UNIVERSITÄT DES SAARLANDES

MAX PLANCK INSTITUTE FOR INFORMATICS

Stable Matching

Mechanism Design Without Money

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Several proofs and examples of this lecture are taken from Thomas Kesselheim's lecture notes

Matching Markets and Stability

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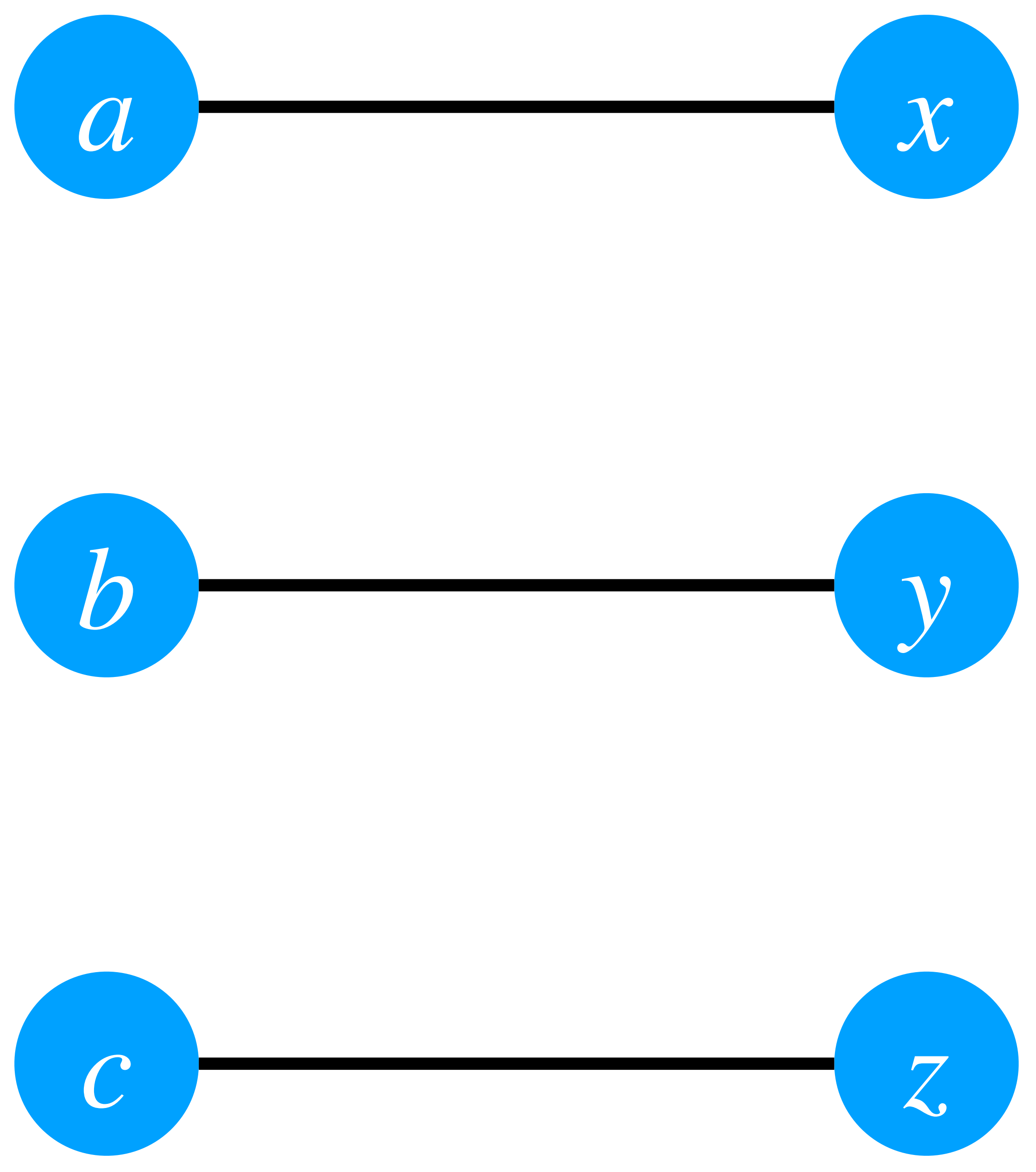

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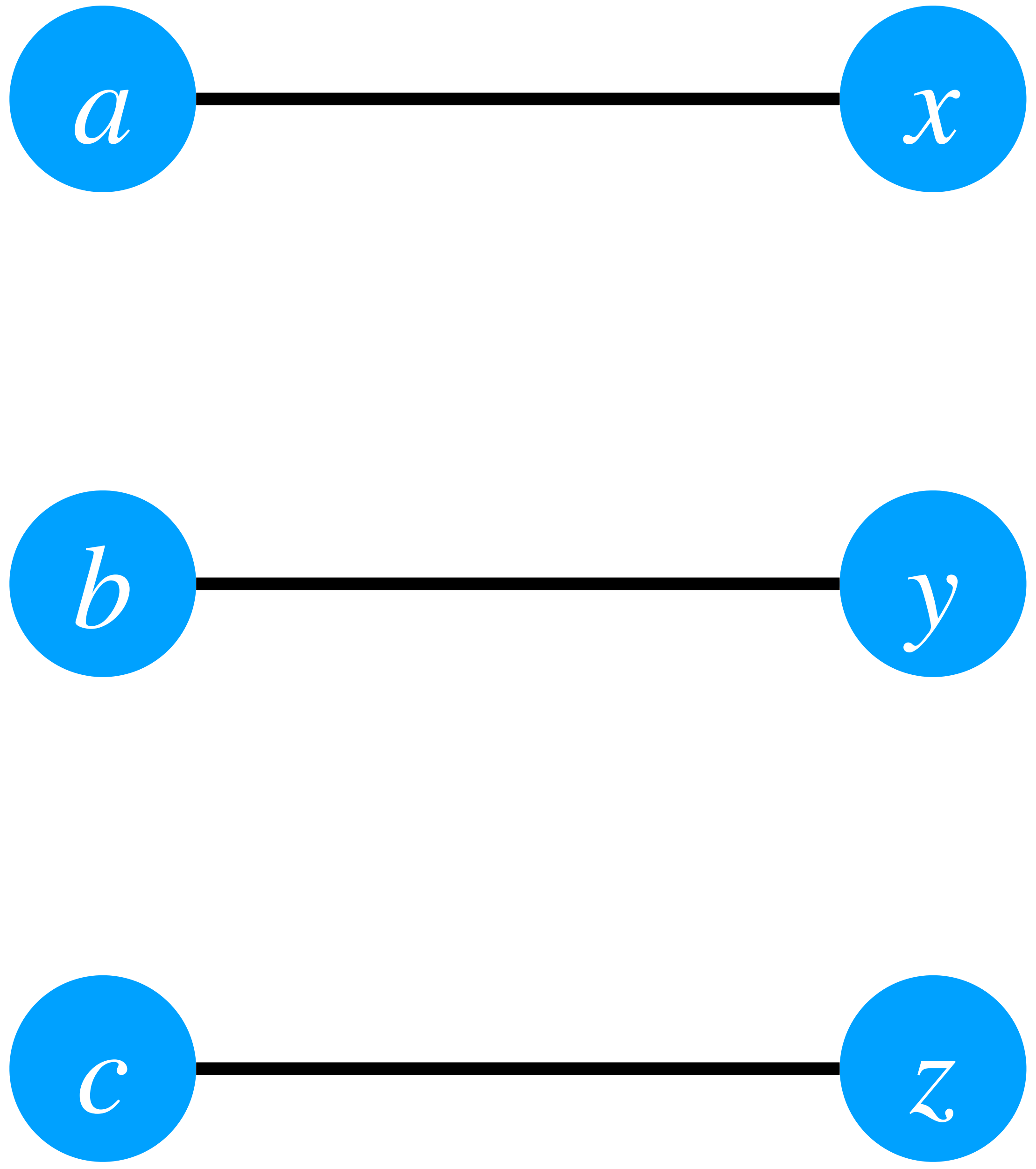

