UNIVERSITÄT DES SAARLANDES

Gibbard-Satterthwaite Theorem

(and a bit about Revelation Principle)

Mechanism Design Without Money

Kurt Mehlhorn, **Javier Cembrano**, Golnoosh Shahkarami

April 29, 2025

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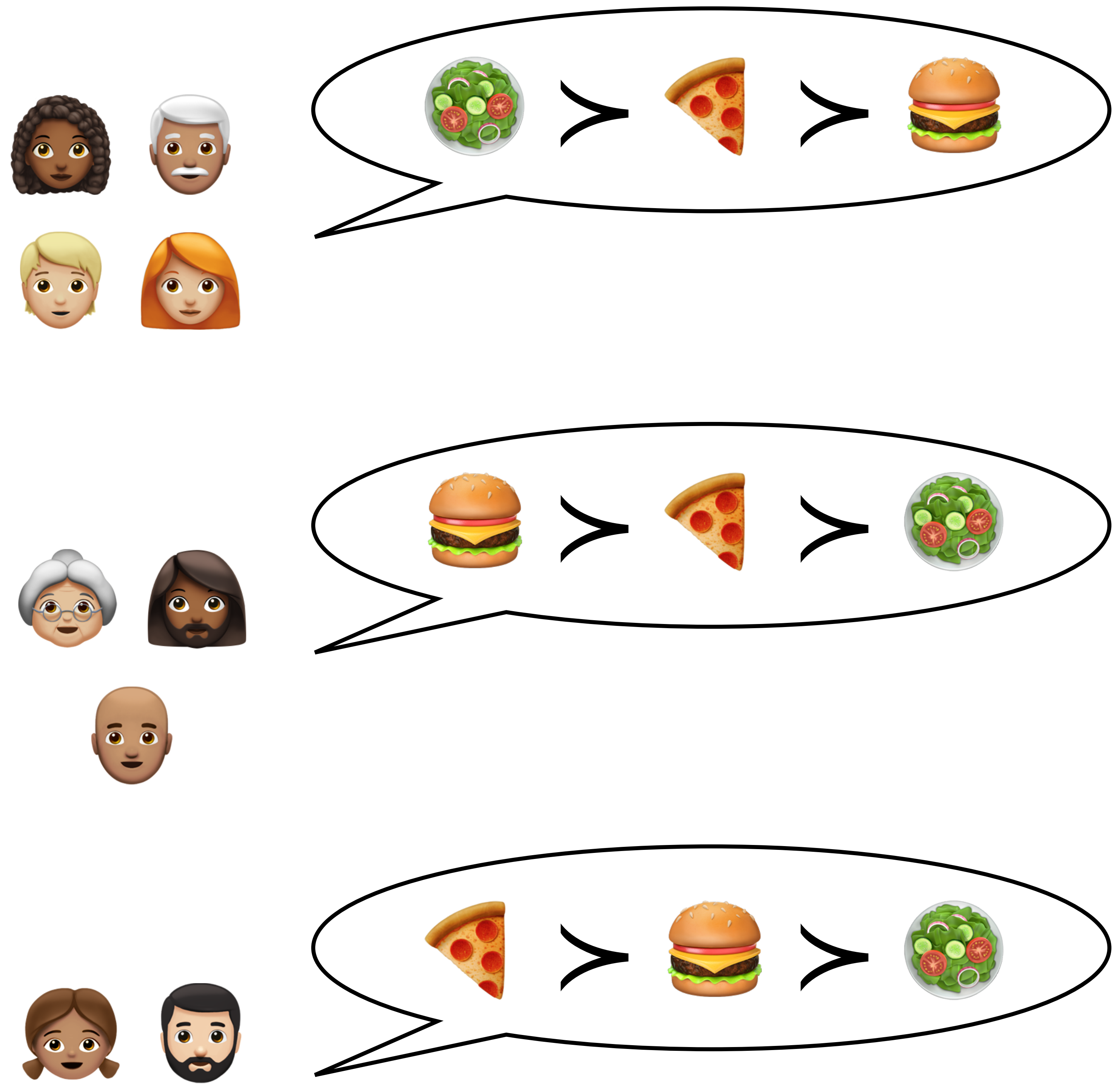

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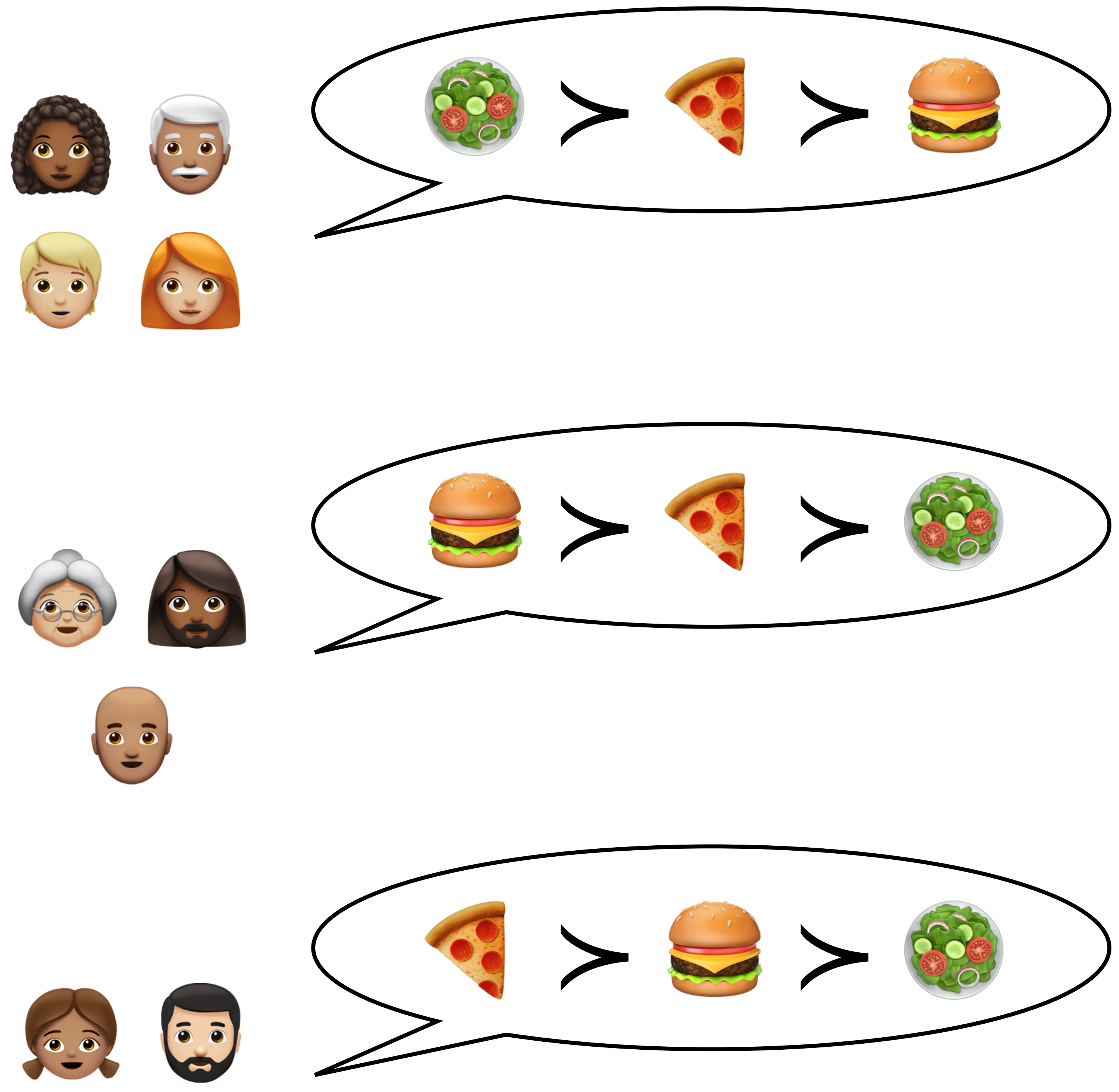

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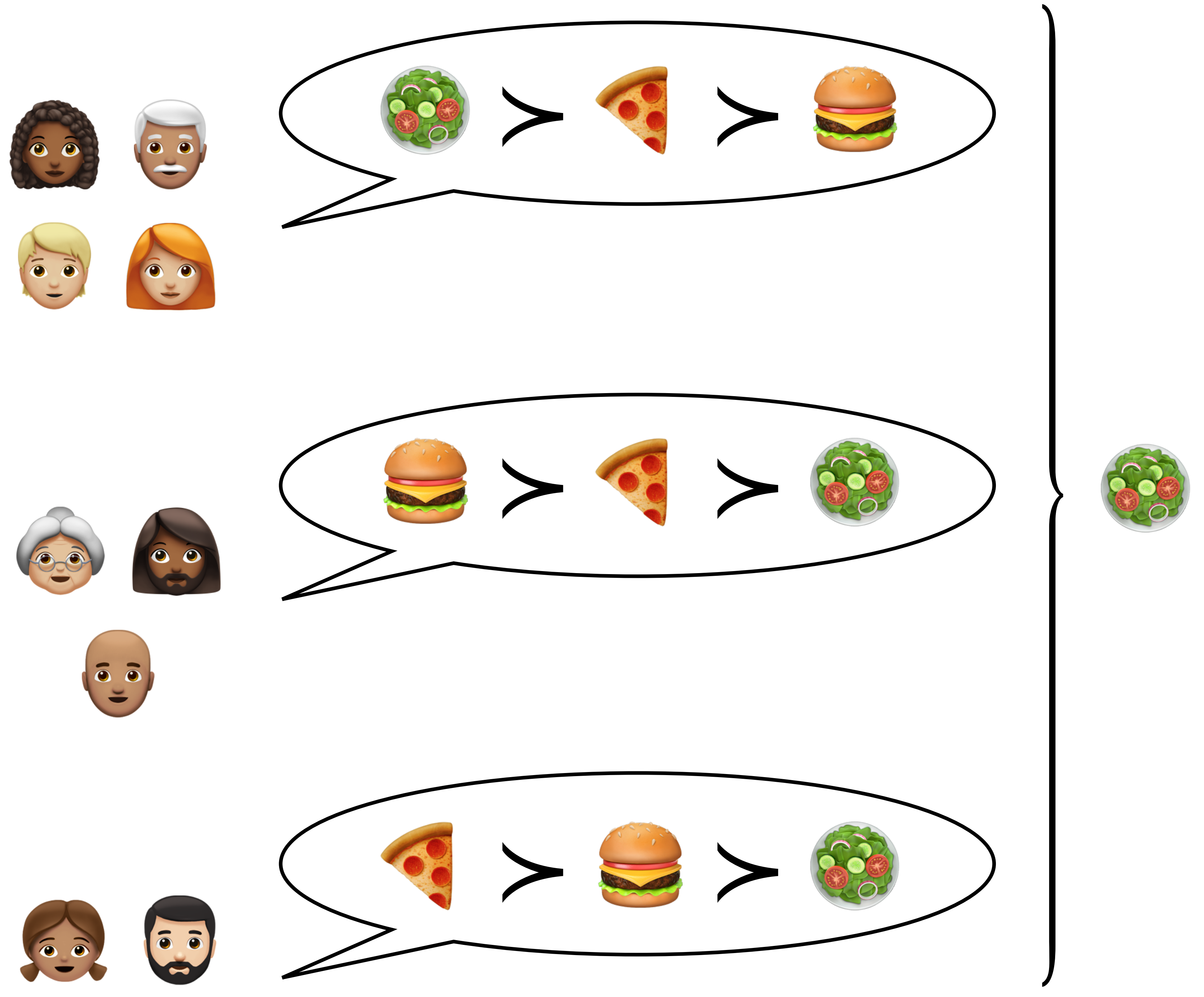

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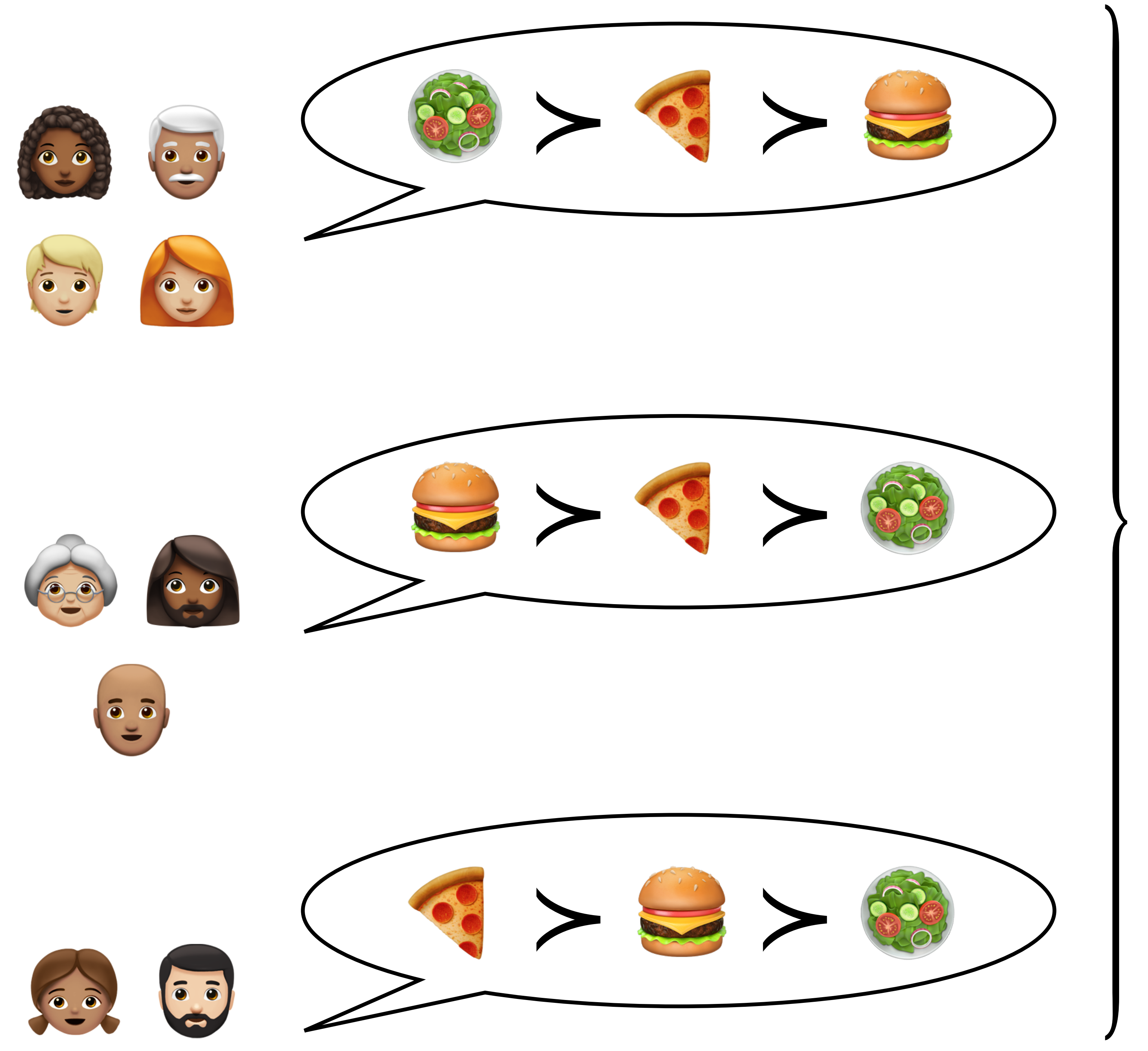

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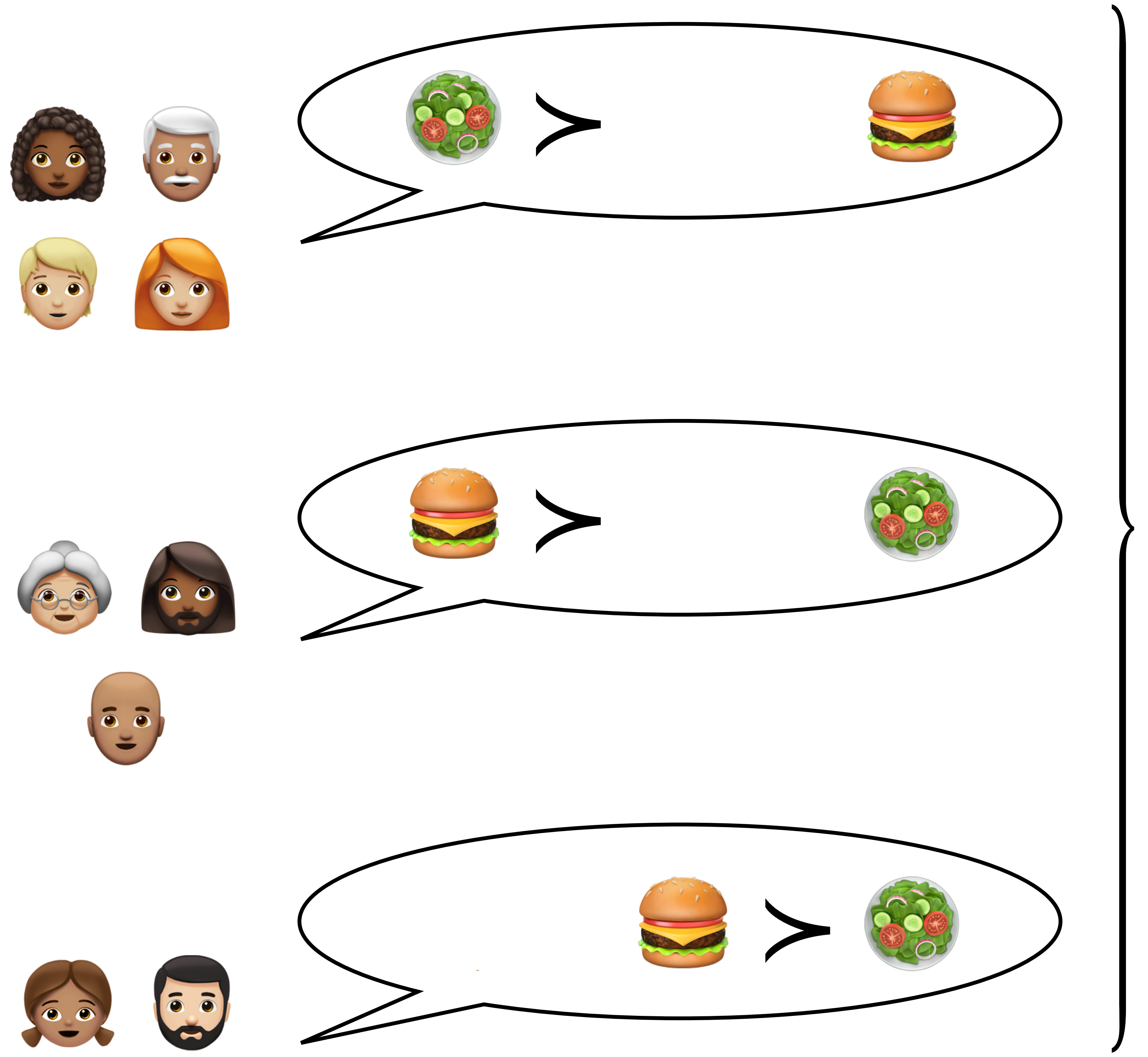

