q(D^{\mathcal{E}}_a(S)) \geq p(S)$.

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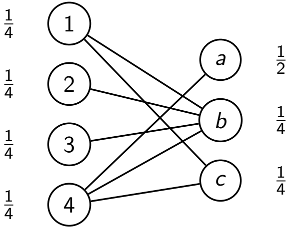
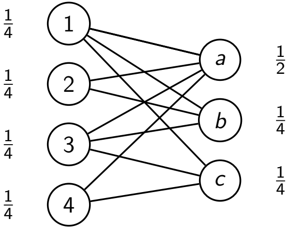
Notice $D^{\mathcal{E}}_a(S)$ is the neighbourhood of S in $G^{\mathcal{E}}_{p,q}(a)$

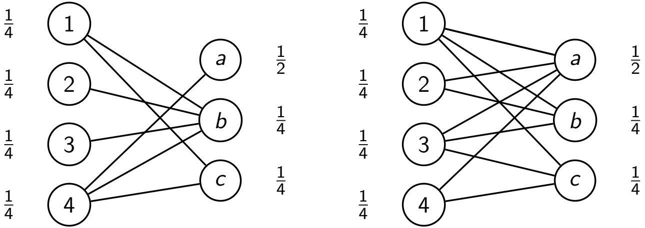
$G^{\mathcal{E}}_{p,q}(b)$



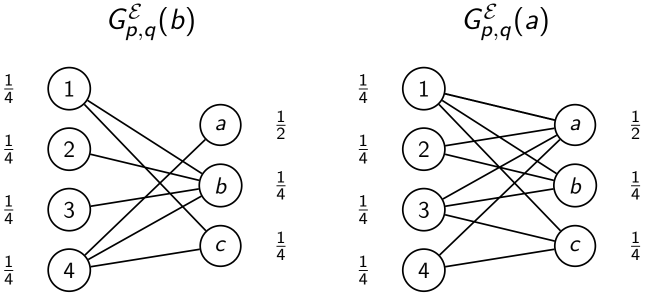
$G^{\mathcal{E}}_{p,q}(a)$



$G_{p,q}^{\mathcal{E}}(b)$ $\qquad$ $G_{p,q}^{\mathcal{E}}(a)$



For **b**:
Let $S = \{2,3\} \to D_b^{\mathcal{E}}(S) = \{b\}$,
$q(D_b^{\mathcal{E}}(S)) = \frac{1}{4} < \frac{1}{2} = p(S)$



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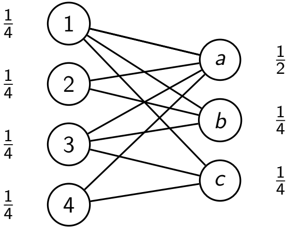
$G^{\mathcal{E}}_{p,q}(b)$ does not allow a fractional perfect matching.

For **a**:

Let $1 \in S$ or $3 \in S \to D_a^{\mathcal{E}}(S) = C$

$\Rightarrow q(C) = 1 \geq p(S)$

$G_{p,q}^{\mathcal{E}}(a)$



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$S_1 = \{2\} \to D_a^{\mathcal{E}}(S_1) = \{a, b\}$

$q(D_a^{\mathcal{E}}(S_1)) = \frac{3}{4} \geq p(S_1) = \frac{1}{4}$

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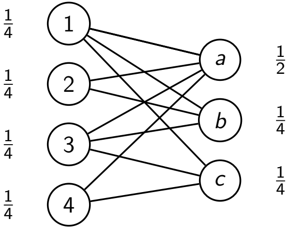

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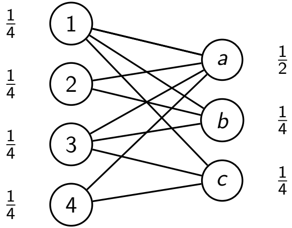

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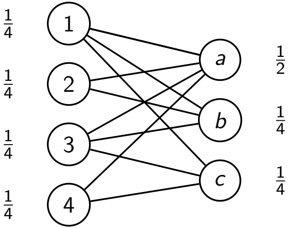
$G_{p,q}^{\mathcal{E}}(a)$



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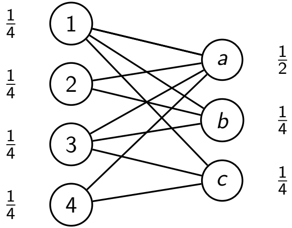
$G_{p,q}^{\mathcal{E}}(a)$ allows a fractional perfect matching.