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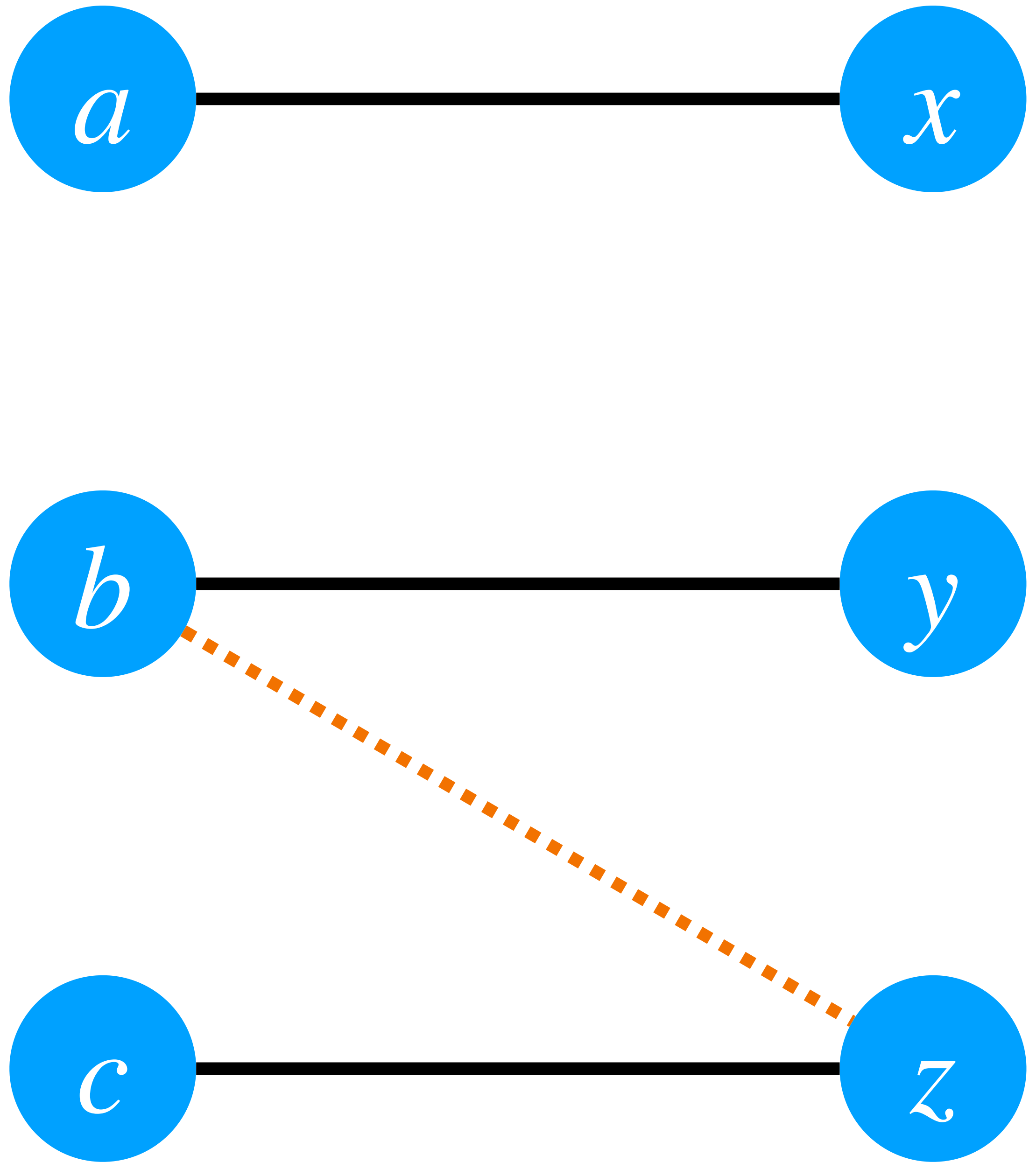

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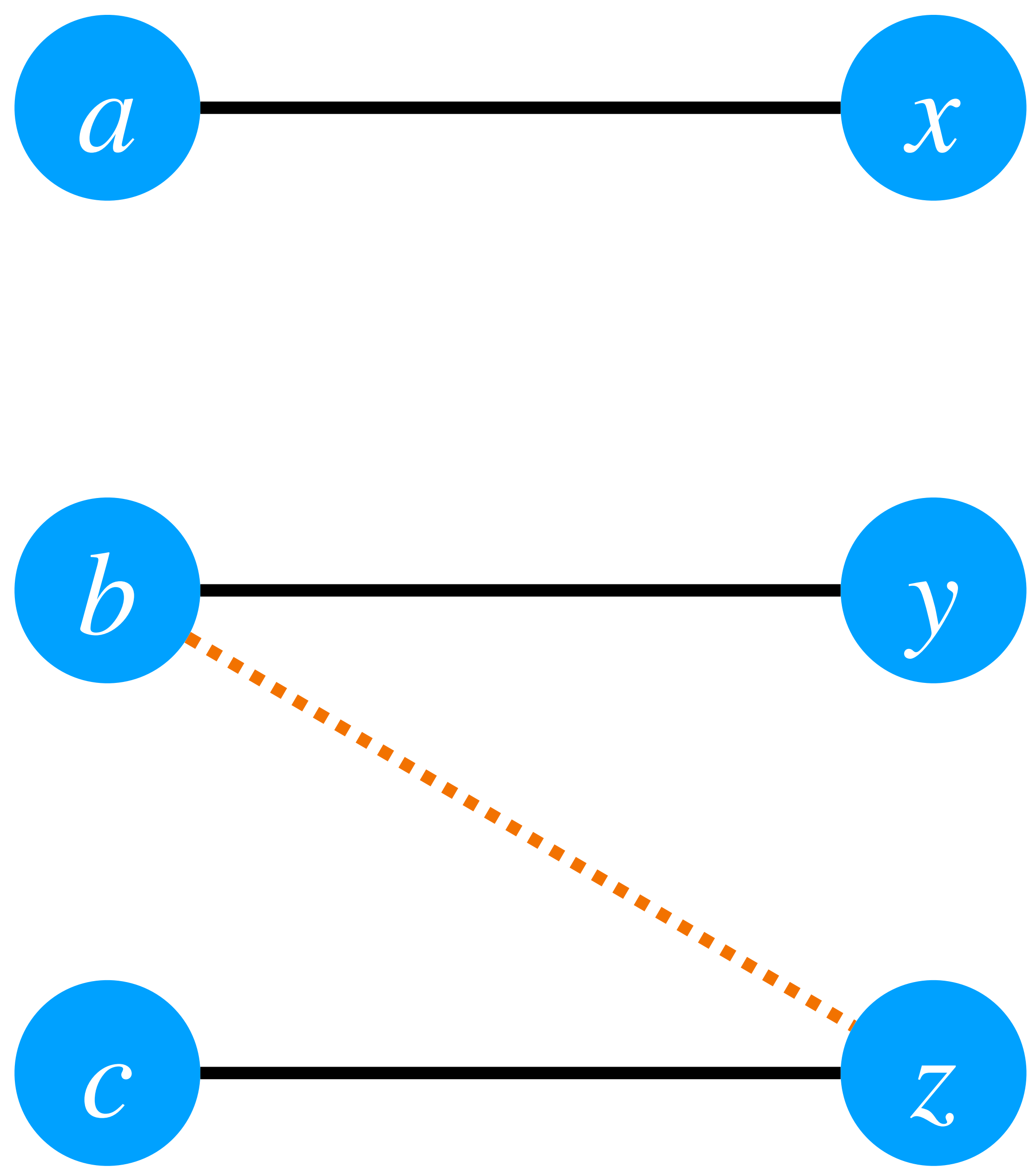