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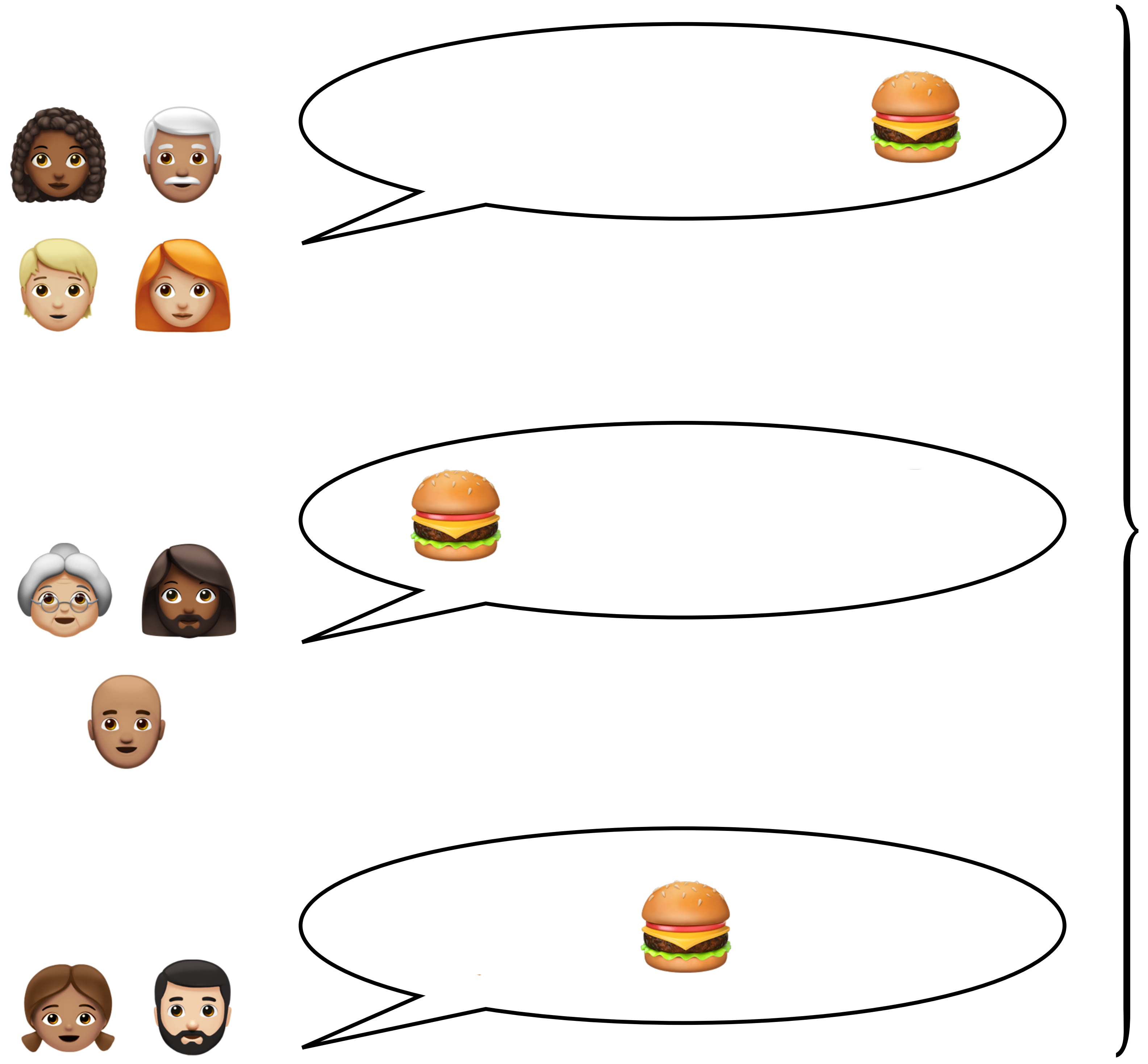

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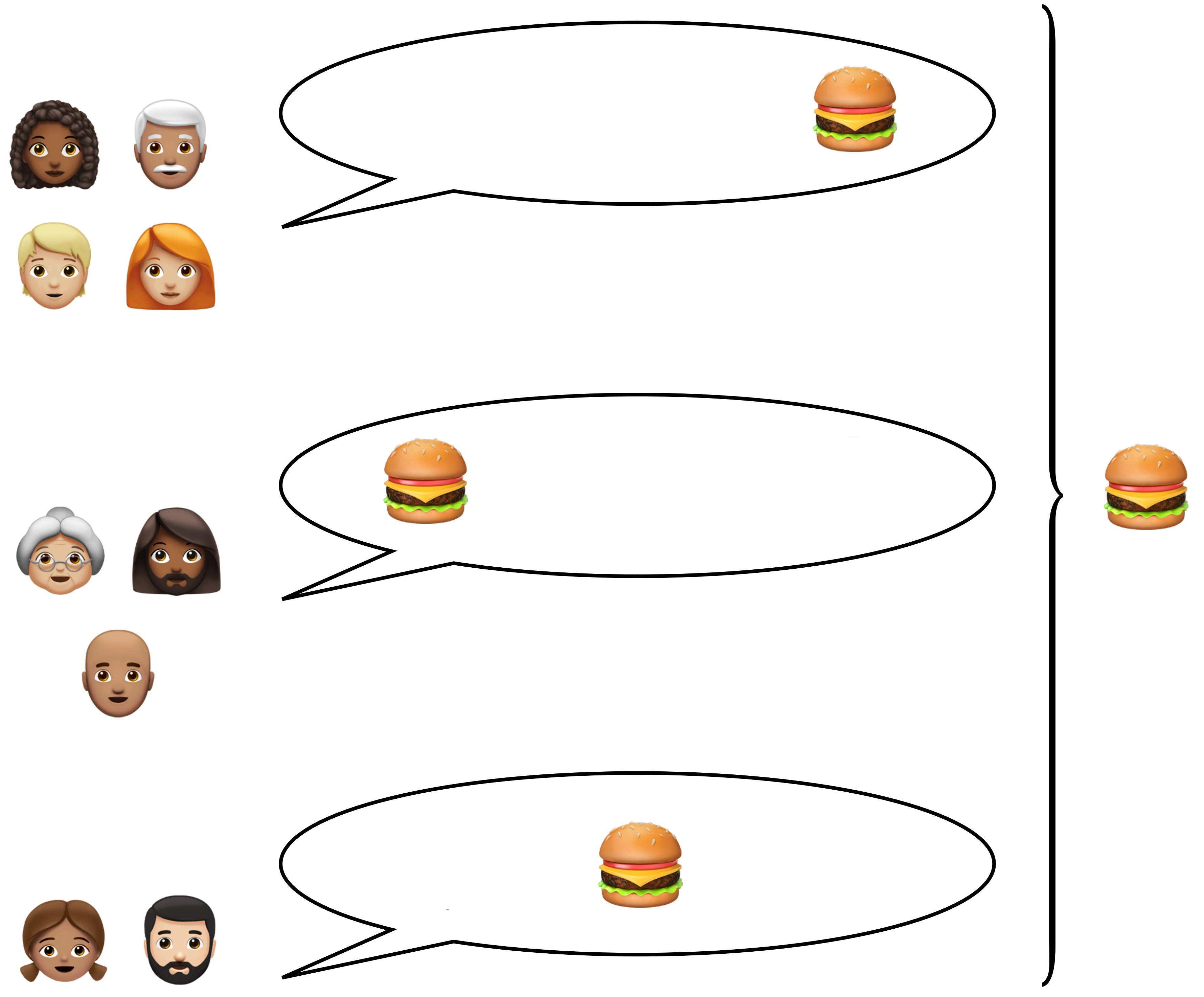

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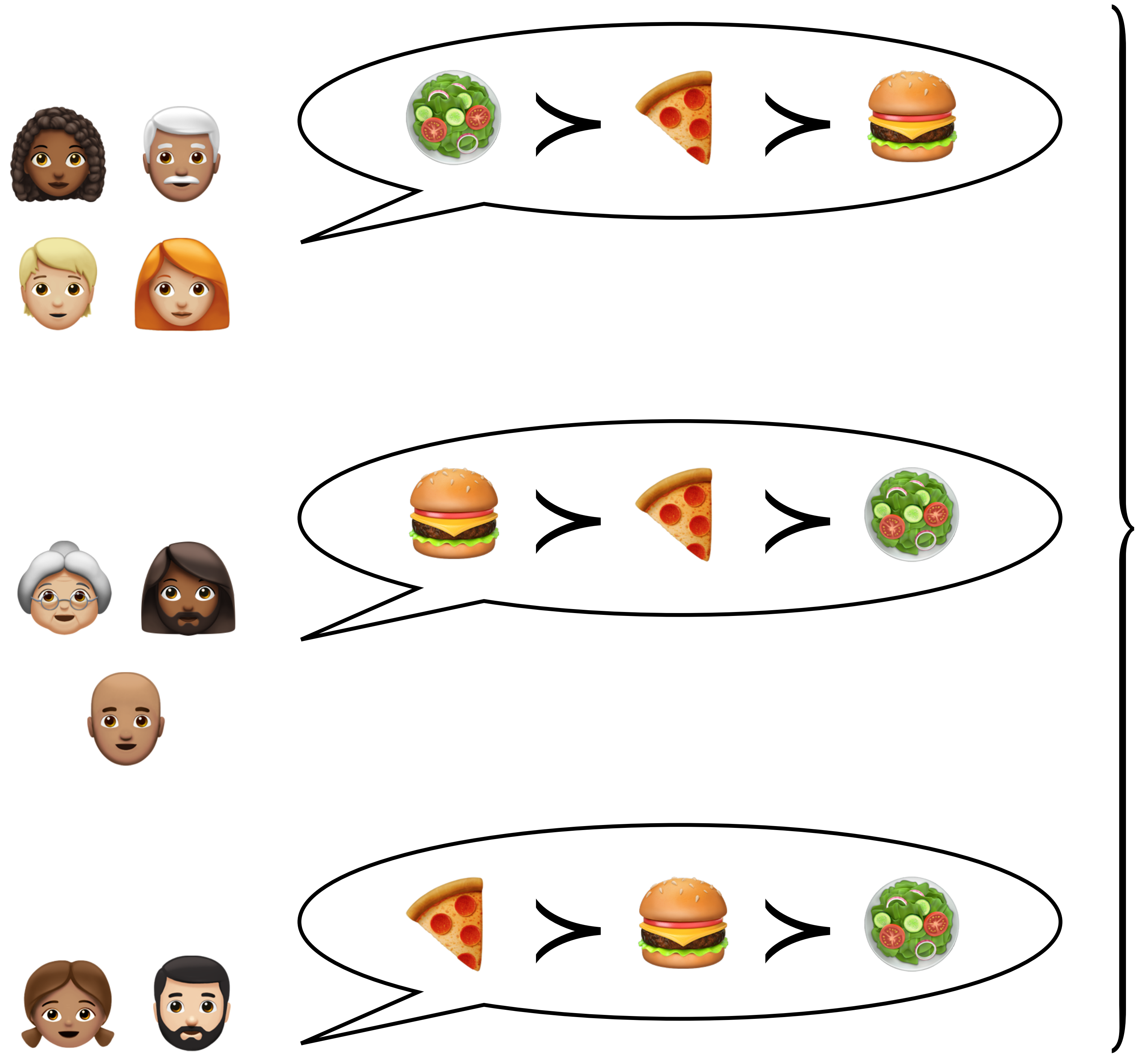

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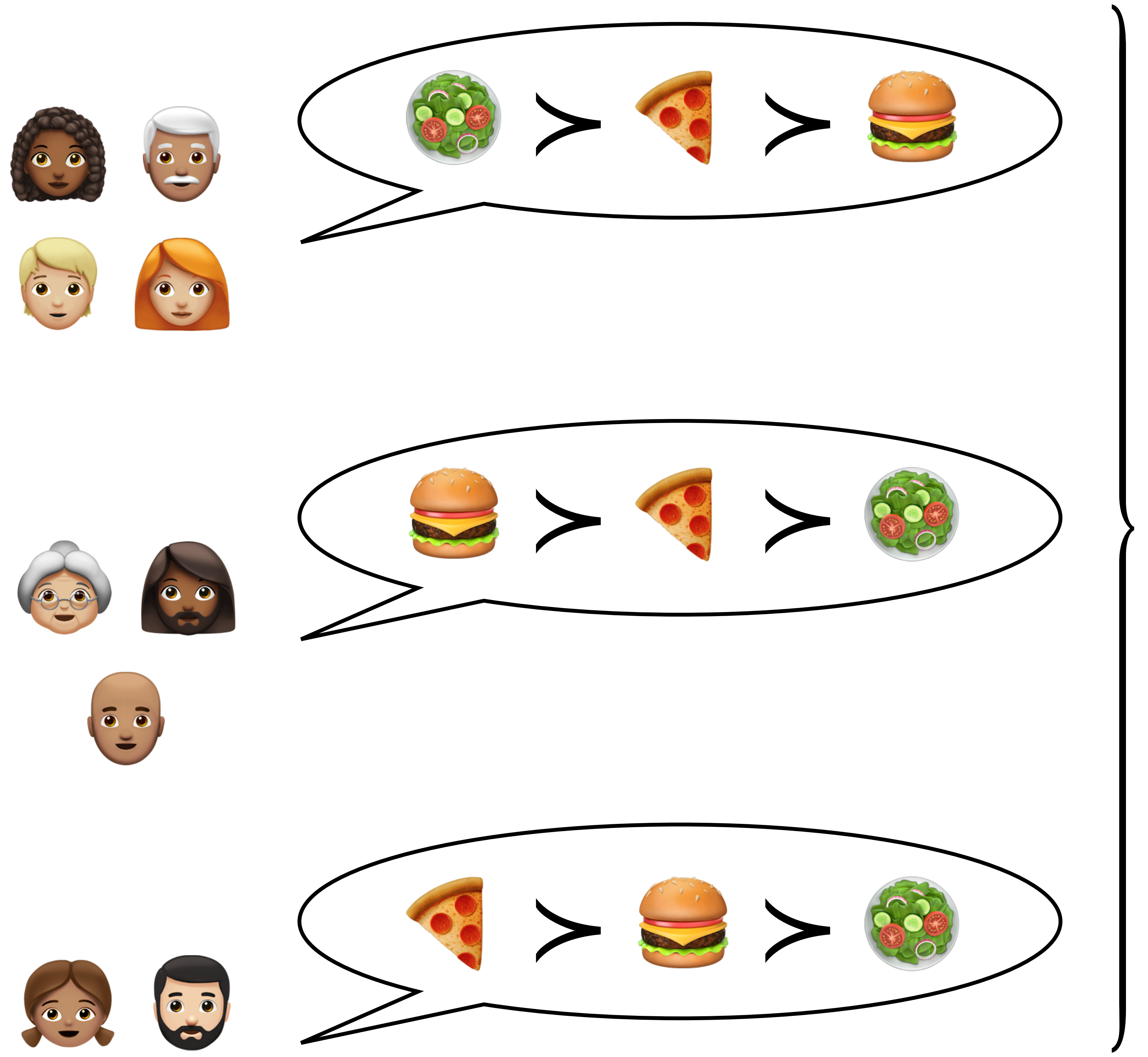

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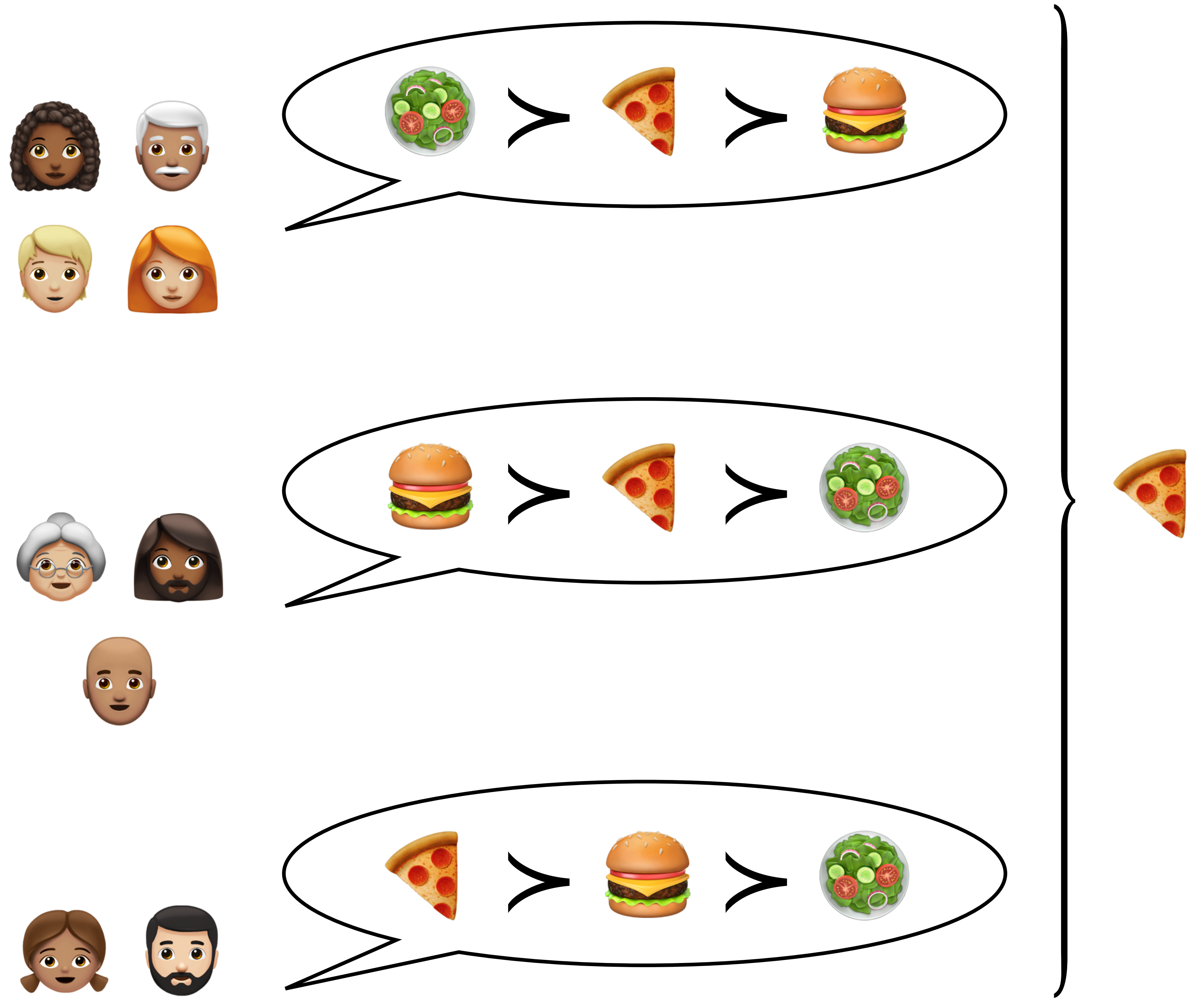