$G_{p,q}^{\mathcal{E}}(a)$



Ranking-Matching Lemma

Lemma *(Ranking-Matching Lemma)*

For any $\mathcal{E} = (V, C, \sigma)$ and weight vectors p, q there exists some candidate $c \in C$ s.t. $G^{\mathcal{E}}_{p,q}(c)$ admits a fractional perfect matching.
Such a candidate can be found in polynomial time.

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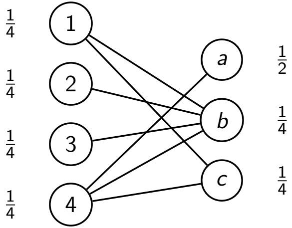
- For every $c \in C$, fix such a minimal counterexample X_c

$G^{\mathcal{E}}_{p,q}(b)$



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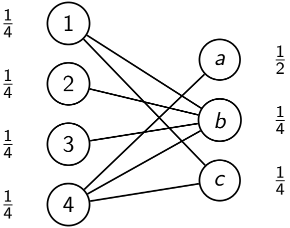

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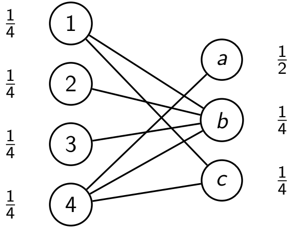

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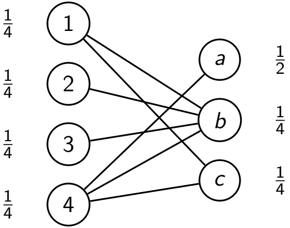

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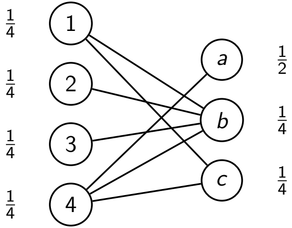

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- fix some maximal a with $\neg \exists c \in C^* : c \succsim a$

We want to show $\forall b \in \overline{D_a^{\mathcal{E}}(X_a)} : X_a \not\subseteq X_b \wedge X_a \not\supseteq X_b$

- fix some b, notice $\forall i \in X_a : b \succ_i a$

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assuming $X_a \subseteq X_b$
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assuming $X_a \supsetneq X_b$
$\rightarrow q(D_b^{\mathcal{E}}(X_b)) \geq p(X_b)$, contradiction!

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We want to show $D_a^{\mathcal{E}}(X_a) \neq \emptyset$, $q(\overline{D_a^{\mathcal{E}}(X_a)}) > 0$, $p(X_a) > 0$ and $\overline{X_a} \neq \emptyset$

We want to show $\forall b \in \overline{D_a^{\mathcal{E}}(X_a)} : X_a \not\subseteq X_b \wedge X_a \not\supseteq X_b$

- fix some b, notice $\forall i \in X_a : b \succ_i a$

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- follows from $\forall b \in \overline{D_a^{\mathcal{E}}(X_a)} : X_a \not\subseteq X_b \wedge X_a \not\supseteq X_b$ and elemental assumptions we made

Using what we have shown before, we get $\forall c \in \overline{D_a^{\mathcal{E}}(X_a)}$:

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which yields the desired contradiction

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Definition: Plurality Matching (PM)

Given $\mathcal{E} = (C, V, \sigma)$, PM returns some $c \in C$ s.t. $G^{\mathcal{E}}_{p^{\text{uni}}, q^{\text{plu}}}(c)$ admits a fractional perfect matching.
Alternatively formulated, PM returns some candidate $c \in C$ s.t. $G^{\mathcal{E}}(c)$ admits a perfect matching.
Ties are broken arbitrarily.

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Theorem 1

PM has distortion 3.

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$$\begin{aligned} \mathrm{SC}(a) &= \sum_{i \in V} d(i, a) \\ &\leq \sum_{i \in V} d(i, \mathrm{top}(M(i))) && \mid a \succcurlyeq_i \mathrm{top}(M(i)) \end{aligned}$$

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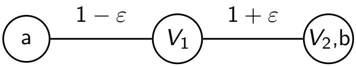
$\Rightarrow \mathrm{SC}(a) \leq 3 \cdot \mathrm{SC}(b)$, i.e. $\mathrm{distortion}(\mathrm{PM}) \leq 3$.