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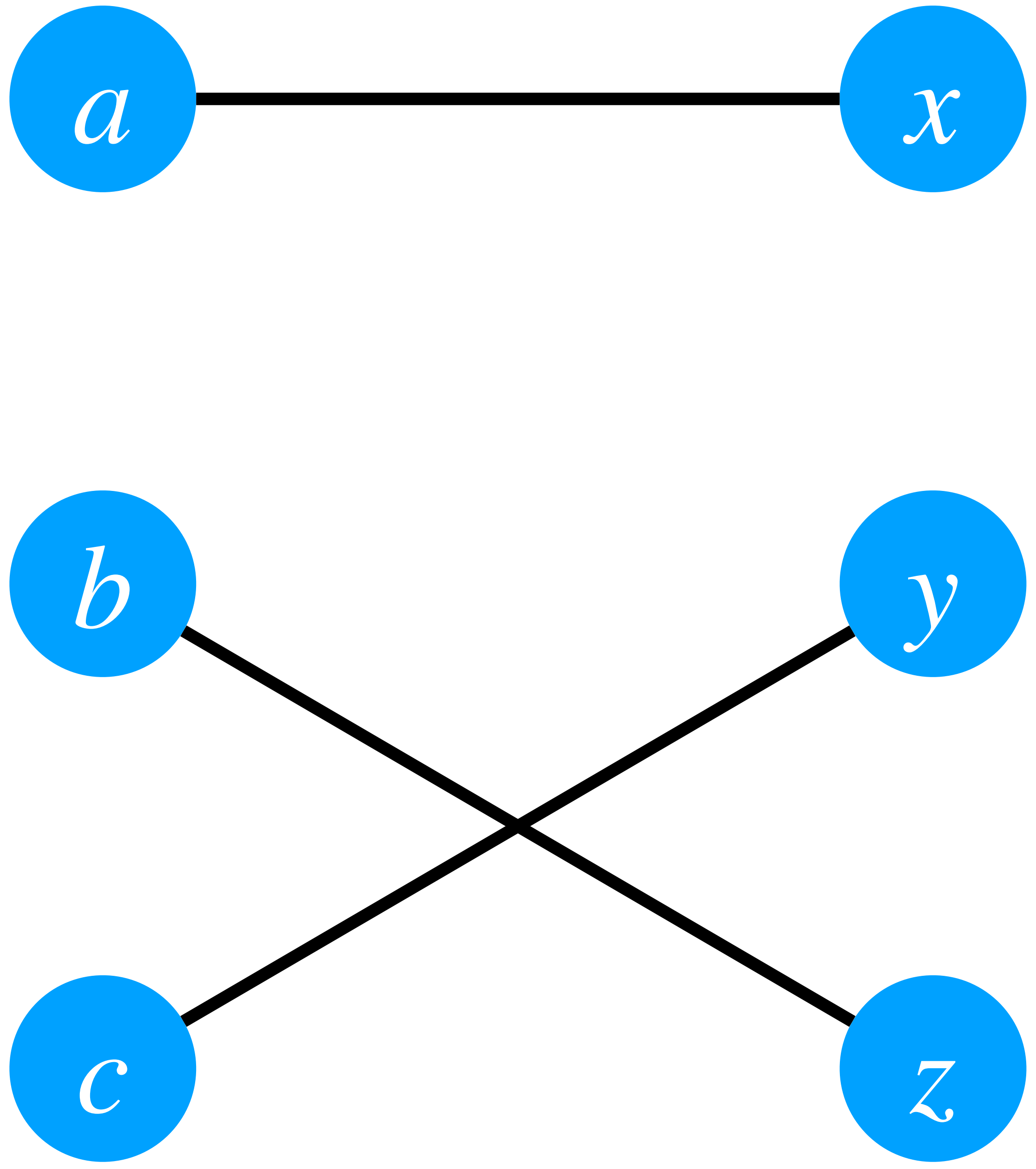

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- a mechanism is stable if it produces stable matchings
  analogously for other properties

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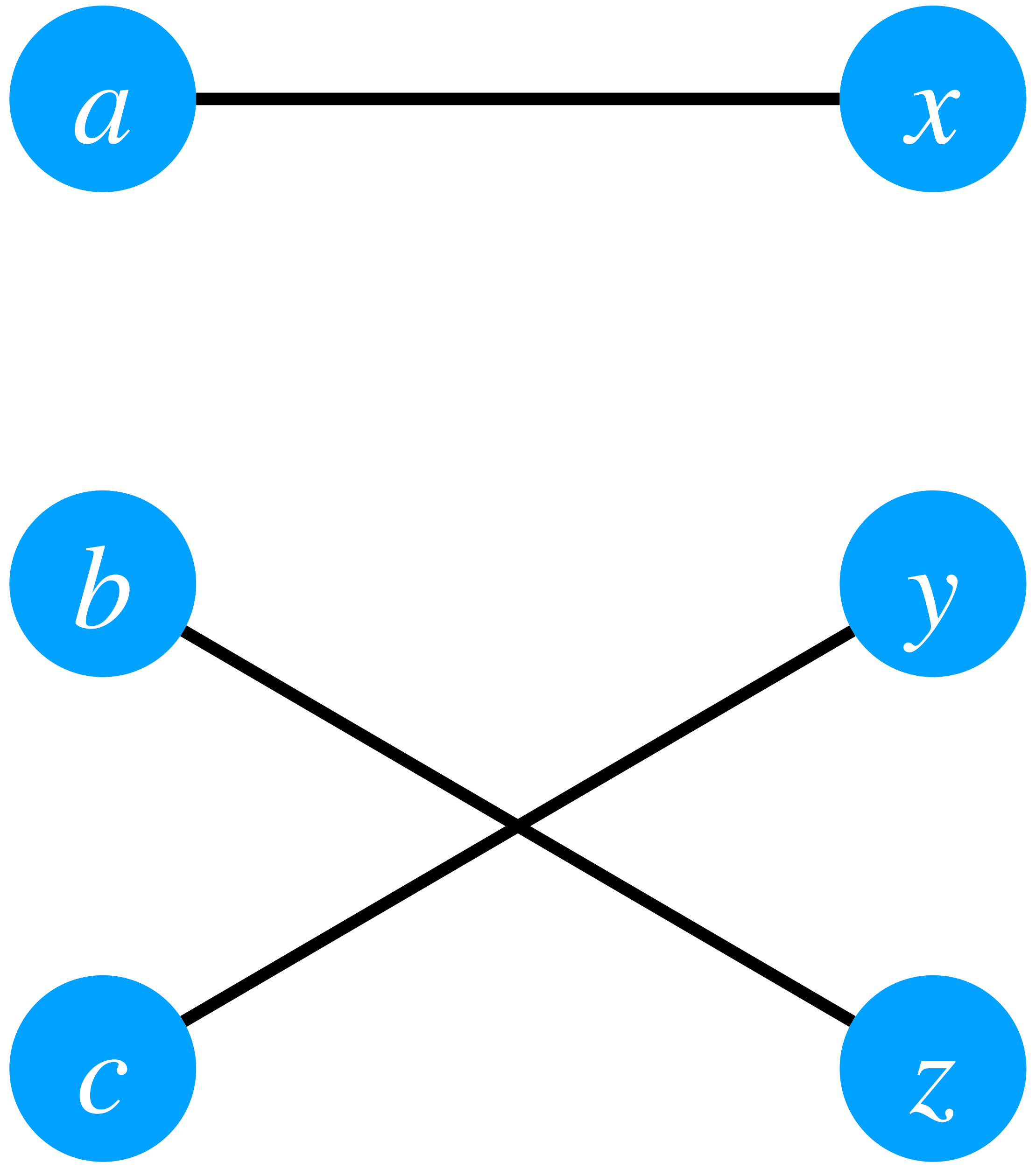