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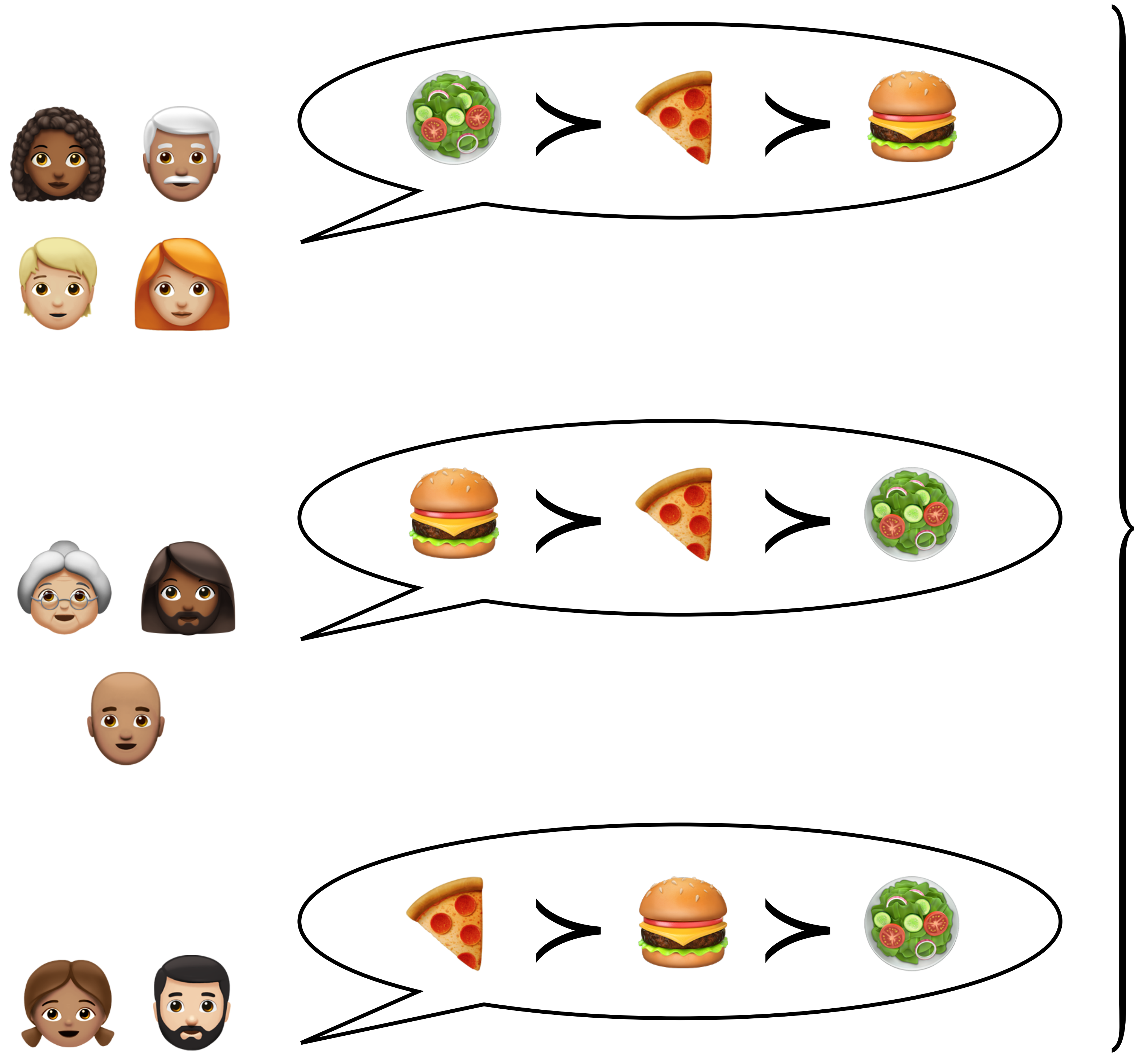

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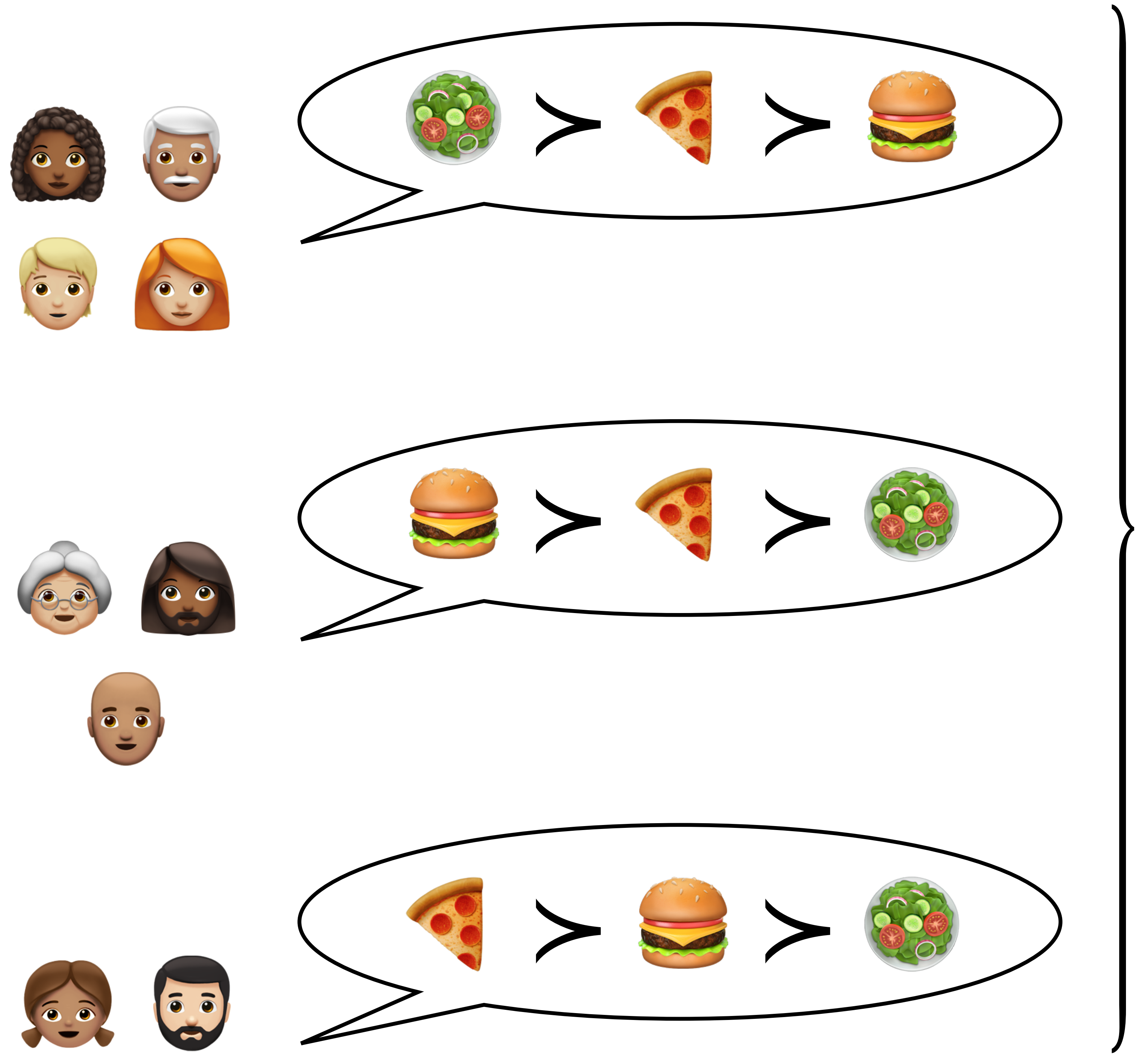

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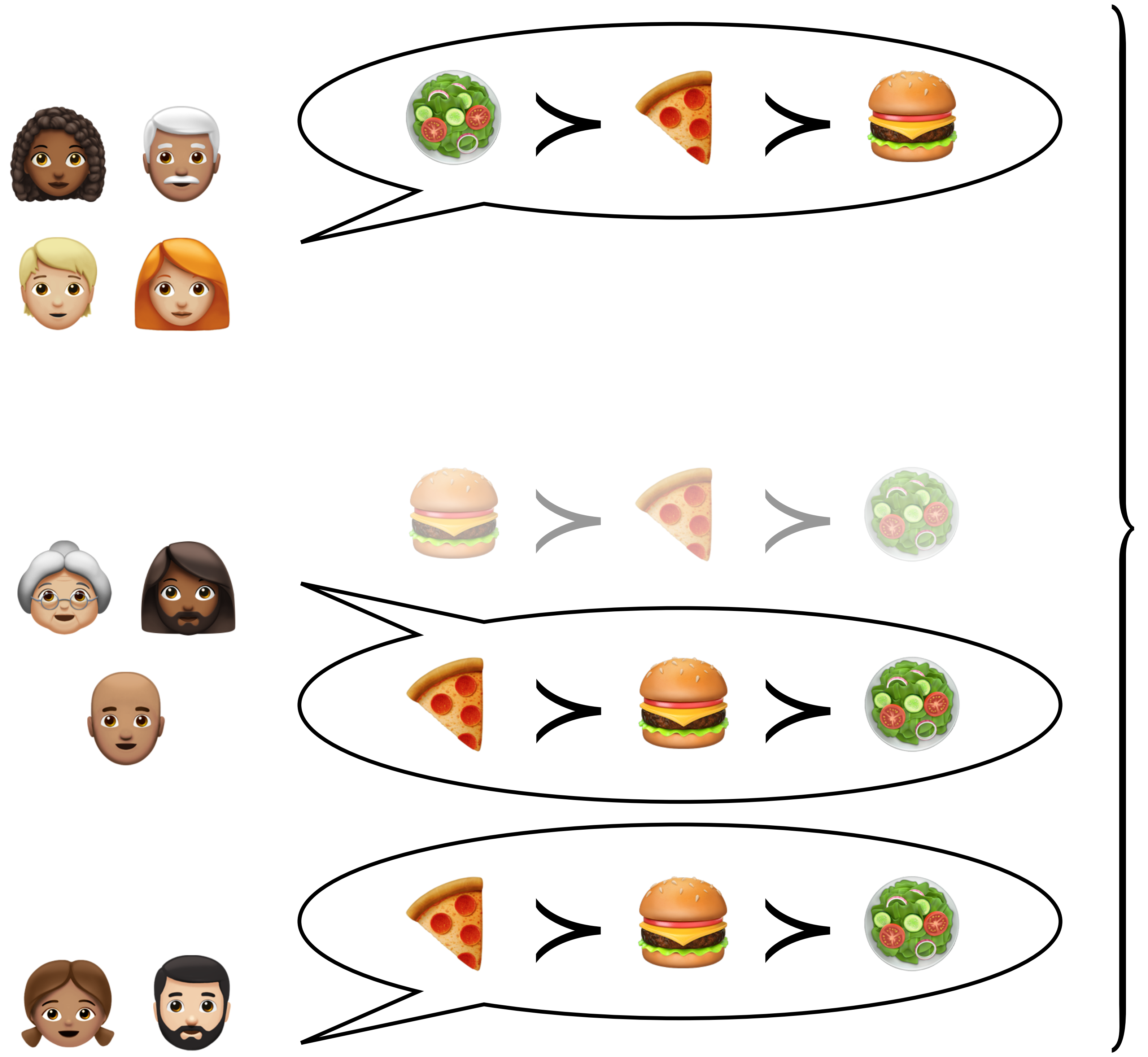

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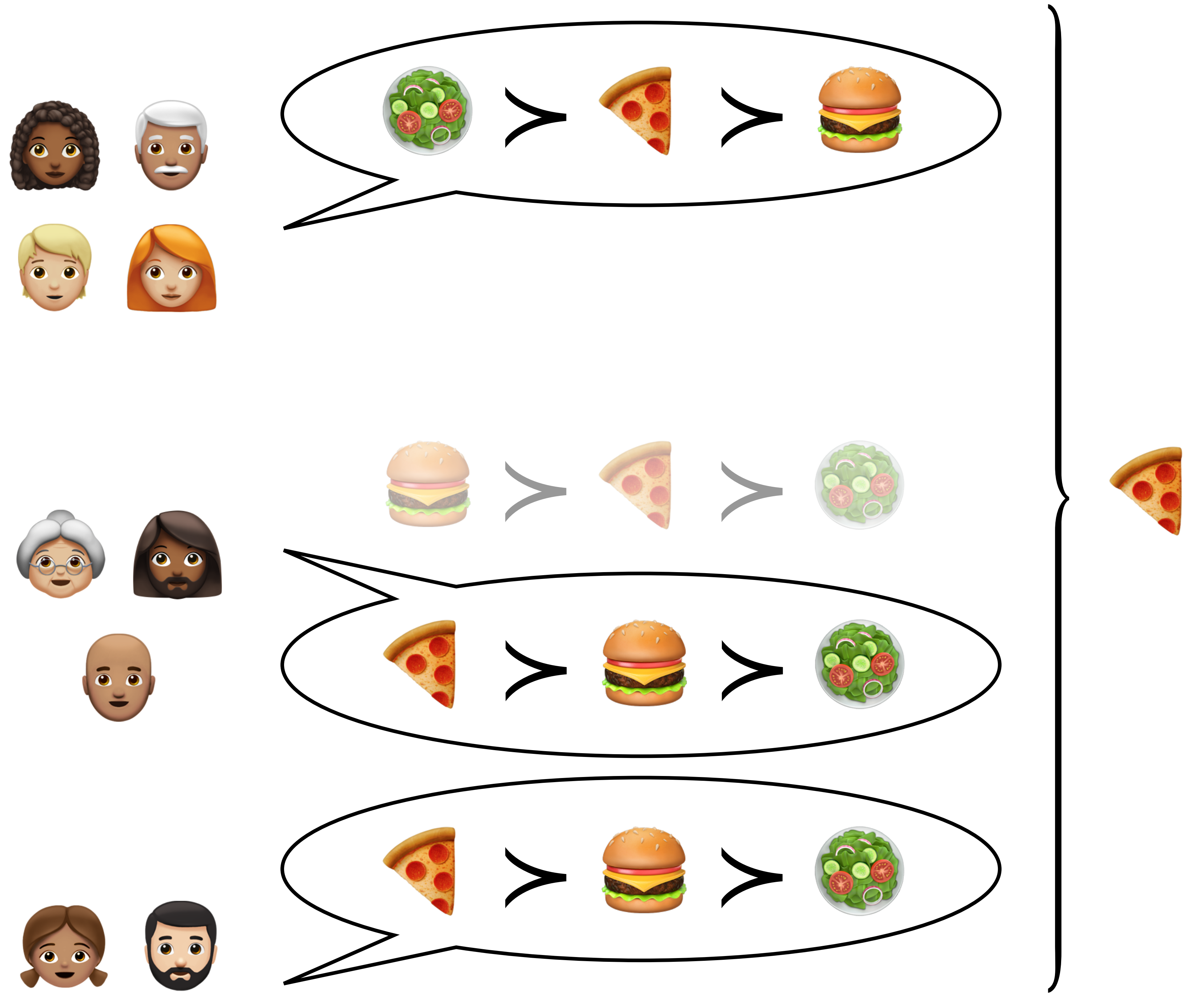

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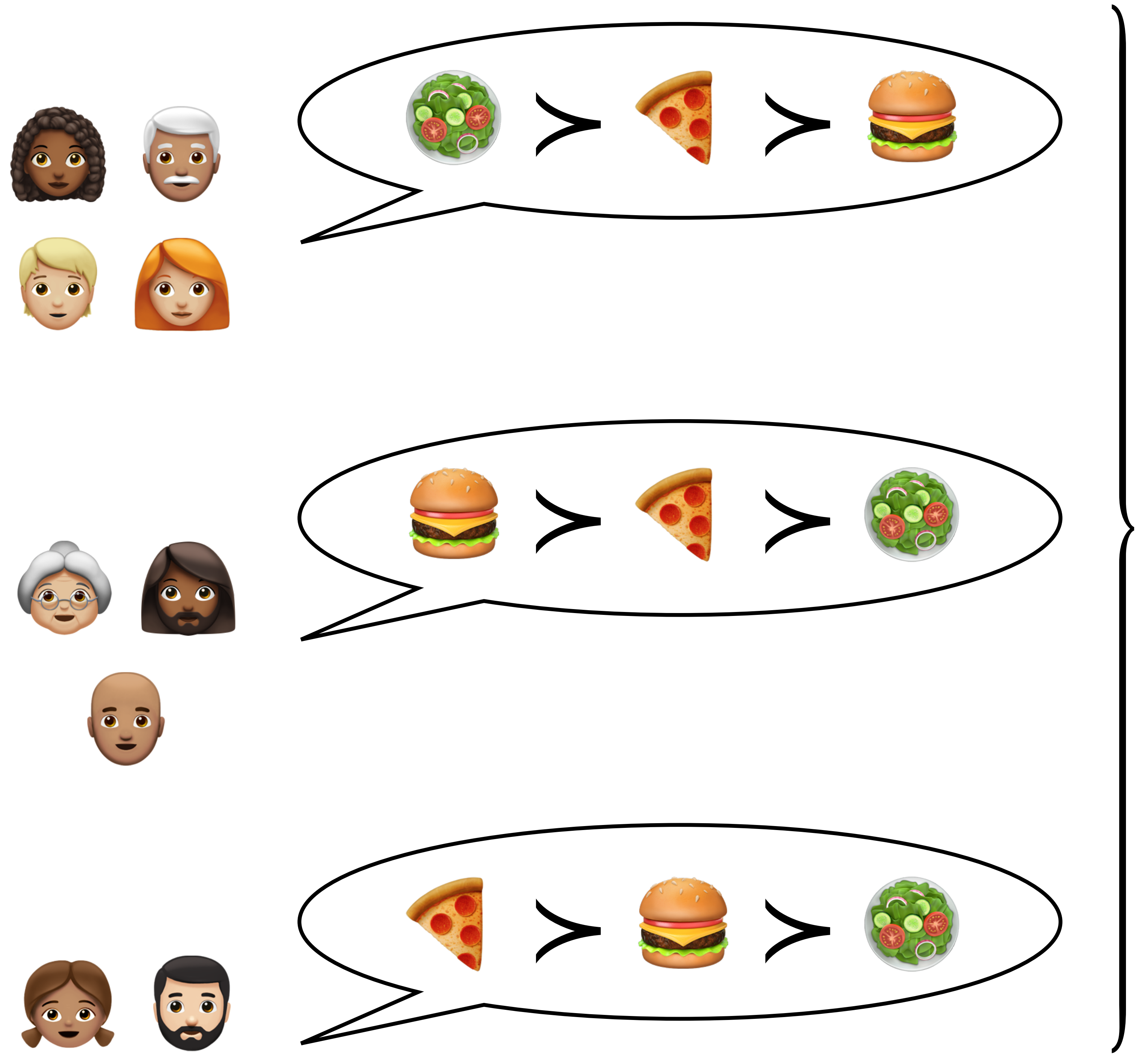