As we already know $\forall f : \mathrm{distortion}(f) \geq 3$ we know $\mathrm{distortion}(\mathrm{PM}) = 3$.

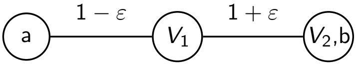
$\square$

α-Decisiveness



α-Decisiveness

example quite unrealistic: half of the voters practically indifferent



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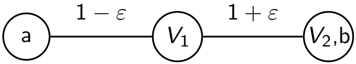

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Let $(V \cup C, d)$ be a metric space.
We call a voter $i \in V$ $\boldsymbol{\alpha}$**-decisive** if $\forall c \neq top(i) : d(i, top(i)) \leq \alpha \cdot d(i, c)$.
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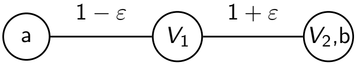

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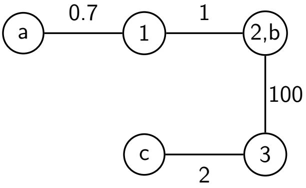

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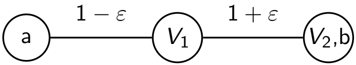

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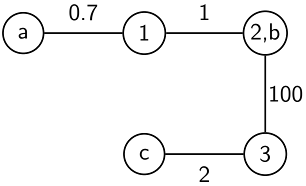

1 : 0.7-decisive
2 : 0-decisive
3 : 0.02-decisive

$\Rightarrow$ metric space is 0.7-decisive

Improvements for PM

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\mathrm{SC}(a) &= \sum_{i \in V} d(i, a) && \\
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\begin{aligned}
\mathrm{SC}(a) &= \sum_{i\in V} d(i,a) \\
&\leq \sum_{i\in V} d(i,\mathrm{top}(M(i))) && \mid a \succcurlyeq_i \mathrm{top}(M(i)) \\
&\leq \sum_{i\in V} (d(i,b) + d(b,\mathrm{top}(M(i)))) && \mid \text{triangle inequality} \\
&= \mathrm{SC}(b) + \sum_{i\in V} d(b,\mathrm{top}(M(i))) \\
&= \mathrm{SC}(b) + \sum_{i\in V} d(b,\mathrm{top}(i)) && \mid M \text{ is a perfect matching} \\
&\leq \mathrm{SC}(b) + \sum_{i\in V} (d(b,i) + d(i,\mathrm{top}(i))) && \mid \text{triangle inequality} \\
&\leq \mathrm{SC}(b) + \sum_{i\in V} (d(b,i) + \alpha \cdot d(i,b)) && \mid \alpha\text{-decisiveness} \\
&= (2+\alpha)\cdot \mathrm{SC}(b)
\end{aligned}
$$

$\Rightarrow \mathrm{SC}(a) \leq (2+\alpha)\cdot\mathrm{SC}(b)$, i.e. $\mathrm{distortion}(\mathrm{PM}) \leq 2+\alpha$. $\square$

A Lower Bound for α-Decisive Metric Spaces

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For every $m \geq 2$ and $\alpha \in [0, 1]$, the distortion of every social choice rule is at least $2 + \alpha - 2\frac{2(1-\alpha)}{\lfloor m \rfloor_{even}}$ for α-decisive metric spaces, where $\lfloor m \rfloor_{even}$ is the largest even integer $\leq m$.

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m many voters and candidates
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split V evenly into V_A and V_B

$$\forall i \in V_A : \quad a_i \succ A \succ B$$

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wlog, return some $a \in A$

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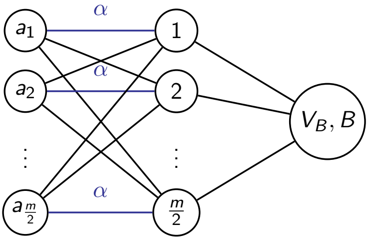

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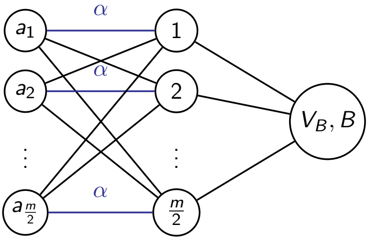

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$\forall a \in A : \mathrm{SC}(a) = \alpha + (\frac{m}{2} - 1) + \frac{m}{2}(1+\alpha)$ and $\forall b \in B : \mathrm{SC}(b) = \frac{m}{2}$ $\square$

Thank you for your attention!