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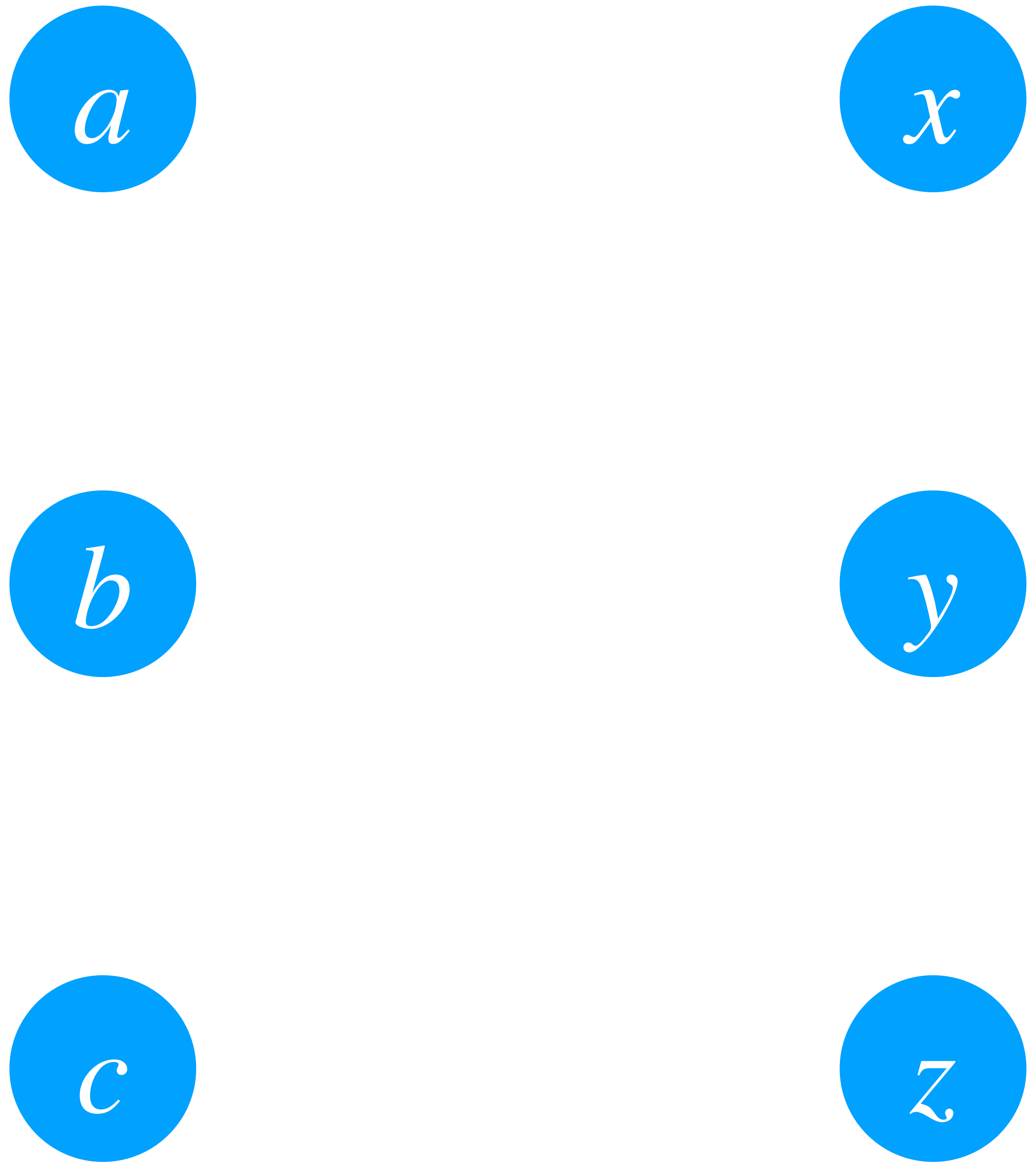

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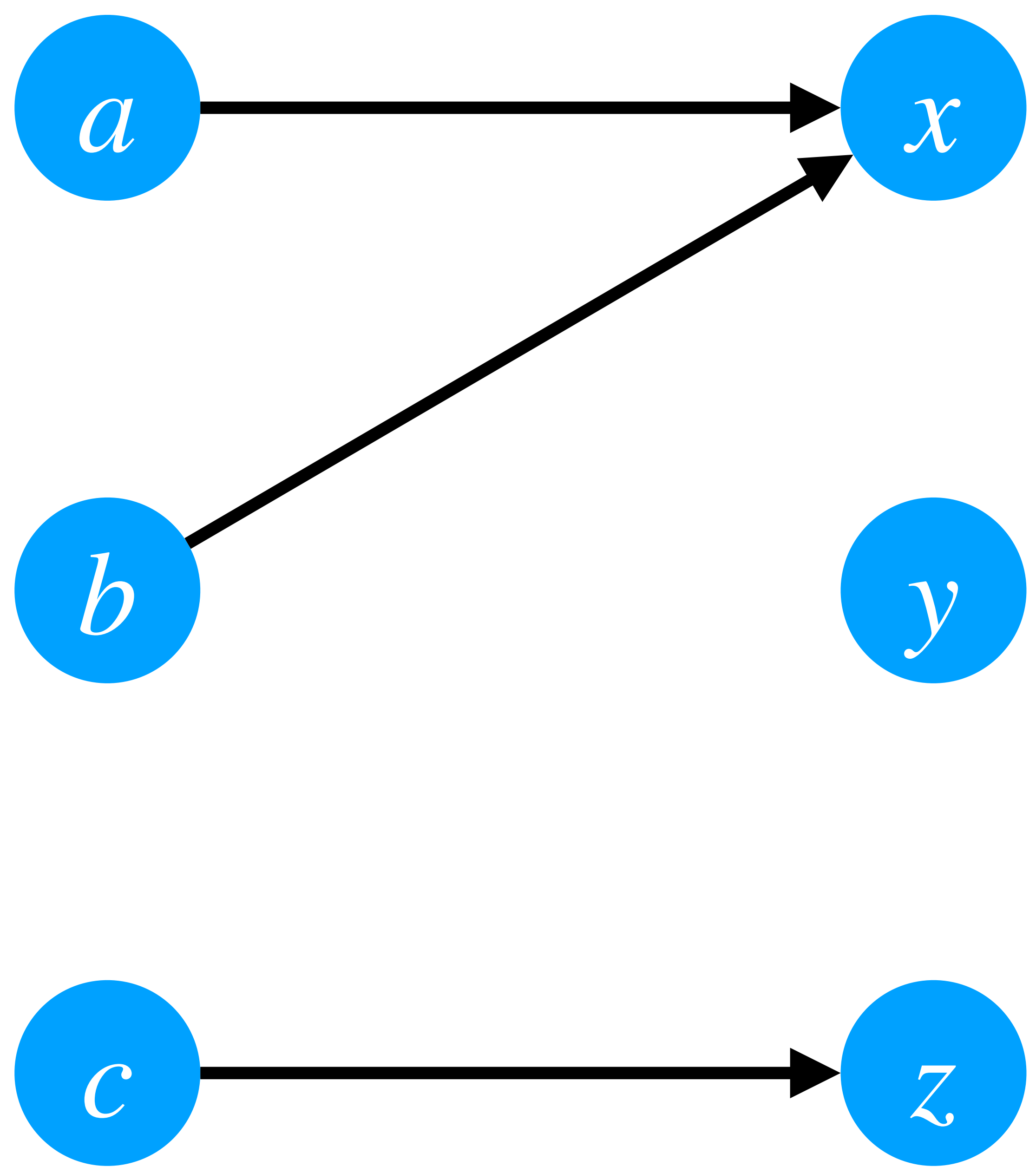