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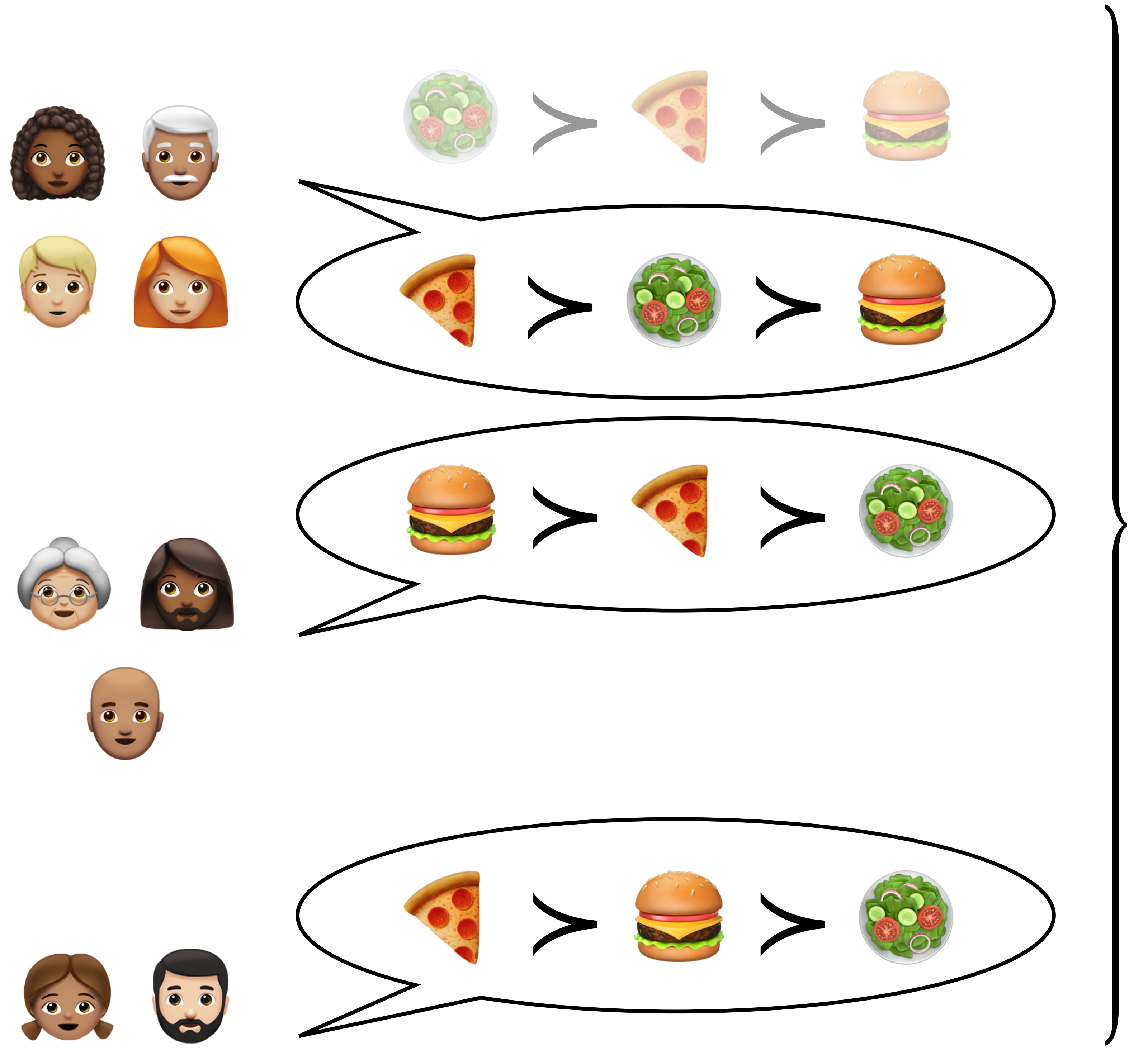

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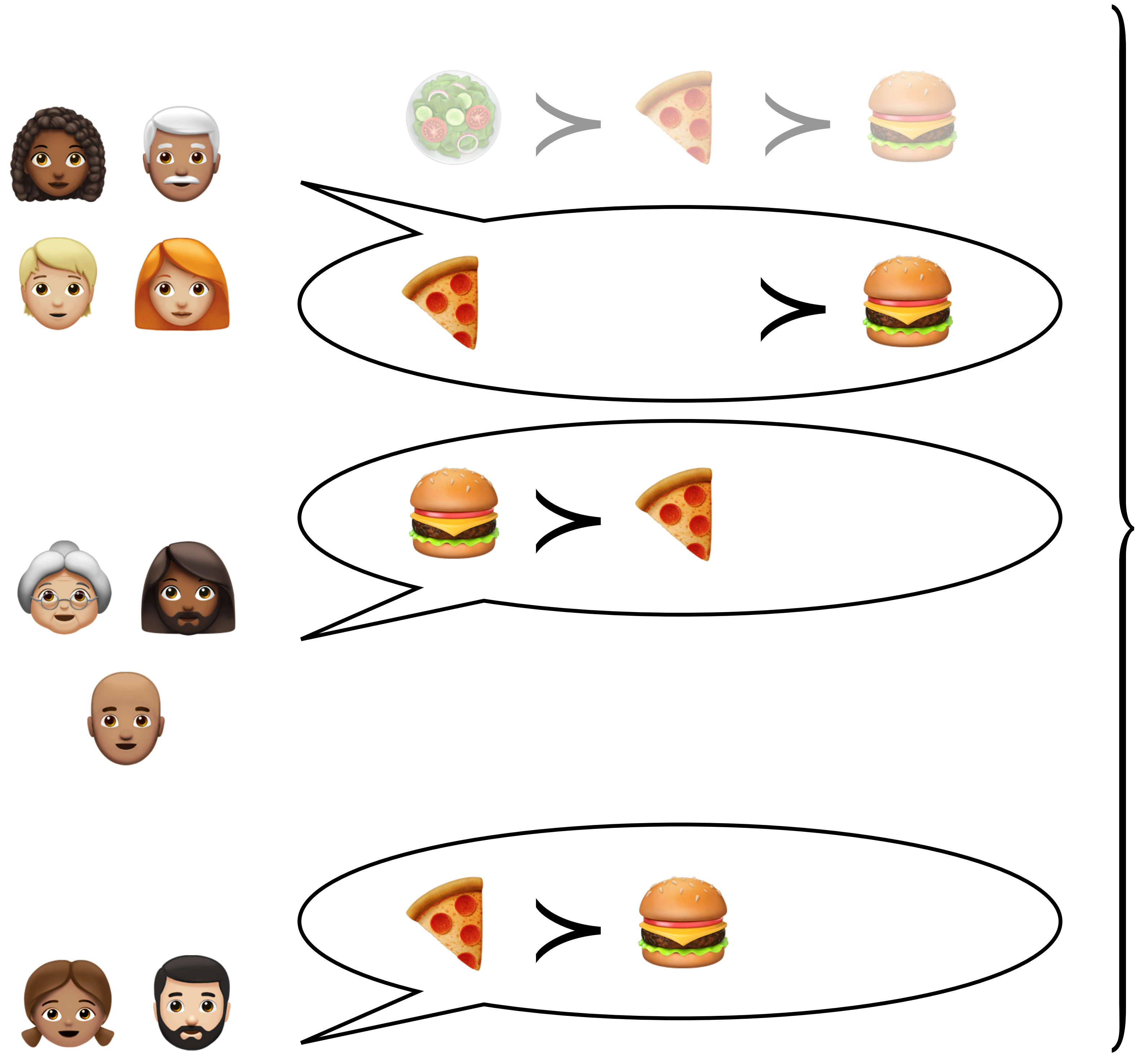

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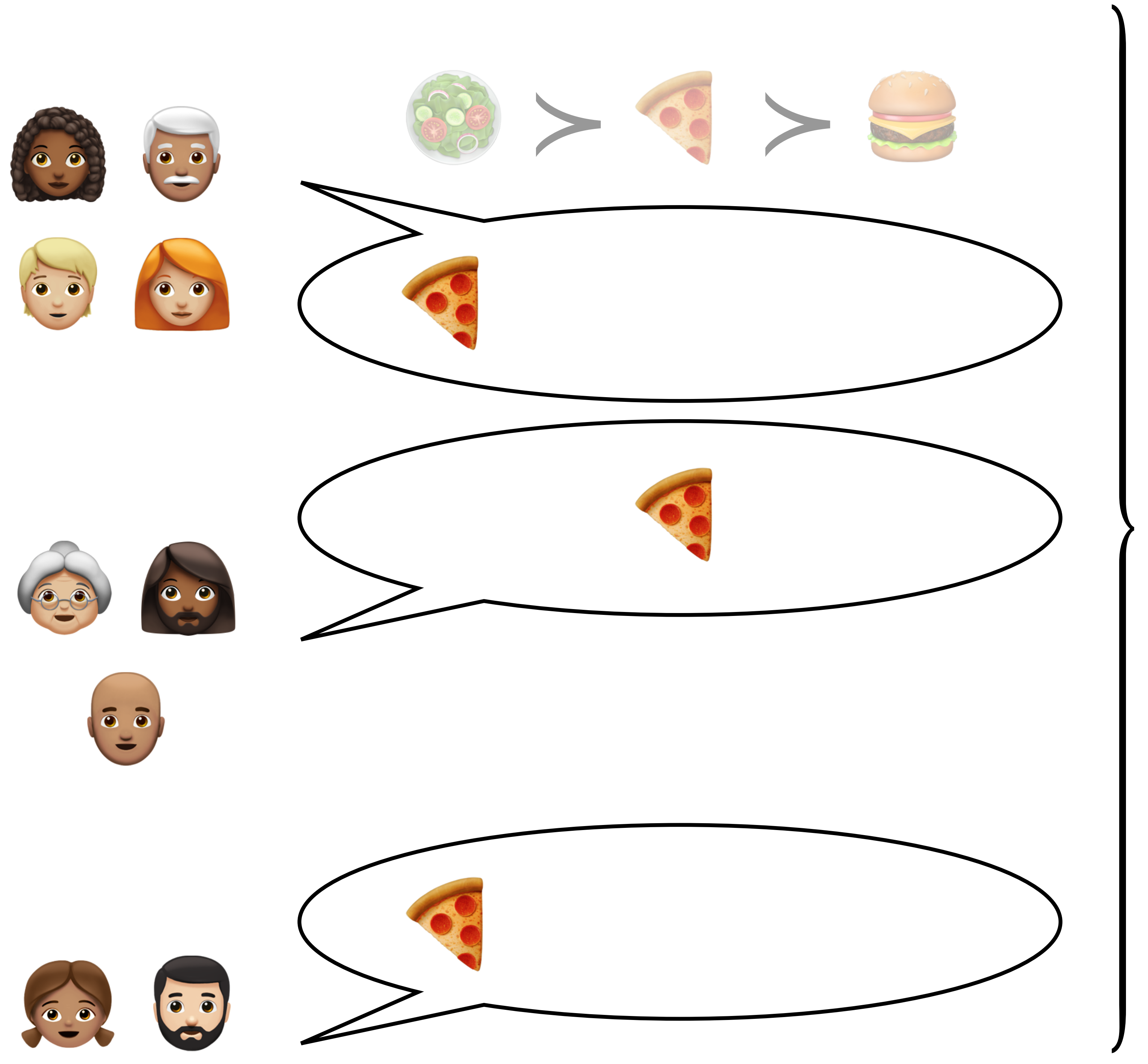

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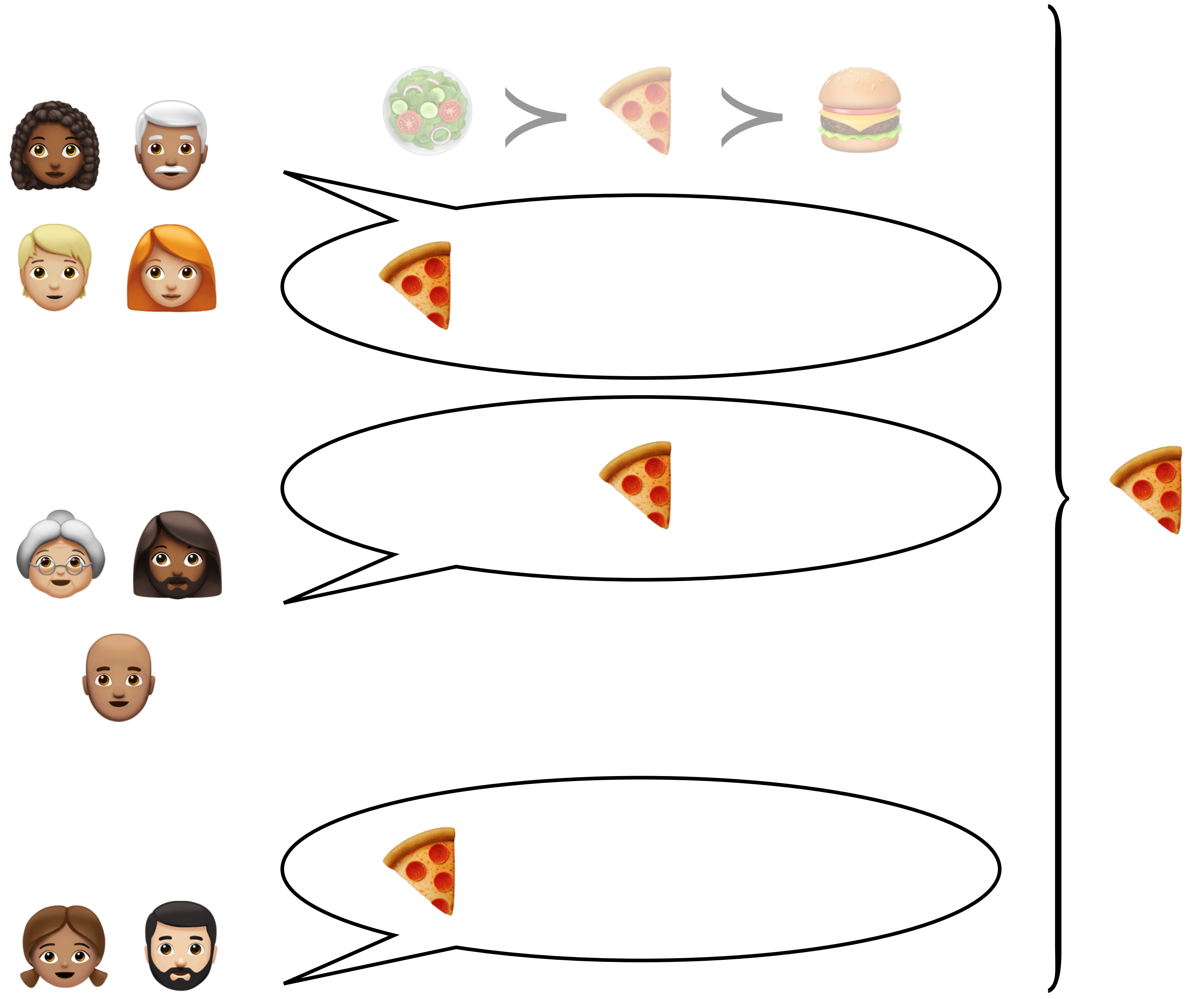