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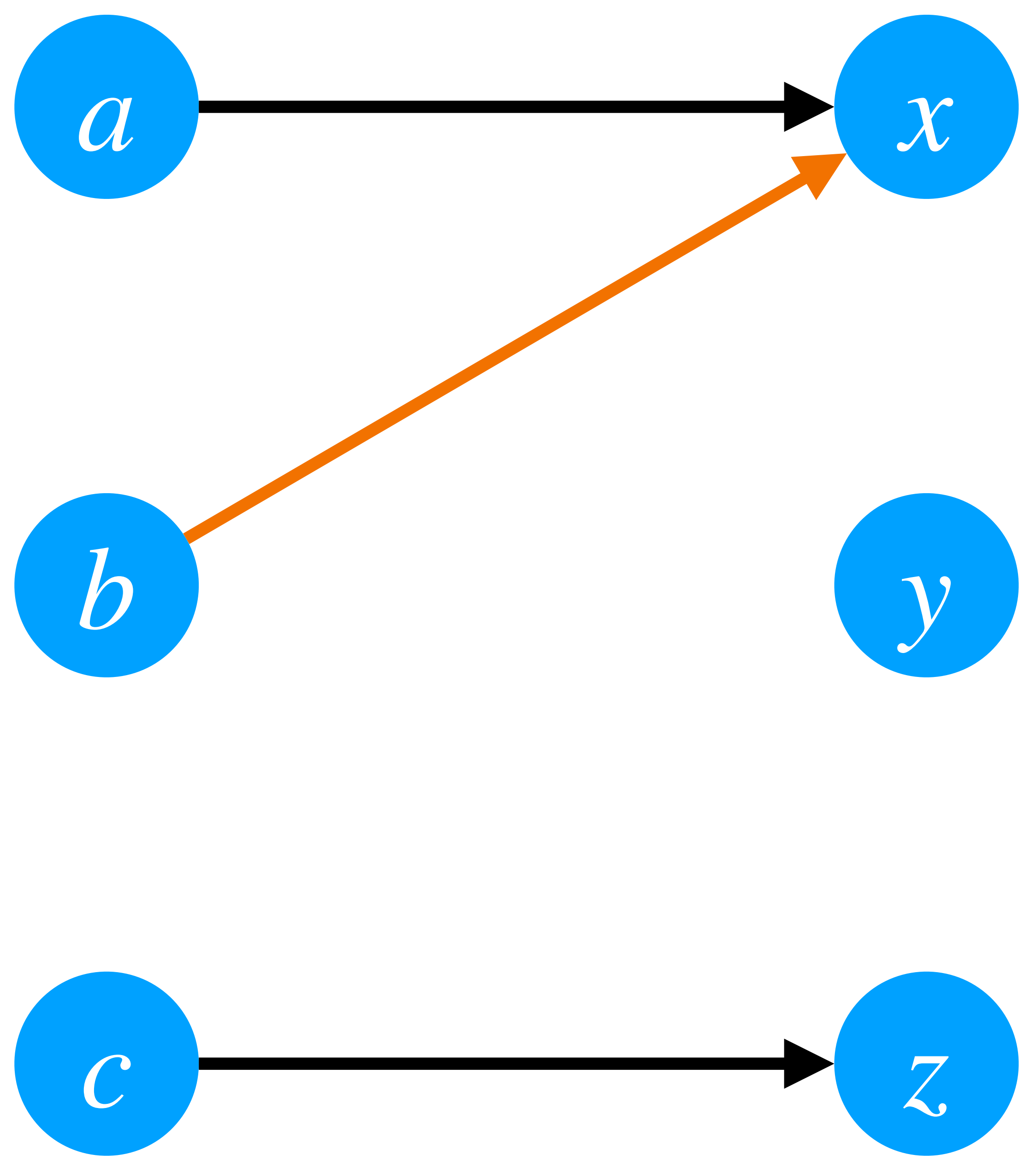

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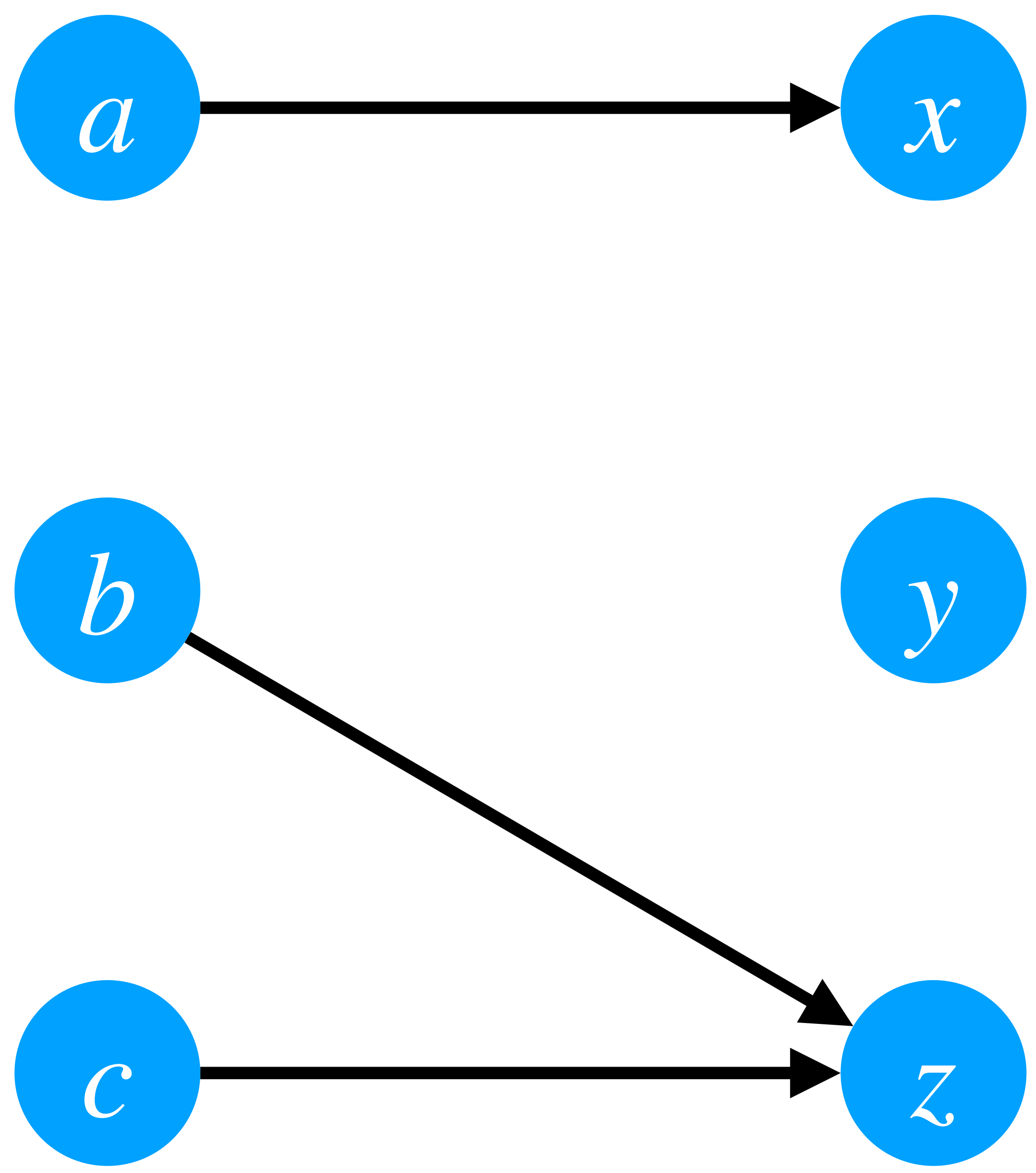

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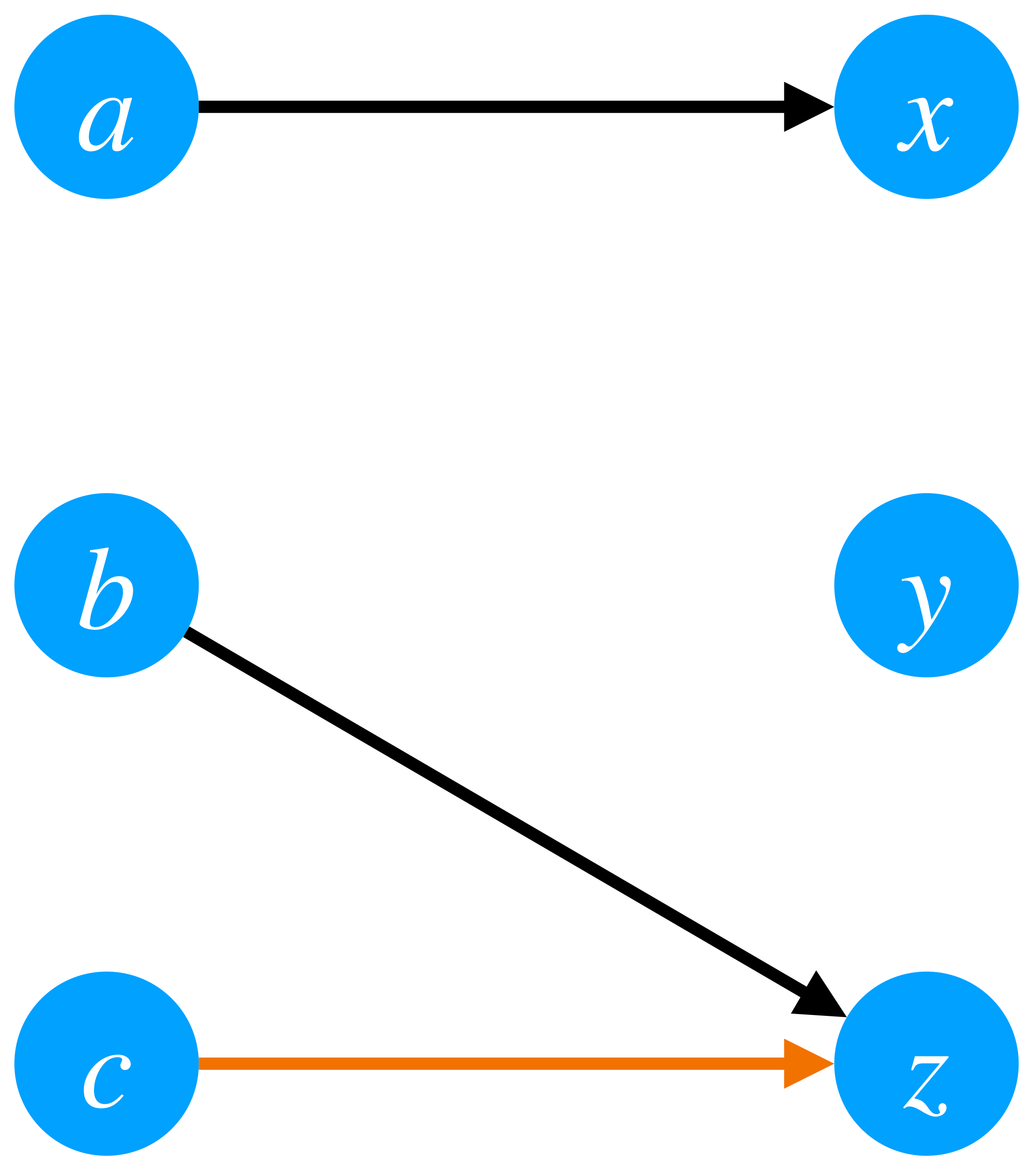