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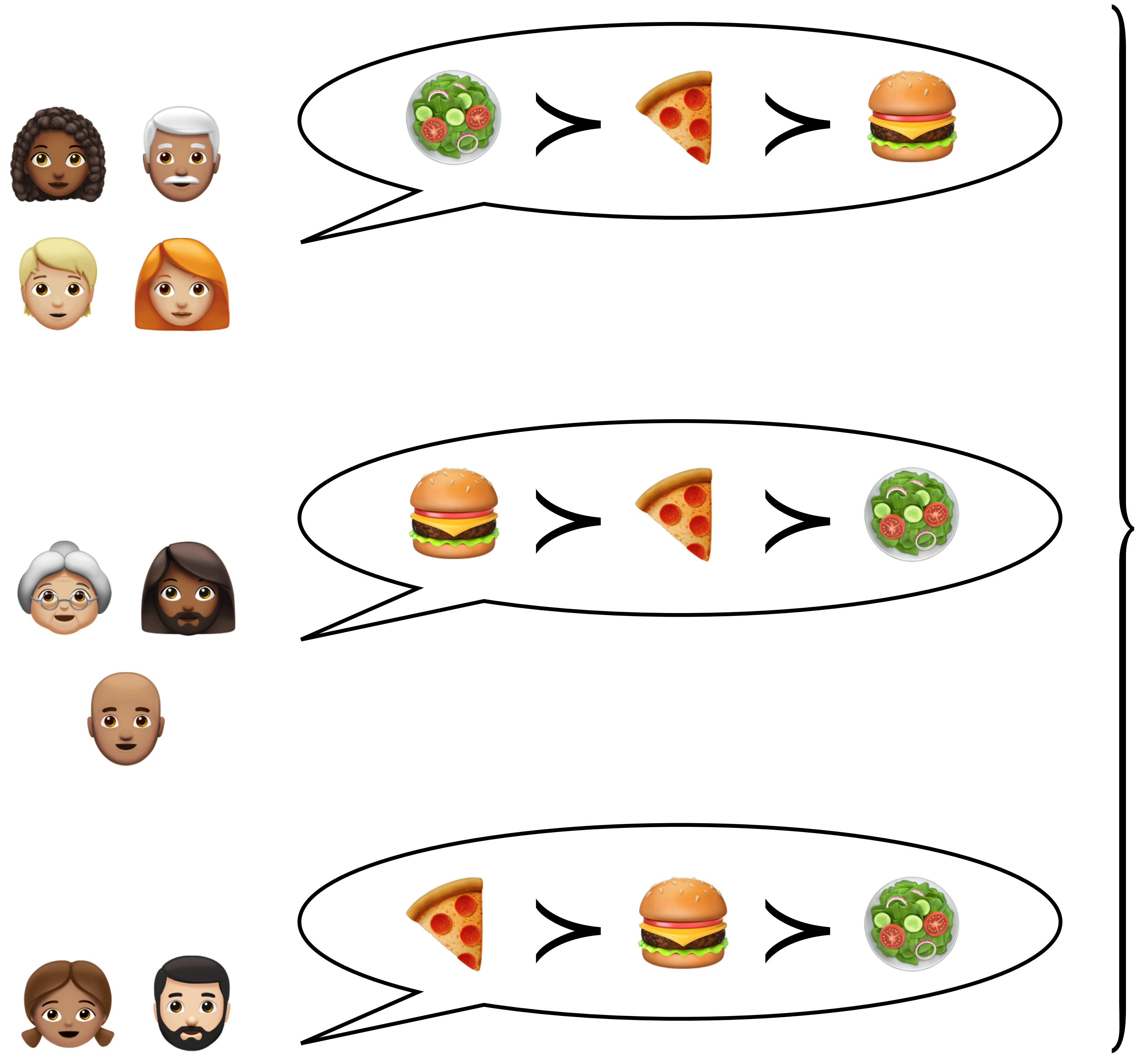

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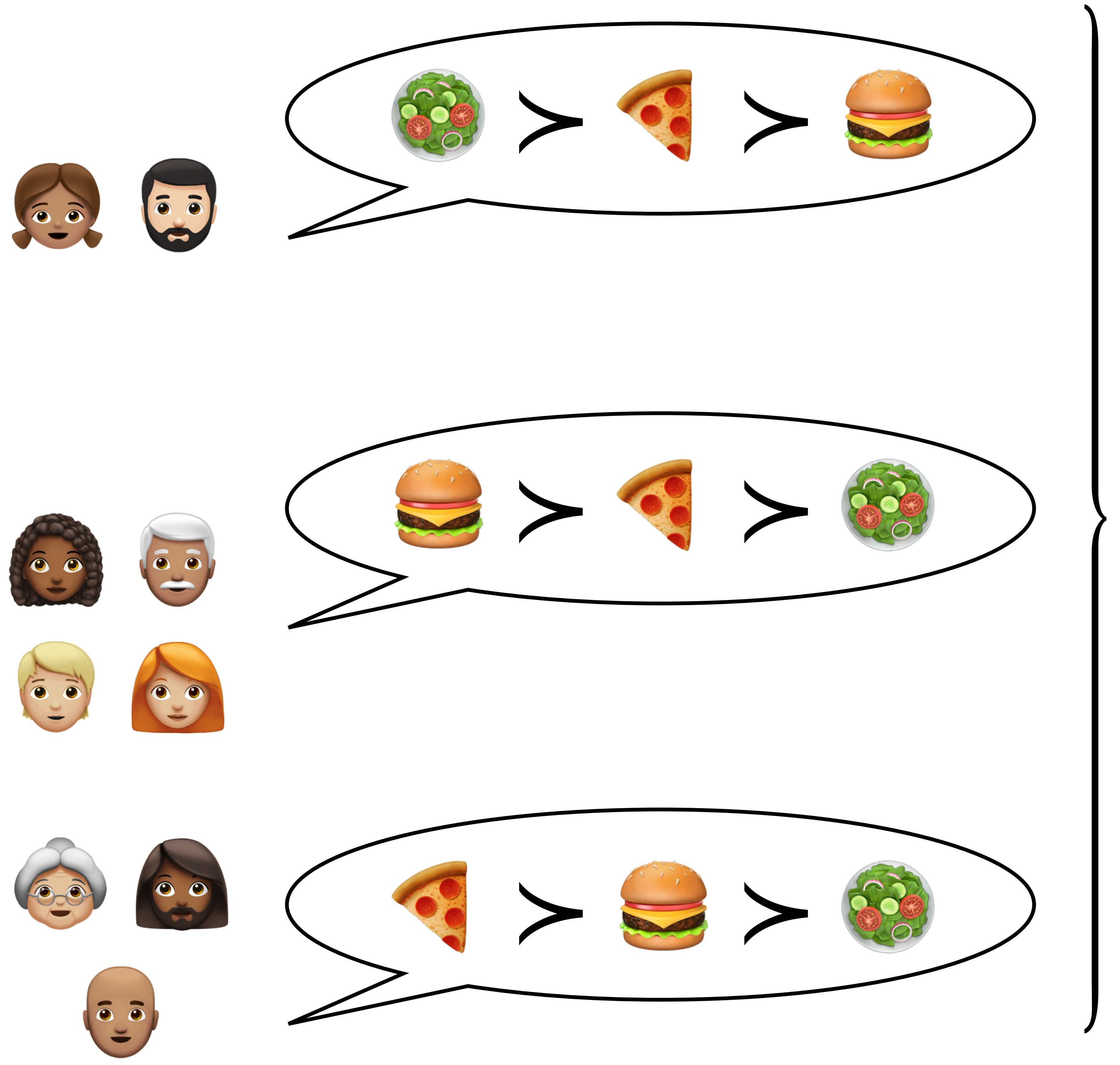

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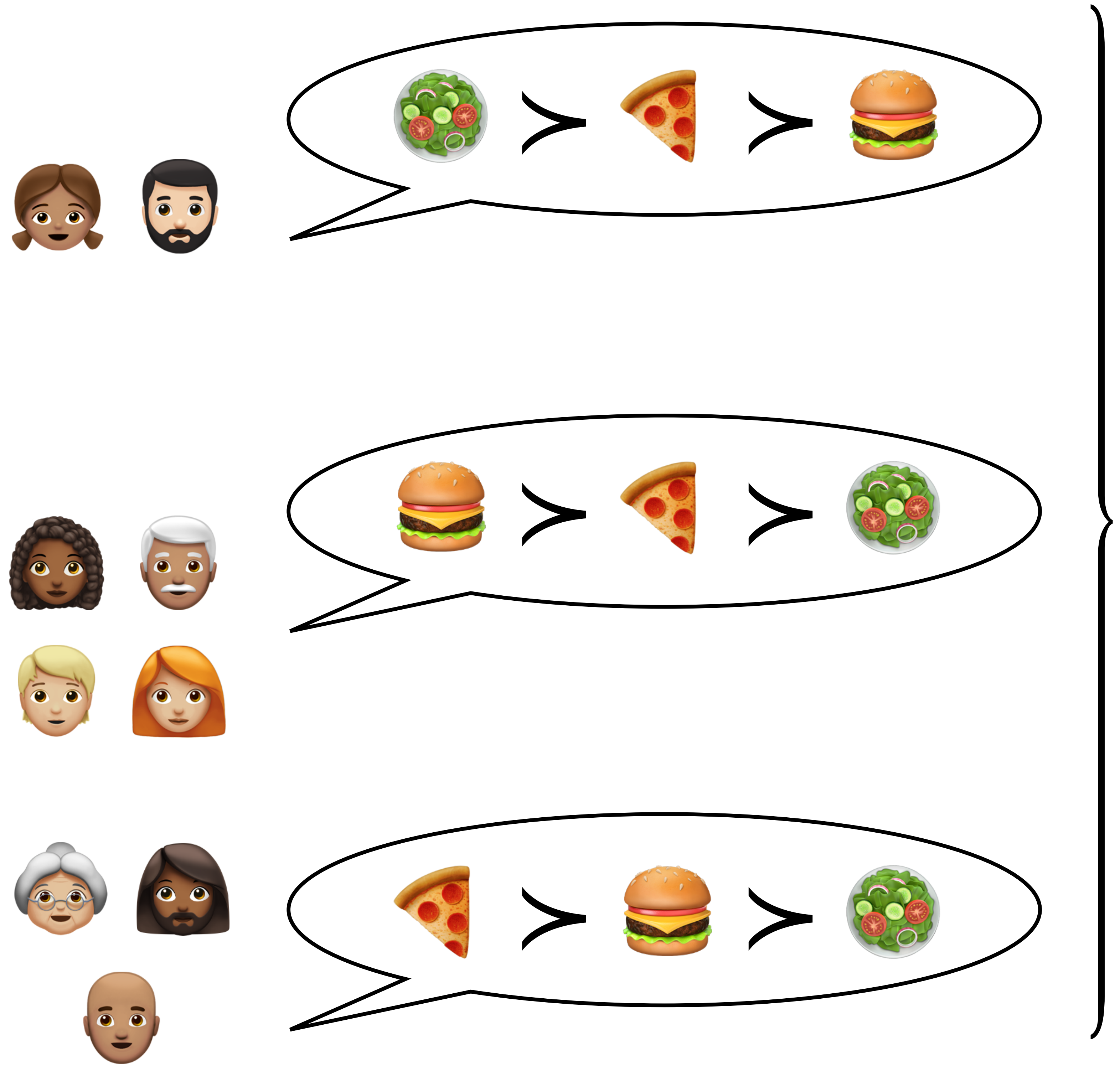

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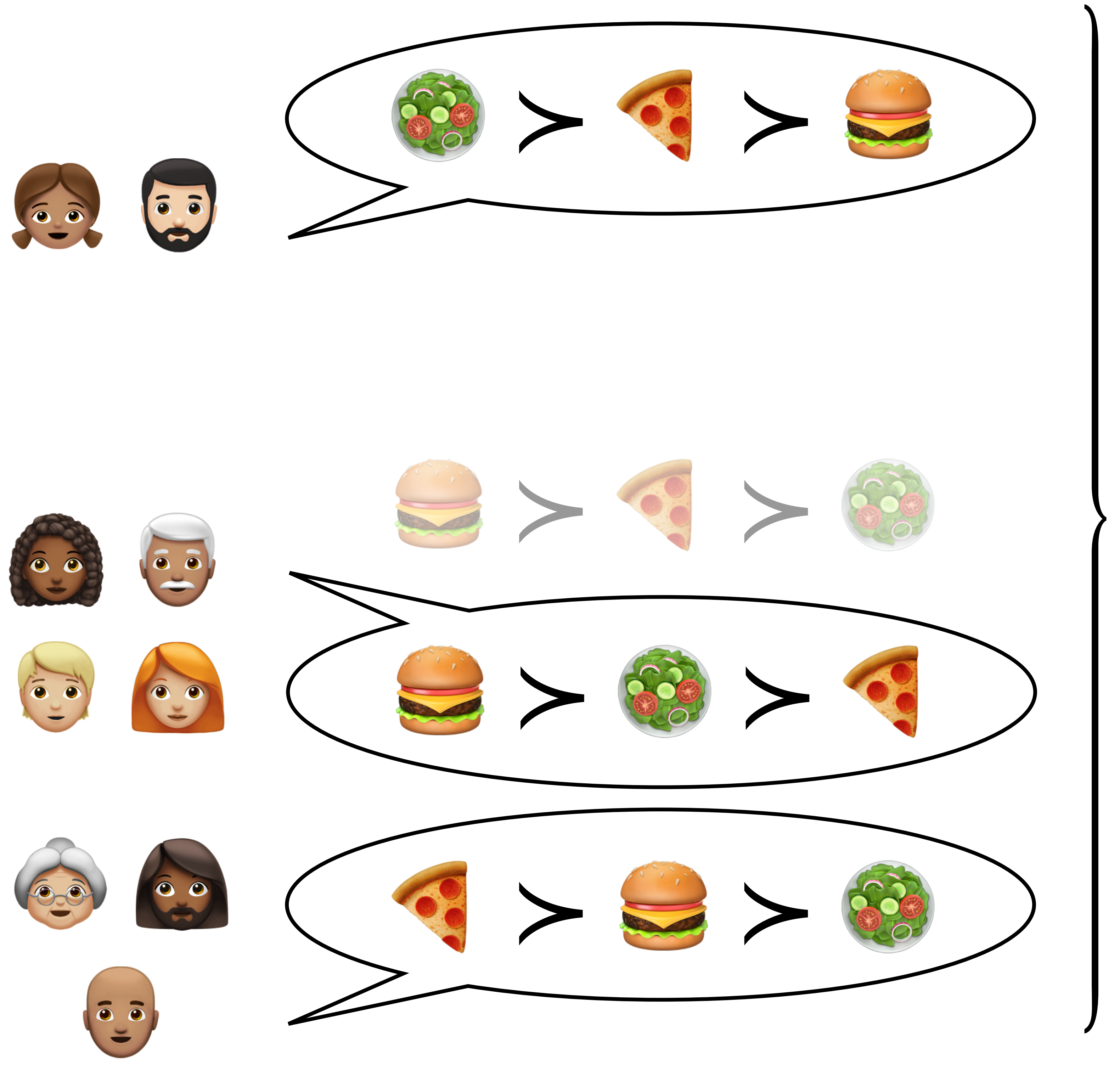

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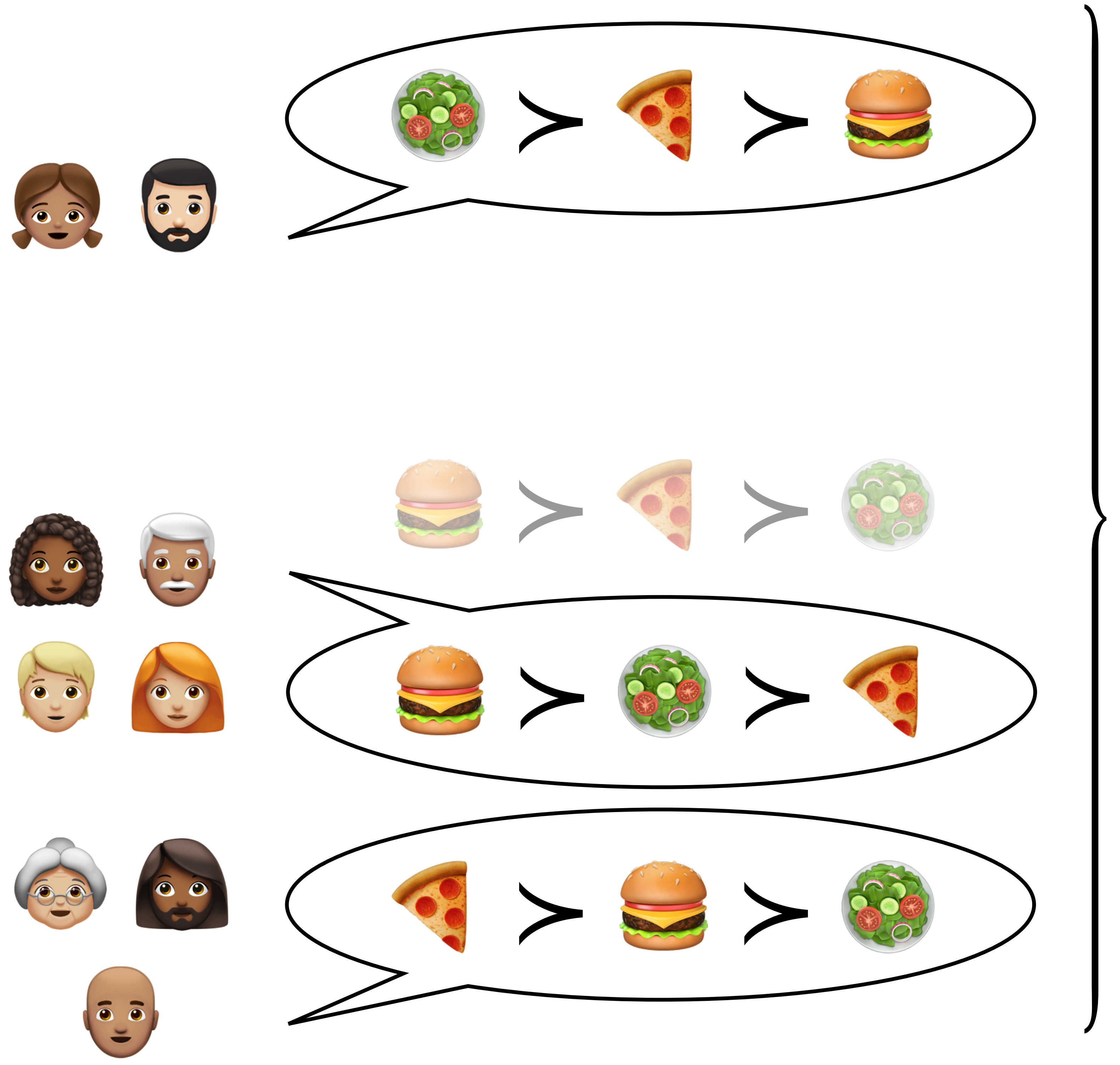


🍕 **Borda:** select the alternative with largest total score, where position $j \in \{1,\ldots,m\}$ gives score $m - j$

- can (a group of) voters misreport their preferences and obtain a better outcome?

Social Choice Functions

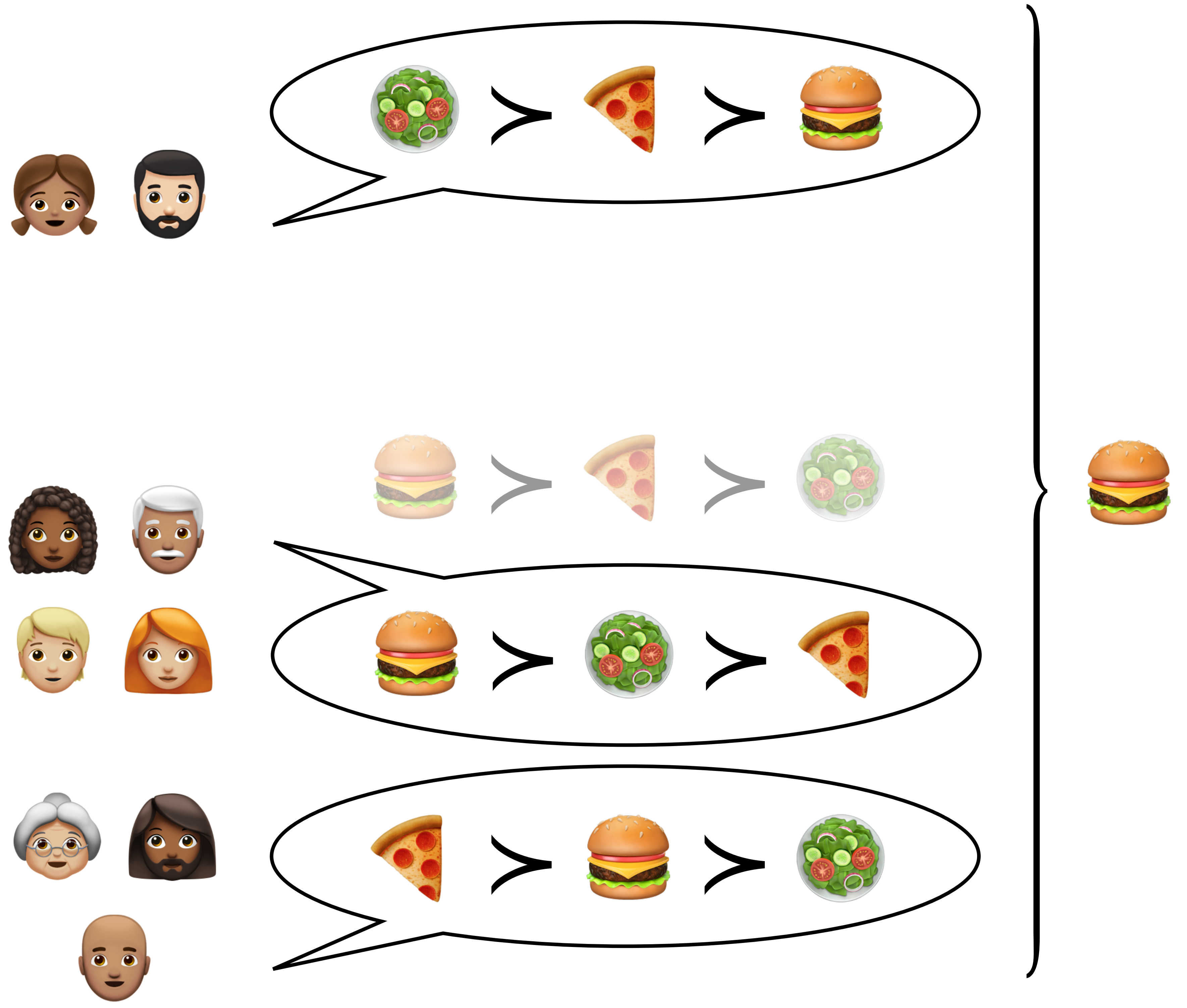
- voters $V = \{1,\ldots,n\}$, alternatives A with $|A| = m$
- each $i \in V$ has strict preference $\succ_i \in \mathscr{L}(A)$ over A
- a **social choice function** is a function $f\colon \mathscr{L}(A)^n \to A$

$\mathscr{L}(A)$: set of binary relations $\succ$ satisfying

- either $a \succ b$ or $b \succ a$ for every $a, b \in A$ with $a \neq b$
- $a \succ c$ whenever $a \succ b$ and $b \succ c$

🥗 $= 2 \cdot 2 + 4 \cdot 0 + 3 \cdot 0 = 4$
🍕 $= 2 \cdot 1 + 4 \cdot 1 + 3 \cdot 2 = 12$
🍔 $= 2 \cdot 0 + 4 \cdot 2 + 3 \cdot 1 = 11$

🥗 $= 2 \cdot 2 + 4 \cdot 1 + 3 \cdot 0 = 8$
🍕 $= 2 \cdot 1 + 4 \cdot 0 + 3 \cdot 2 = 8$
🍔 $= 2 \cdot 0 + 4 \cdot 2 + 3 \cdot 1 = 11$



🍕 **Borda:** select the alternative with largest total score, where position $j \in \{1,\ldots,m\}$ gives score $m - j$

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The Gibbard-Satterthwaite Theorem

- f is **strategyproof** if $f(\succ) \succeq_i f(\succ'_i, \succ_{-i})$ holds for every $\succ \in \mathscr{L}(A)^n$, $i \in V$, and $\succ'_i \in \mathscr{L}(A)$

  no voter has incentives to misreport

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- f is **dictatorial** if there is $i \in V$ such that $f(\succ) \succ_i a$ for every $\succ \in \mathscr{L}(A)^n$ and $a \in A \setminus f(\succ)$

  top choice of i is always selected

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$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
$\cdot$	$\cdots$	$\cdot$	a	b	$\cdots$	$\cdot$
$\cdot$	$\cdots$	b	$\cdot$	a	$\cdots$	b
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
a	$\cdots$	$\cdot$	b	$\cdot$	$\cdots$	$\cdot$
b	$\cdots$	a	$\cdot$	$\cdot$	$\cdots$	$\cdot$

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$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
.	$\cdots$	.	a	b	$\cdots$	.
.	$\cdots$	b	.	a	$\cdots$	b
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
a	$\cdots$	.	b	.	$\cdots$	.
b	$\cdots$	a	.	.	$\cdots$	.

↓

a

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  top choice of i is always selected

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	b	a	b	$\cdots$	b
$\cdot$	$\cdots$	$\cdot$	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$
a	$\cdots$	a	$\cdot$	a	$\cdots$	a

$\downarrow$

a

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- f is **dictatorial** if there is $i \in V$ such that $f(\succ) \succ_i a$ for every $\succ \in \mathscr{L}(A)^n$ and $a \in A \backslash f(\succ)$
  top choice of i is always selected
- any dictatorship is surjective and strategyproof

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	b	a	b	$\cdots$	b
$\cdot$	$\cdots$	$\cdot$	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$
a	$\cdots$	a	$\cdot$	a	$\cdots$	a

$\downarrow$

a

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  top choice of i is always selected

- any dictatorship is surjective and strategyproof

- the converse is also true!

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	b	a	b	$\cdots$	b
$\cdot$	$\cdots$	$\cdot$	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$
a	$\cdots$	a	$\cdot$	a	$\cdots$	a

a

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- any dictatorship is surjective and strategyproof

- the converse is also true!

- **Theorem** [Gibbard '73, Satterthwaite '75]

  Let $f\colon \mathscr{L}(A)^n \to A$ be a surjective and strategyproof social choice function, where $|A| \geq 3$.
  Then, f is dictatorial.

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	b	a	b	$\cdots$	b
$\cdot$	$\cdots$	$\cdot$	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$
a	$\cdots$	a	$\cdot$	a	$\cdots$	a

$\downarrow$

a

The Muller-Satterthwaite Theorem

- f is **unanimous** if $f(\succ) = a$ whenever $a \succ_i b$ for every $i \in V$ and $b \in A \setminus \{a\}$

  when all voters have the same top choice, it is selected

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$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
a	$\cdots$	a	a	a	$\cdots$	a
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$

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$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
a	$\cdots$	a	a	a	$\cdots$	a
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$

a

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  a selected alternative remains selected

  if dominated alternatives in all rankings remain dominated

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
a	$\cdots$	a	a	a	$\cdots$	a
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$

$\downarrow$

a

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$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
a	$\cdots$	a	a	a	$\cdots$	a
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$

$\downarrow$

a

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$
a	c	b	c
b	d	a	a
c	a	d	d
d	b	c	b

$\downarrow$

a

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$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
a	$\cdots$	a	a	a	$\cdots$	a
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$

a

$\succ'_1$	$\succ_2$	$\succ_3$	$\succ_4$
a	c	b	c
d	d	a	a
b	a	d	d
c	b	c	b

a

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$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
a	$\cdots$	a	a	a	$\cdots$	a
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$

$\downarrow$

a

$\succ'_1$	$\succ'_2$	$\succ_3$	$\succ_4$
a	c	b	c
d	a	a	a
b	b	d	d
c	d	c	b

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  a selected alternative remains selected

  if dominated alternatives in all rankings remain dominated

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
a	$\cdots$	a	a	a	$\cdots$	a
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$

a

$\succ'_1$	$\succ'_2$	$\succ'_3$	$\succ_4$
a	c	b	c
d	a	a	a
b	b	c	d
c	d	d	b

a

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$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
a	$\cdots$	a	a	a	$\cdots$	a
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$

$\downarrow$

a

$\succ'_1$	$\succ'_2$	$\succ'_3$	$\succ'_4$
a	c	b	a
d	a	a	b
b	b	c	d
c	d	d	c

$\downarrow$

a

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$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
a	$\cdots$	a	a	a	$\cdots$	a
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$

$\downarrow$

a

$\succ'_1$	$\succ'_2$	$\succ'_3$	$\succ'_4$
a	c	b	a
d	a	a	b
b	b	c	d
c	d	d	c

$\downarrow$

a

Theorem [Muller, Satterthwaite '77]

Let $f\colon \mathscr{L}(A)^n \to A$ be a unanimous and monotone social choice function, where $|A| \geq 3$.
Then, f is dictatorial.

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  if dominated alternatives in all rankings remain dominated

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
a	$\cdots$	a	a	a	$\cdots$	a
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$

$\downarrow$

a

$\succ'_1$	$\succ'_2$	$\succ'_3$	$\succ'_4$
a	c	b	a
d	a	a	b
b	b	c	d
c	d	d	c

$\downarrow$

a

Theorem [Muller, Satterthwaite '77]
Let $f\colon \mathscr{L}(A)^n \to A$ be a unanimous and monotone social choice function, where $|A| \geq 3$.
Then, f is dictatorial.

- we give a proof of this theorem due to Reny ['00], so let f be as in the statement

Pivotal Voter

- we consider two fixed alternatives $a, b \in A$

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- we move from a profile where a is ranked highest by all voters to a profile where b is
  moving b one position at a time and one voter at a time

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$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
a	$\cdots$	a	a	a	$\cdots$	a
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
b	$\cdots$	b	b	b	$\cdots$	b

$\downarrow$

a

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$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
a	$\cdots$	a	a	a	$\cdots$	a
$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
b	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	b	b	b	$\cdots$	b

$\downarrow$ a

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$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
a	$\cdots$	a	a	a	$\cdots$	a
b	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	b	b	b	$\cdots$	b

$\downarrow$ a

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  moving b one position at a time and one voter at a time

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	a	a	a	$\cdots$	a
a	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	b	b	b	$\cdots$	b

$\downarrow$

a

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$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	a	a	a	$\cdots$	a
a	$\cdots$	b	$\cdot$	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	b	b	$\cdots$	b

$\downarrow$ a

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$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	b	a	a	$\cdots$	a
a	$\cdots$	a	$\cdot$	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	b	b	$\cdots$	b

$\downarrow$

a

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$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	b	a	a	$\cdots$	a
a	$\cdots$	a	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	b	$\cdots$	b

$\downarrow$

a

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$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	b	a	a	$\cdots$	a
a	$\cdots$	a	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	b	$\cdots$	b

a

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	b	b	a	$\cdots$	a
a	$\cdots$	a	a	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	b	$\cdots$	b

b

- we consider two fixed alternatives $a, b \in A$
- we move from a profile where a is ranked highest by all voters to a profile where b is
  moving b one position at a time and one voter at a time

P1

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	b	a	a	$\cdots$	a
a	$\cdots$	a	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	b	$\cdots$	b

$\downarrow$ a

P2

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	b	b	a	$\cdots$	a
a	$\cdots$	a	a	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	b	$\cdots$	b

$\downarrow$ b

Pivotal Voter

- we consider two fixed alternatives $a, b \in A$
- we move from a profile where a is ranked highest by all voters to a profile where b is
  moving b one position at a time and one voter at a time
- by unanimity, the outcome must change from a to b
  voter i changes profile P1 to P2

P1

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	b	a	a	$\cdots$	a
a	$\cdots$	a	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	b	$\cdots$	b

a

P2

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	b	b	a	$\cdots$	a
a	$\cdots$	a	a	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	b	$\cdots$	b

b

Pivotal Voter

- we consider two fixed alternatives $a, b \in A$
- we move from a profile where a is ranked highest by all voters to a profile where b is
  moving b one position at a time and one voter at a time
- by unanimity, the outcome must change from a to b
  voter i changes profile P1 to P2
- start from P2 and move a below for all voters but i, without changing pairwise relationships with b

P1

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	b	a	a	$\cdots$	a
a	$\cdots$	a	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	b	$\cdots$	b

$\downarrow$ a

P2

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	b	b	a	$\cdots$	a
a	$\cdots$	a	a	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	b	$\cdots$	b

$\downarrow$ b

Pivotal Voter

- we consider two fixed alternatives $a, b \in A$
- we move from a profile where a is ranked highest by all voters to a profile where b is
  moving b one position at a time and one voter at a time
- by unanimity, the outcome must change from a to b
  voter i changes profile P1 to P2
- start from P2 and move a below for all voters but i, without changing pairwise relationships with b

P1

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	b	a	a	$\cdots$	a
a	$\cdots$	a	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	b	$\cdots$	b

$\downarrow$ a

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	b	b	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	a	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	a	$\cdots$	a
a	$\cdots$	a	$\cdot$	b	$\cdots$	b

$\downarrow$ b

Pivotal Voter

- we consider two fixed alternatives $a, b \in A$
- we move from a profile where a is ranked highest by all voters to a profile where b is

  moving b one position at a time and one voter at a time
- by unanimity, the outcome must change from a to b

  voter i changes profile P1 to P2
- start from P2 and move a below for all voters but i, without changing pairwise relationships with b
  - by monotonicity, b must remain selected

P1

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	b	a	a	$\cdots$	a
a	$\cdots$	a	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	b	$\cdots$	b

a

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	b	b	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	a	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	a	$\cdots$	a
a	$\cdots$	a	$\cdot$	b	$\cdots$	b

b

- we consider two fixed alternatives $a, b \in A$
- we move from a profile where a is ranked highest by all voters to a profile where b is
  moving b one position at a time and one voter at a time
- by unanimity, the outcome must change from a to b
  voter i changes profile P1 to P2
- start from P2 and move a below for all voters but i, without changing pairwise relationships with b
  - by monotonicity, b must remain selected
- flip a and b in i's ranking and call the resulting profile P3

P1

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	b	a	a	$\cdots$	a
a	$\cdots$	a	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	b	$\cdots$	b

a

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	b	b	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	a	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	a	$\cdots$	a
a	$\cdots$	a	$\cdot$	b	$\cdots$	b

b

Pivotal Voter

- we consider two fixed alternatives $a, b \in A$
- we move from a profile where a is ranked highest by all voters to a profile where b is
  moving b one position at a time and one voter at a time
- by unanimity, the outcome must change from a to b
  voter i changes profile P1 to P2
- start from P2 and move a below for all voters but i, without changing pairwise relationships with b
  - by monotonicity, b must remain selected
- flip a and b in i's ranking and call the resulting profile P3

P1

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	b	a	a	$\cdots$	a
a	$\cdots$	a	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	b	$\cdots$	b

a

P3

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	b	a	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	a	$\cdots$	a
a	$\cdots$	a	$\cdot$	b	$\cdots$	b

Pivotal Voter

- we consider two fixed alternatives $a, b \in A$
- we move from a profile where a is ranked highest by all voters to a profile where b is
  moving b one position at a time and one voter at a time
- by unanimity, the outcome must change from a to b
  voter i changes profile P1 to P2
- start from P2 and move a below for all voters but i, without changing pairwise relationships with b
  - by monotonicity, b must remain selected
- flip a and b in i's ranking and call the resulting profile P3
  - by monotonicity, either a or b is selected

P1

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	b	a	a	$\cdots$	a
a	$\cdots$	a	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	b	$\cdots$	b

↓ a

P3

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	b	a	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	a	$\cdots$	a
a	$\cdots$	a	$\cdot$	b	$\cdots$	b

↓

Pivotal Voter

- we consider two fixed alternatives $a, b \in A$
- we move from a profile where a is ranked highest by all voters to a profile where b is
  moving b one position at a time and one voter at a time
- by unanimity, the outcome must change from a to b
  voter i changes profile P1 to P2
- start from P2 and move a below for all voters but i, without changing pairwise relationships with b
  - by monotonicity, b must remain selected
- flip a and b in i's ranking and call the resulting profile P3
  - by monotonicity, either a or b is selected
  - selecting b would imply that b is selected in P1

P1

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	b	a	a	$\cdots$	a
a	$\cdots$	a	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	b	$\cdots$	b

↓ a

P3

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	b	a	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	a	$\cdots$	a
a	$\cdots$	a	$\cdot$	b	$\cdots$	b

↓

Pivotal Voter

- we consider two fixed alternatives $a, b \in A$
- we move from a profile where a is ranked highest by all voters to a profile where b is
  moving b one position at a time and one voter at a time
- by unanimity, the outcome must change from a to b
  voter i changes profile P1 to P2
- start from P2 and move a below for all voters but i, without changing pairwise relationships with b
  - by monotonicity, b must remain selected
- flip a and b in i's ranking and call the resulting profile P3
  - by monotonicity, either a or b is selected
  - selecting b would imply that b is selected in P1

P1

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	b	a	a	$\cdots$	a
a	$\cdots$	a	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	b	$\cdots$	b

$\downarrow$ a

P3

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	b	a	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	a	$\cdots$	a
a	$\cdots$	a	$\cdot$	b	$\cdots$	b

$\downarrow$ a

From Pivotal to Dictator

- consider $c \in A \setminus \{a, b\}$ and profile P4

P3

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	b	a	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	a	$\cdots$	a
a	$\cdots$	a	$\cdot$	b	$\cdots$	b

→ a

P4

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
$\cdot$	$\cdots$	$\cdot$	a	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	c	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
c	$\cdots$	c	$\cdot$	c	$\cdots$	c
b	$\cdots$	b	$\cdot$	a	$\cdots$	a
a	$\cdots$	a	$\cdot$	b	$\cdots$	b

From Pivotal to Dictator

- consider $c \in A \setminus \{a, b\}$ and profile P4
  - by monotonicity, a remains selected

P3

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	b	a	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	a	$\cdots$	a
a	$\cdots$	a	$\cdot$	b	$\cdots$	b

→ a

P4

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
$\cdot$	$\cdots$	$\cdot$	a	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	c	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
c	$\cdots$	c	$\cdot$	c	$\cdots$	c
b	$\cdots$	b	$\cdot$	a	$\cdots$	a
a	$\cdots$	a	$\cdot$	b	$\cdots$	b

→ a

From Pivotal to Dictator

- consider $c \in A \setminus \{a, b\}$ and profile P4
  - by monotonicity, a remains selected
- flip a and b in the ranking of all voters $j > i$

P3

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	b	a	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	a	$\cdots$	a
a	$\cdots$	a	$\cdot$	b	$\cdots$	b