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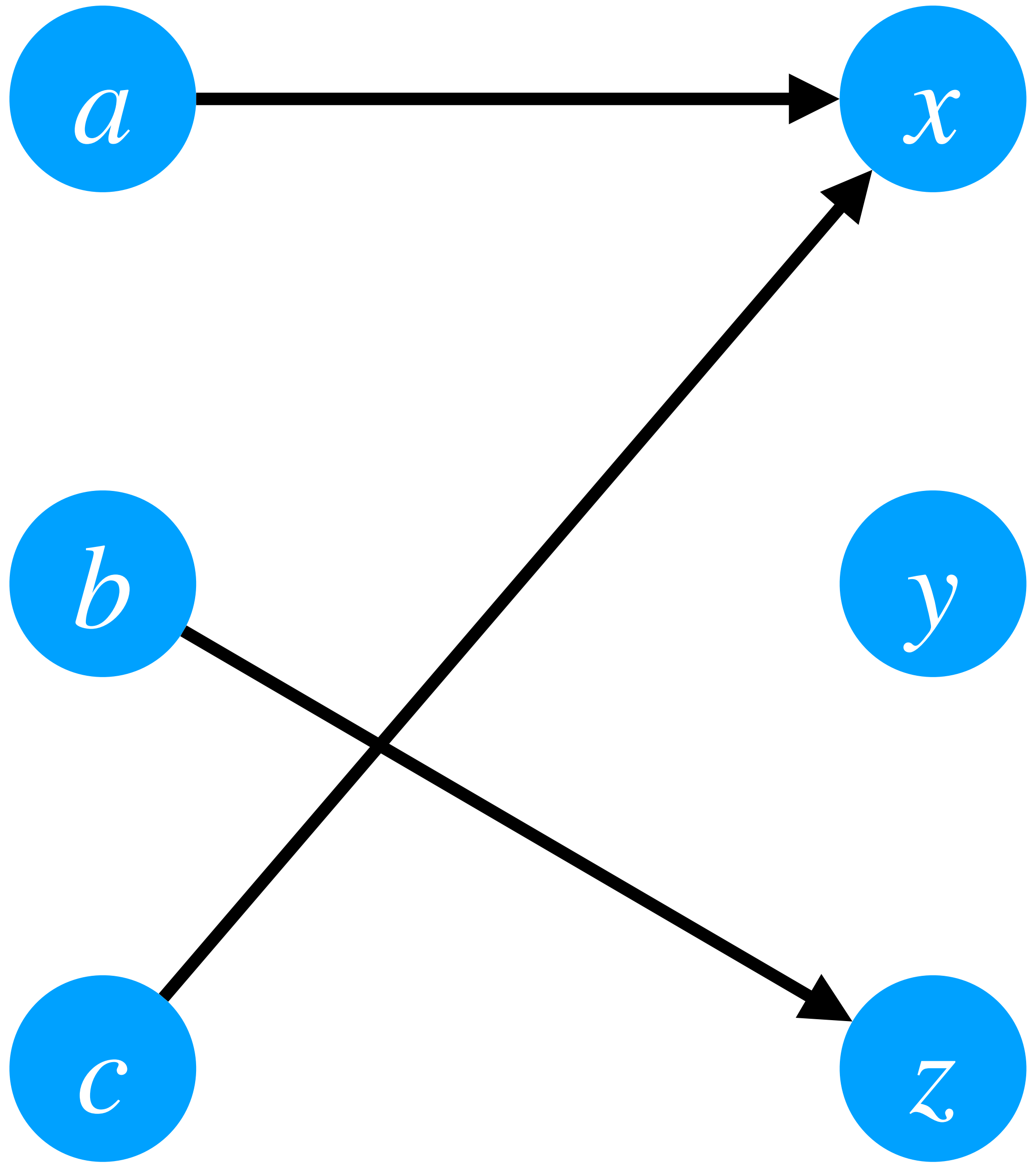

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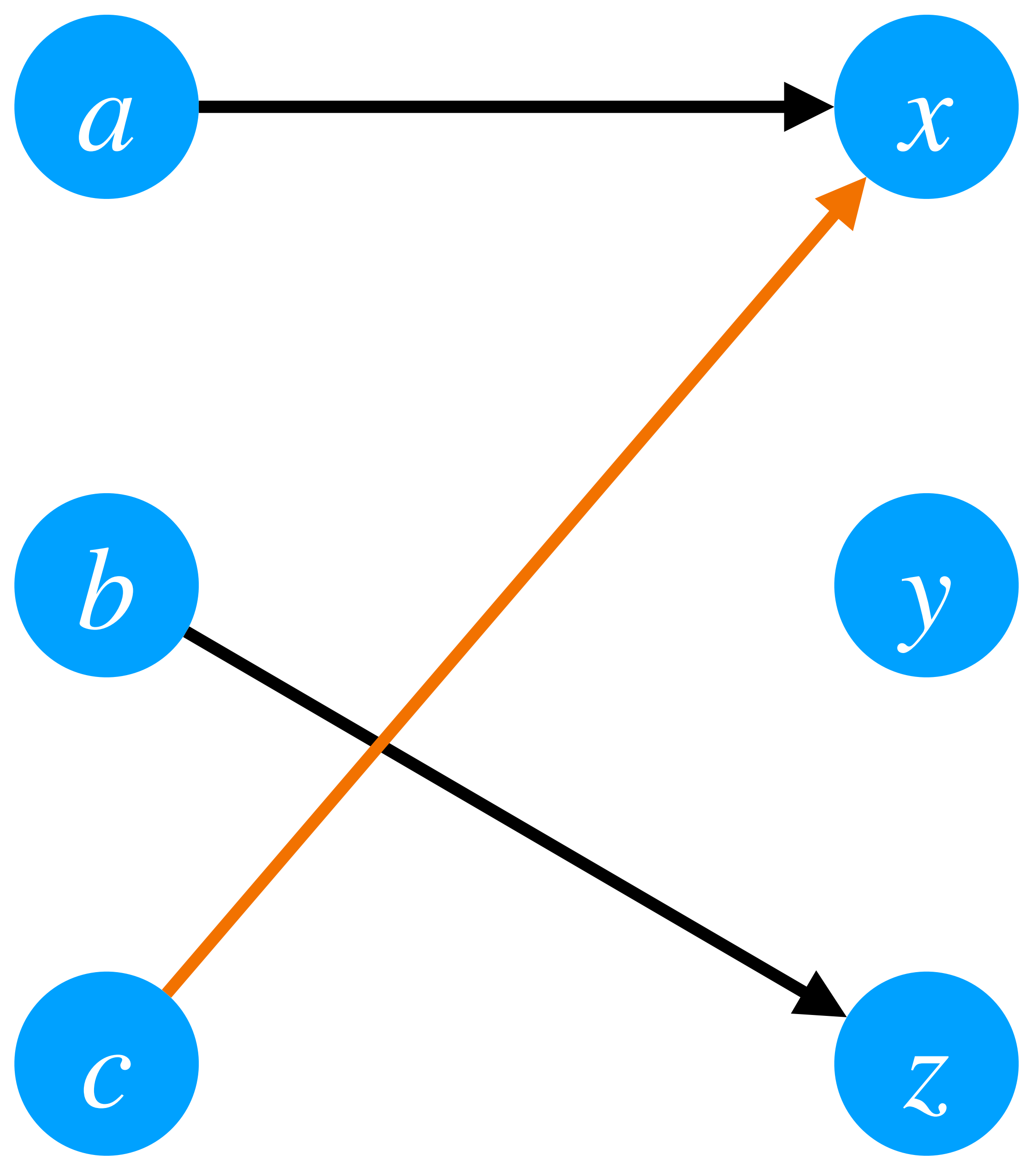