$\rightarrow a$

P4

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
$\cdot$	$\cdots$	$\cdot$	a	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	c	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
c	$\cdots$	c	$\cdot$	c	$\cdots$	c
b	$\cdots$	b	$\cdot$	a	$\cdots$	a
a	$\cdots$	a	$\cdot$	b	$\cdots$	b

$\rightarrow a$

- consider $c \in A \backslash \{a, b\}$ and profile P4
  - by monotonicity, a remains selected
- flip a and b in the ranking of all voters $j > i$

P3

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	b	a	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	a	$\cdots$	a
a	$\cdots$	a	$\cdot$	b	$\cdots$	b

→ a

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
$\cdot$	$\cdots$	$\cdot$	a	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	c	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
c	$\cdots$	c	$\cdot$	c	$\cdots$	c
b	$\cdots$	b	$\cdot$	b	$\cdots$	b
a	$\cdots$	a	$\cdot$	a	$\cdots$	a

From Pivotal to Dictator

- consider $c \in A \setminus \{a, b\}$ and profile P4
  - by monotonicity, a remains selected
- flip a and b in the ranking of all voters $j > i$
  - by monotonicity, either a or b is selected

P3

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	b	a	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	a	$\cdots$	a
a	$\cdots$	a	$\cdot$	b	$\cdots$	b

→ a

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
$\cdot$	$\cdots$	$\cdot$	a	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	c	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
c	$\cdots$	c	$\cdot$	c	$\cdots$	c
b	$\cdots$	b	$\cdot$	b	$\cdots$	b
a	$\cdots$	a	$\cdot$	a	$\cdots$	a

→

From Pivotal to Dictator

- consider $c \in A \setminus \{a, b\}$ and profile P4
  - by monotonicity, a remains selected
- flip a and b in the ranking of all voters $j > i$
  - by monotonicity, either a or b is selected
  - selecting b would imply that b is selected when c moves to the top of all rankings, contradicting unanimity

P3

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	b	a	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	a	$\cdots$	a
a	$\cdots$	a	$\cdot$	b	$\cdots$	b

$\rightarrow a$

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
$\cdot$	$\cdots$	$\cdot$	a	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	c	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
c	$\cdots$	c	$\cdot$	c	$\cdots$	c
b	$\cdots$	b	$\cdot$	b	$\cdots$	b
a	$\cdots$	a	$\cdot$	a	$\cdots$	a

$\rightarrow$

From Pivotal to Dictator

- consider $c \in A \setminus \{a, b\}$ and profile P4
  - by monotonicity, a remains selected
- flip a and b in the ranking of all voters $j > i$
  - by monotonicity, either a or b is selected
  - selecting b would imply that b is selected when c moves to the top of all rankings, contradicting unanimity

P3

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	b	a	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	a	$\cdots$	a
a	$\cdots$	a	$\cdot$	b	$\cdots$	b

$\rightarrow a$

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
$\cdot$	$\cdots$	$\cdot$	a	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	c	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
c	$\cdots$	c	$\cdot$	c	$\cdots$	c
b	$\cdots$	b	$\cdot$	b	$\cdots$	b
a	$\cdots$	a	$\cdot$	a	$\cdots$	a

$\rightarrow a$

From Pivotal to Dictator

- consider $c \in A \setminus \{a, b\}$ and profile P4
  - by monotonicity, a remains selected
- flip a and b in the ranking of all voters $j > i$
  - by monotonicity, either a or b is selected
  - selecting b would imply that b is selected when c moves to the top of all rankings, contradicting unanimity
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P3

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	b	a	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	a	$\cdots$	a
a	$\cdots$	a	$\cdot$	b	$\cdots$	b

$\rightarrow a$

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
$\cdot$	$\cdots$	$\cdot$	a	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	c	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
c	$\cdots$	c	$\cdot$	c	$\cdots$	c
b	$\cdots$	b	$\cdot$	b	$\cdots$	b
a	$\cdots$	a	$\cdot$	a	$\cdots$	a

$\rightarrow a$

From Pivotal to Dictator

- consider $c \in A \backslash \{a, b\}$ and profile P4
  - by monotonicity, a remains selected
- flip a and b in the ranking of all voters $j > i$
  - by monotonicity, either a or b is selected
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- by monotonicity, a is selected for any profile where it is at the top of i's ranking
- there is a dictator for any alternative

P3

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b	$\cdots$	b	a	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	a	$\cdots$	a
a	$\cdots$	a	$\cdot$	b	$\cdots$	b

$\rightarrow a$

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
$\cdot$	$\cdots$	$\cdot$	a	$\cdot$	$\cdots$	$\cdot$
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$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
c	$\cdots$	c	$\cdot$	c	$\cdots$	c
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- there is a dictator for any alternative
  - there is a unique dictator for all alternatives

P3

$\succ_1$	$\cdots$	$\succ_{i-1}$	$\succ_i$	$\succ_{i+1}$	$\cdots$	$\succ_n$
b	$\cdots$	b	a	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdots$	$\cdot$	b	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\cdot$	$\cdots$	$\cdot$	$\cdot$	a	$\cdots$	a
a	$\cdots$	a	$\cdot$	b	$\cdots$	b

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$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
c	$\cdots$	c	$\cdot$	c	$\cdots$	c
b	$\cdots$	b	$\cdot$	b	$\cdots$	b
a	$\cdots$	a	$\cdot$	a	$\cdots$	a

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Lemma [Muller, Satterthwaite ’77]

If $f\colon \mathscr{L}(A)^n \to A$ is surjective and strategyproof, then it is unanimous and monotone.

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  - by monotonicity, this holds when we move a to the top of all rankings and shuffle the rest

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General Mechanisms

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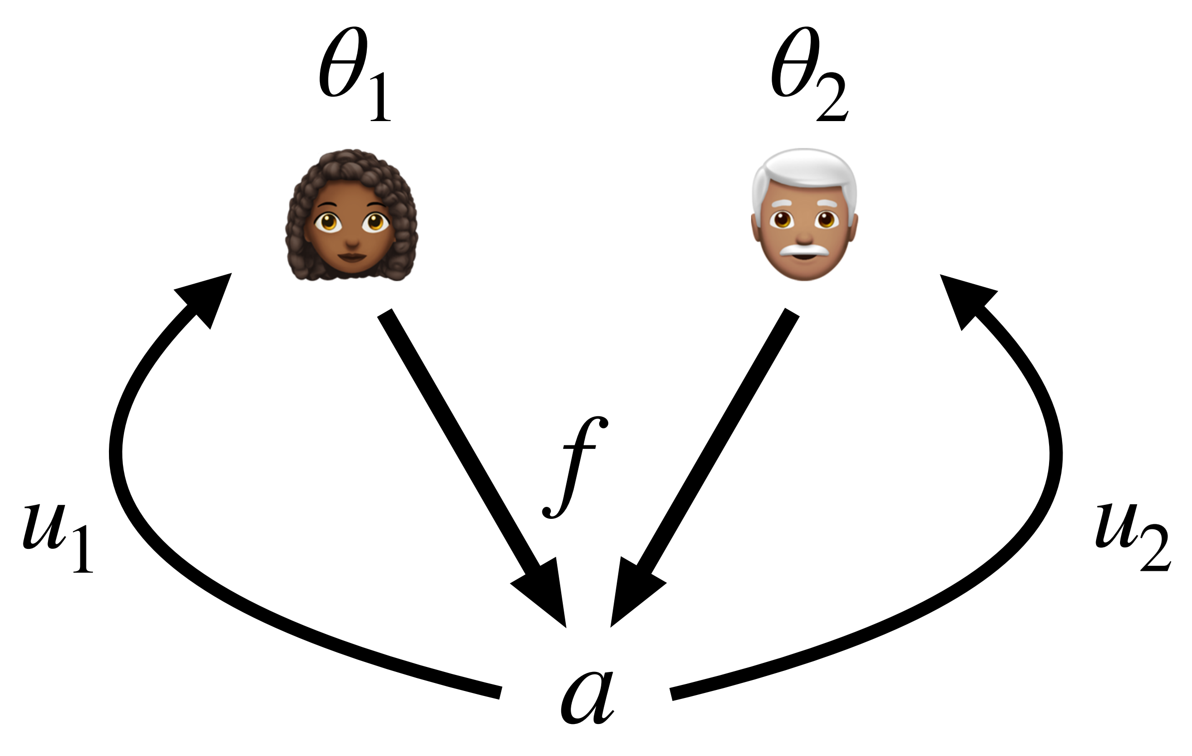

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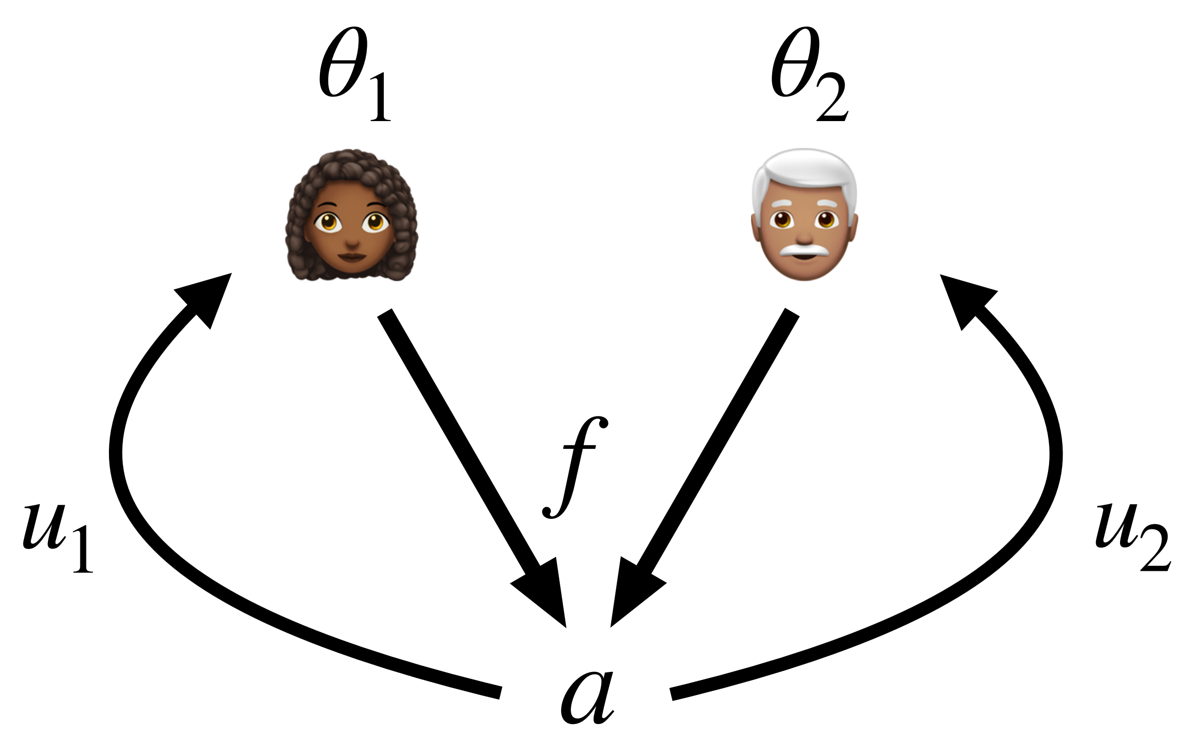

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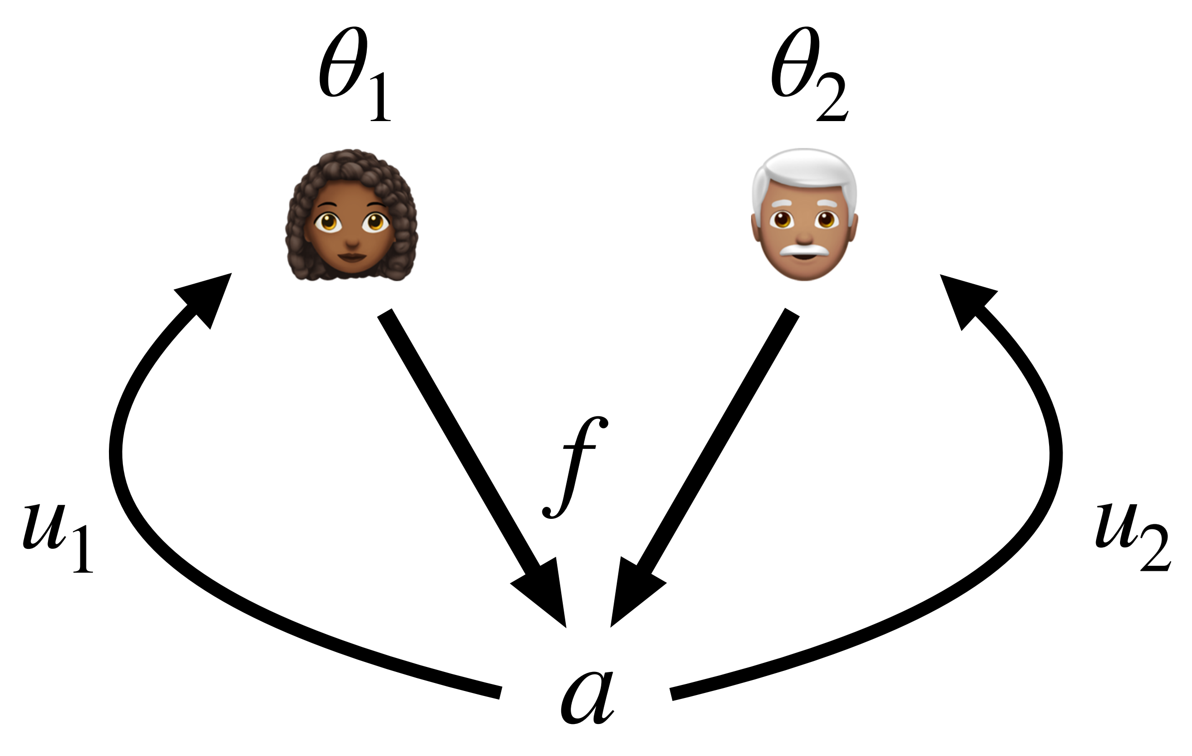