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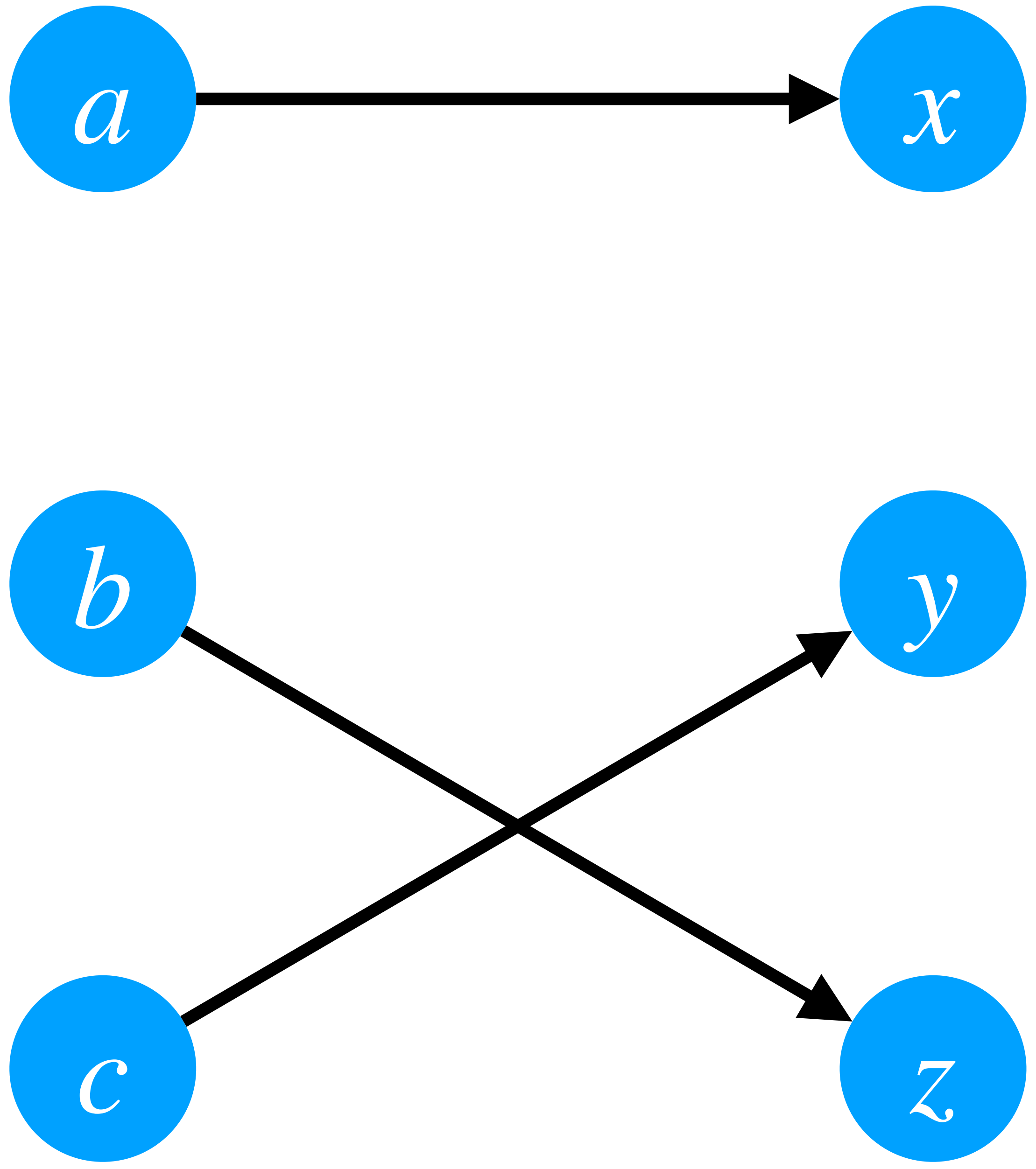

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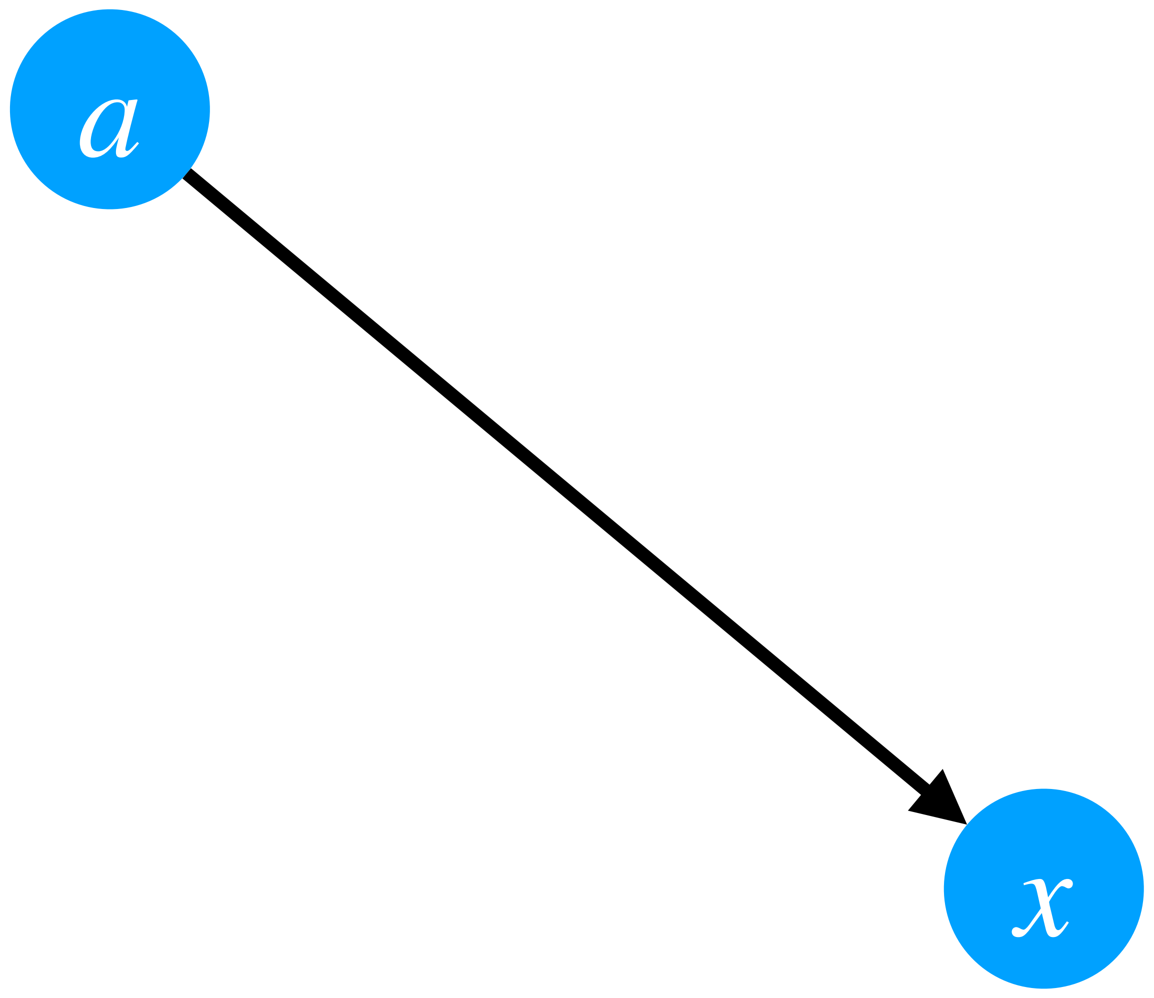