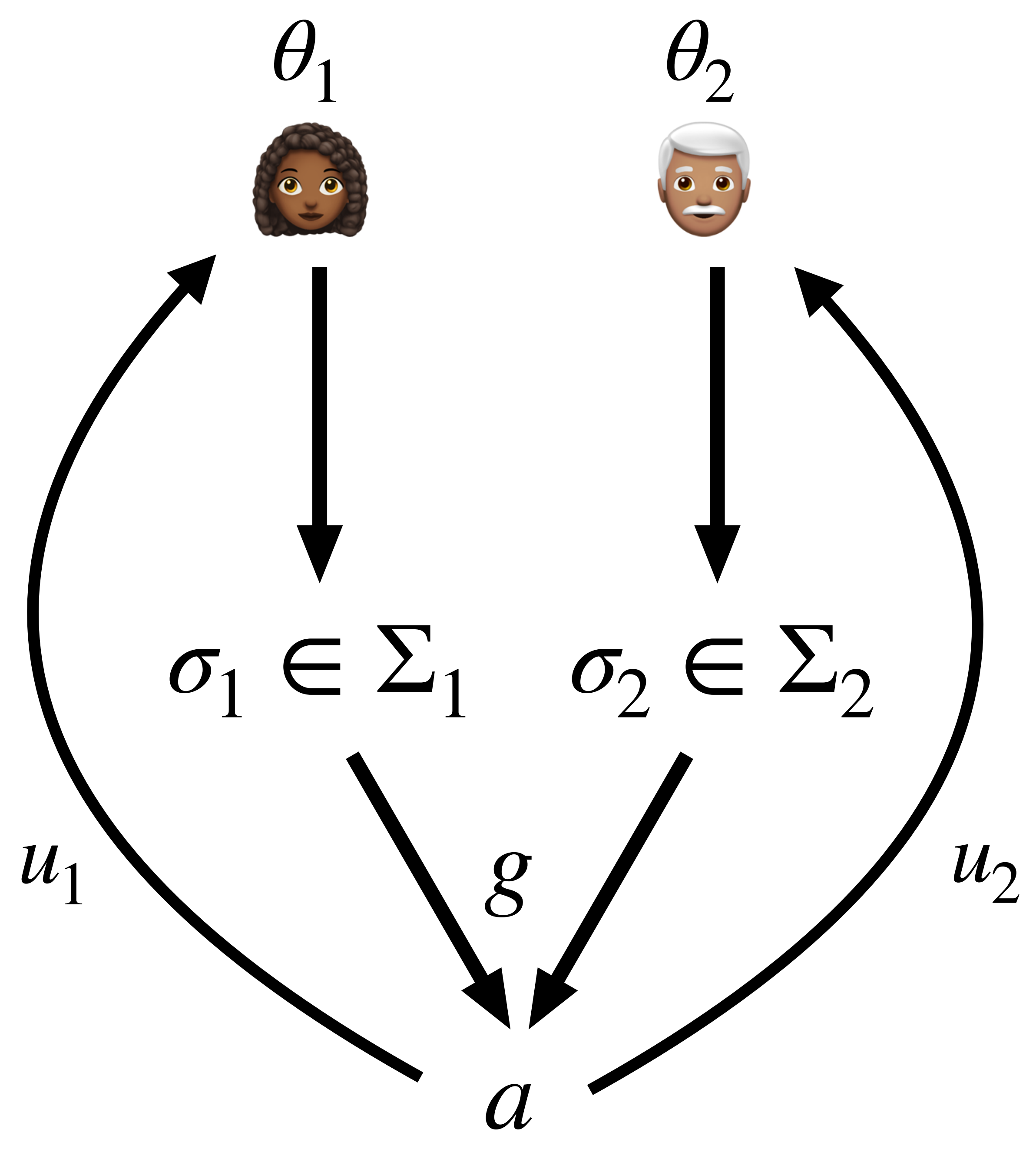

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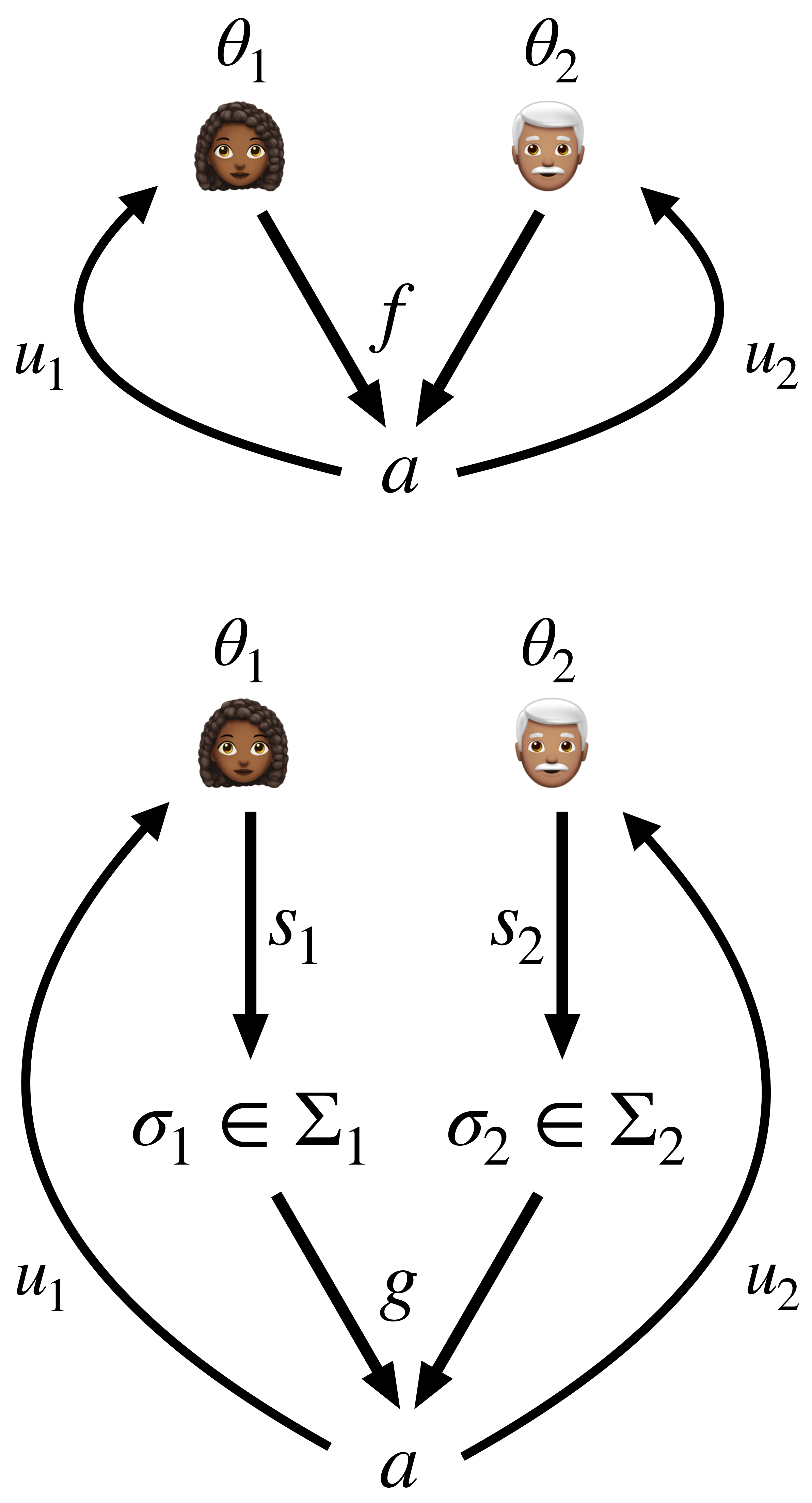

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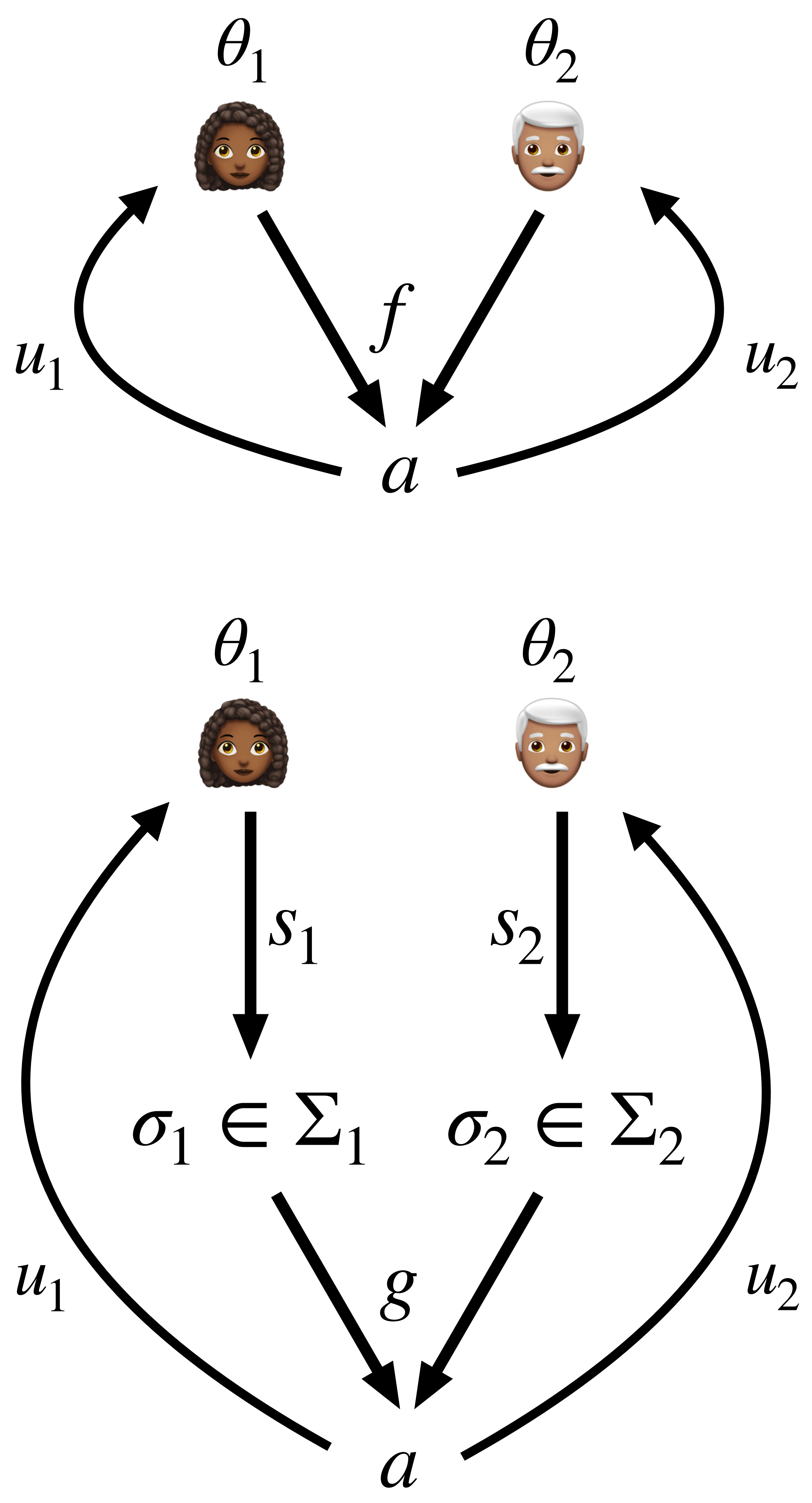

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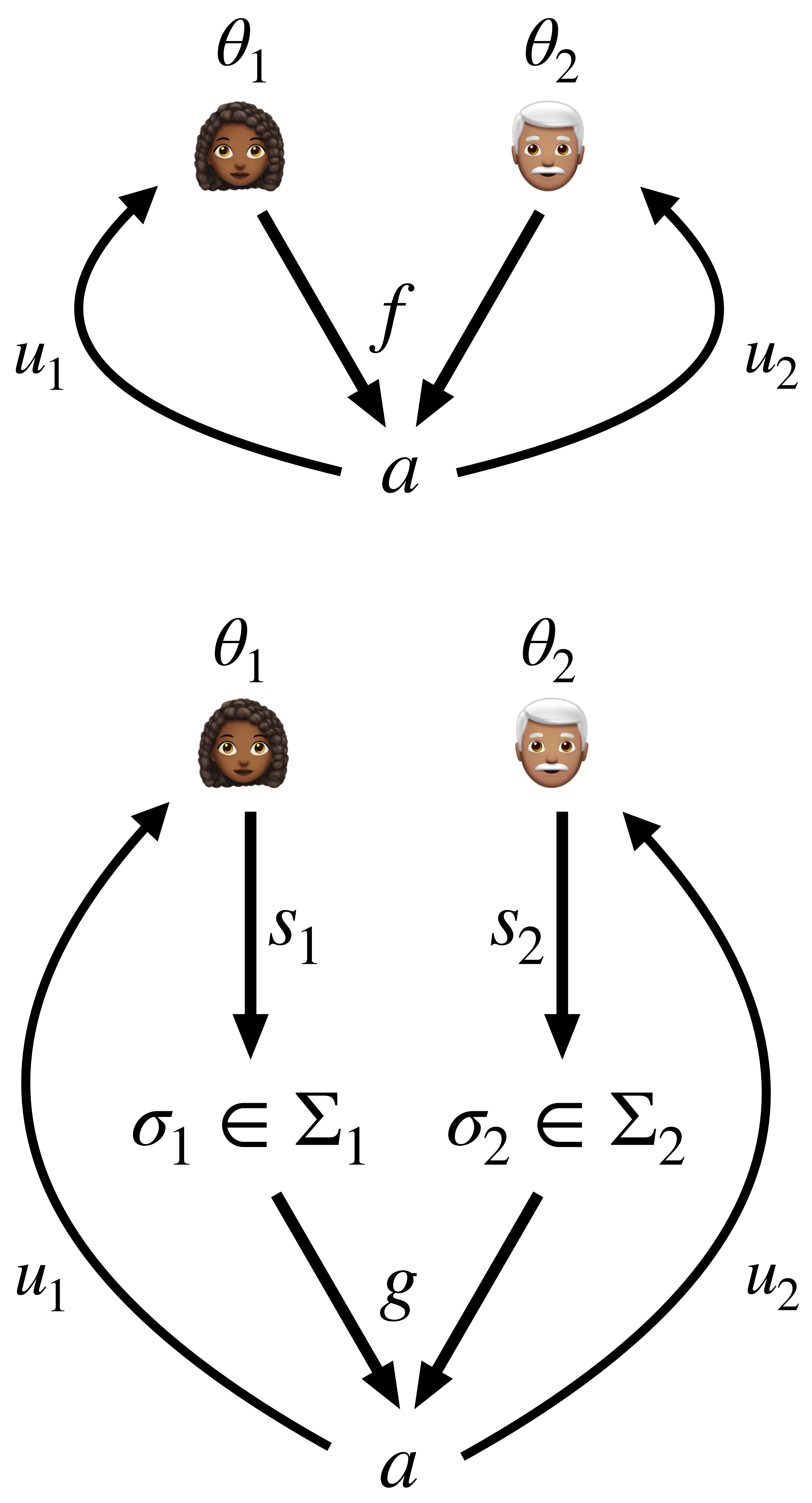

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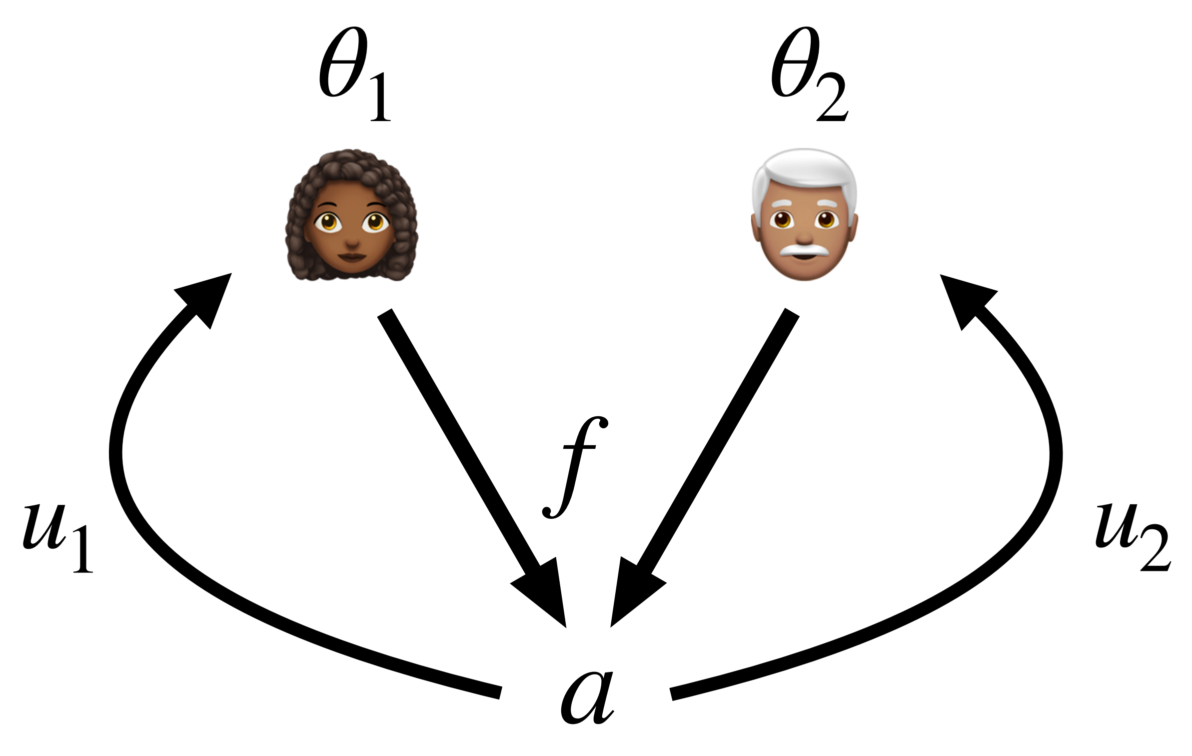
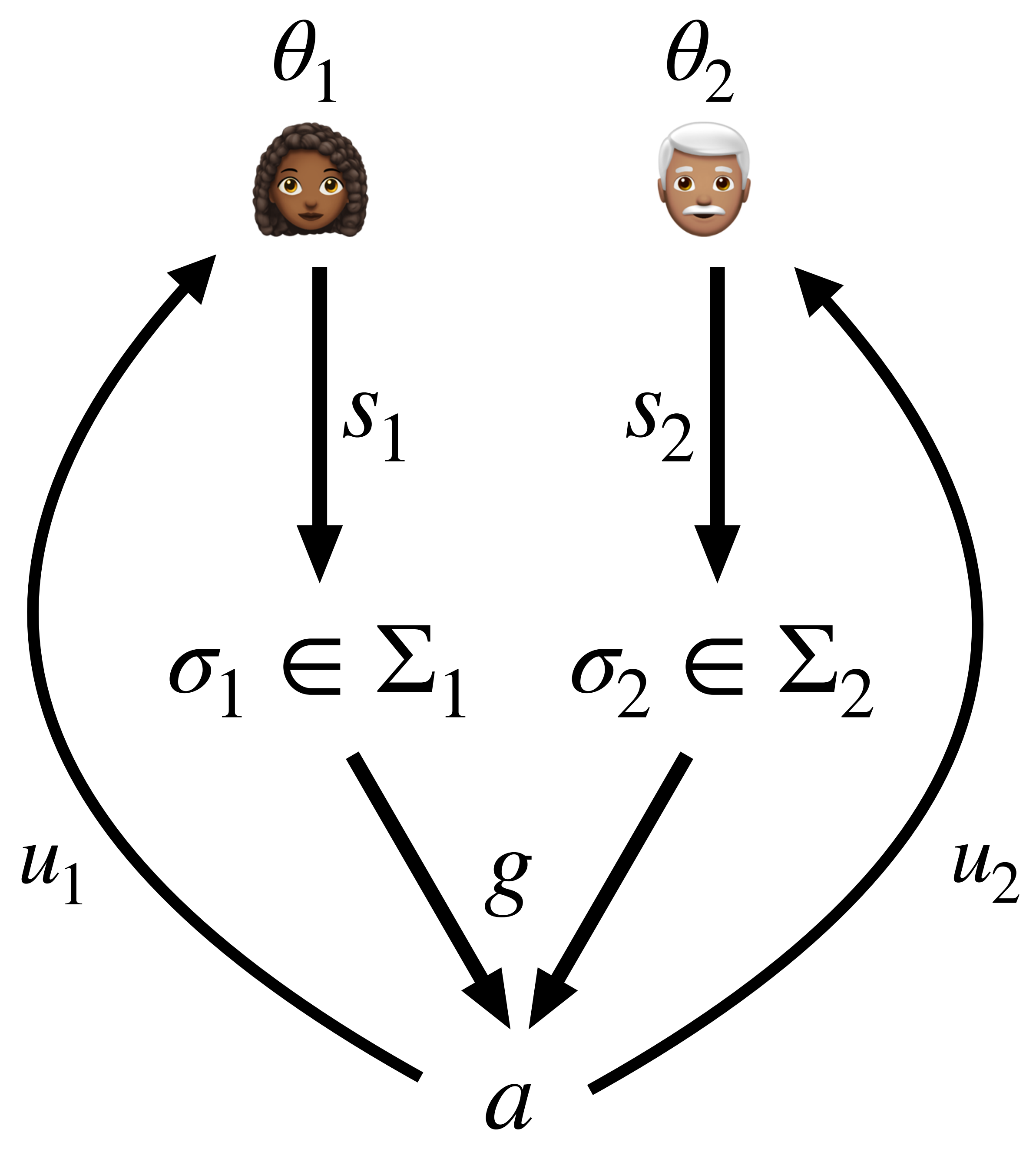
  - $u_i(g(s_i(\theta_i), \sigma_{-i}), \theta_i) \geq u_i(g(\sigma), \theta_i)$ for all $i \in V$ and $\sigma \in \Sigma$
    s_i is a dominant strategy for every player i





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$$u_i(h(\theta), \theta_i) = u_i(g(s_1(\theta_1), \ldots, s_n(\theta_n)), \theta_i)$$
$$\geq u_i(g(s_1(\theta_1), \ldots, s_i(\theta_i'), \ldots, s_n(\theta_n)), \theta_i) = u_i(h(\theta_i', \theta_{-i}), \theta_i)$$

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- G-S: when $|A| \geq 3$, $\Theta = \mathscr{L}(A)^n$, and $u_i \colon A \times \Theta_i \to \mathbb{R}$ is s.t. $u_i(a, \theta_i) > u_i(b, \theta_i) \iff a \succ_i b$, the only surjective social choice functions that can be implemented are dictatorships
- stronger assumptions on Θ_i and/or u_i are required to obtain positive results

The G-S theorem seems to quash any hope of designing incentive-compatible social-choice functions. The whole field of Mechanism Design attempts escaping from this impossibility result using various modifications in the model. [Nisan '07]

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