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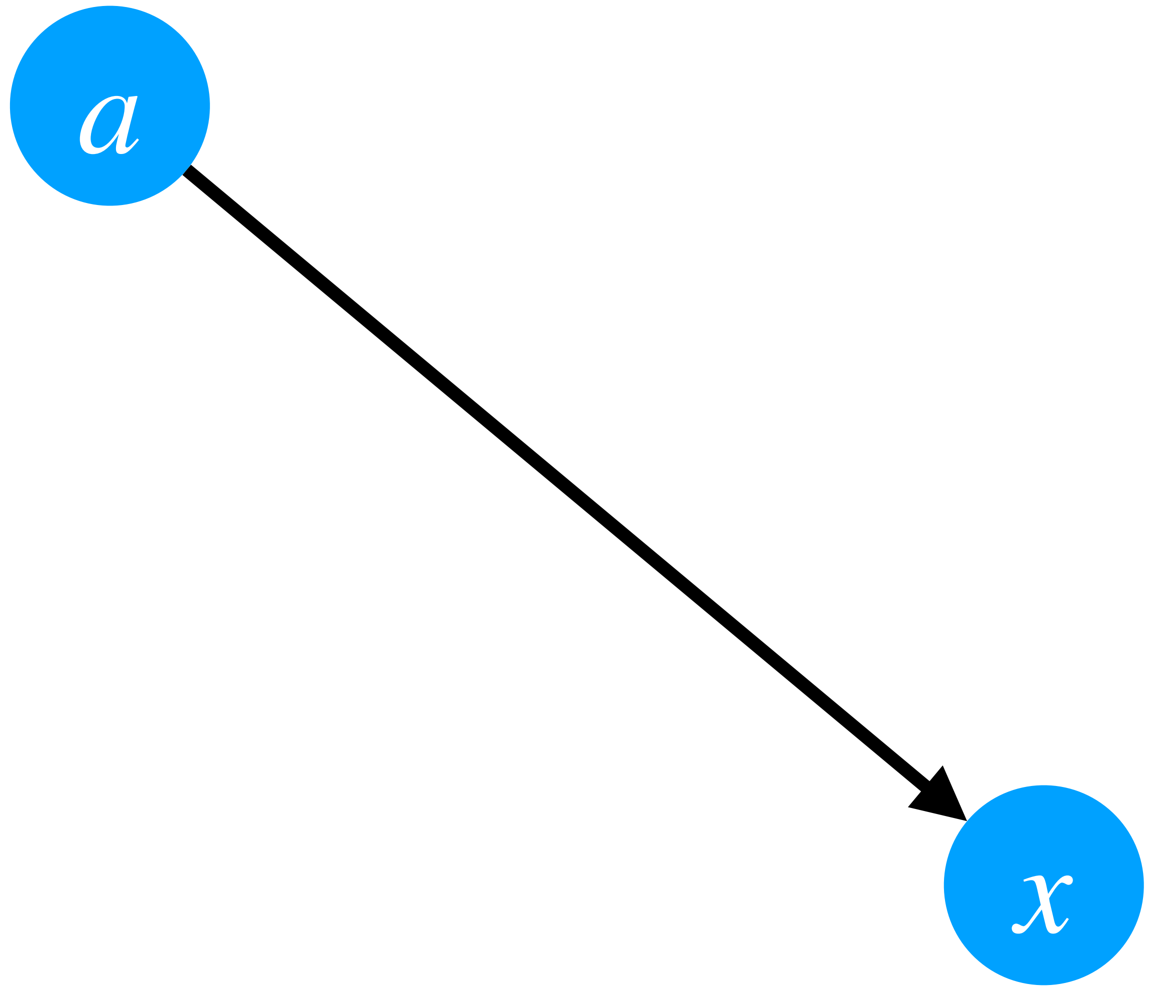

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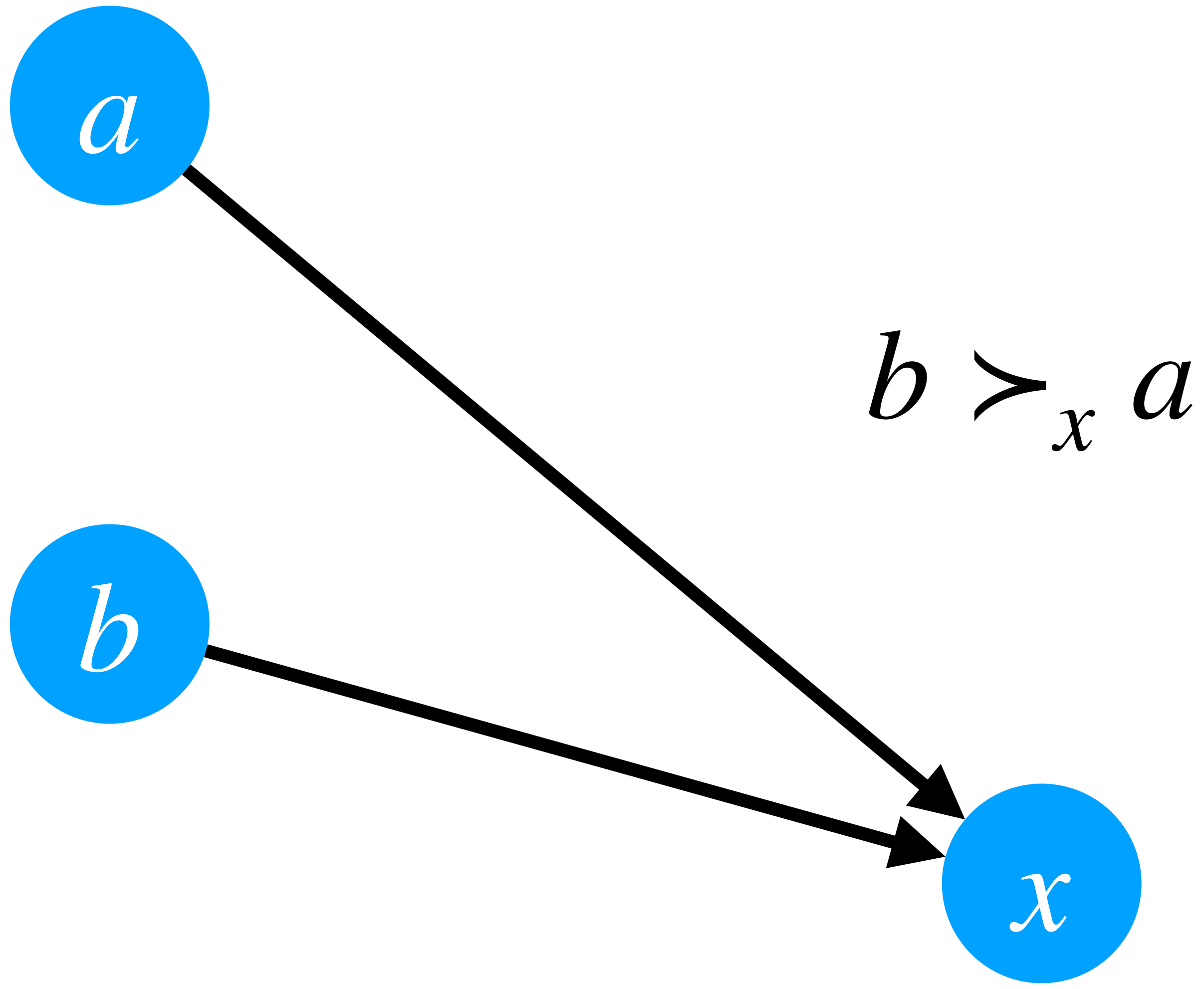

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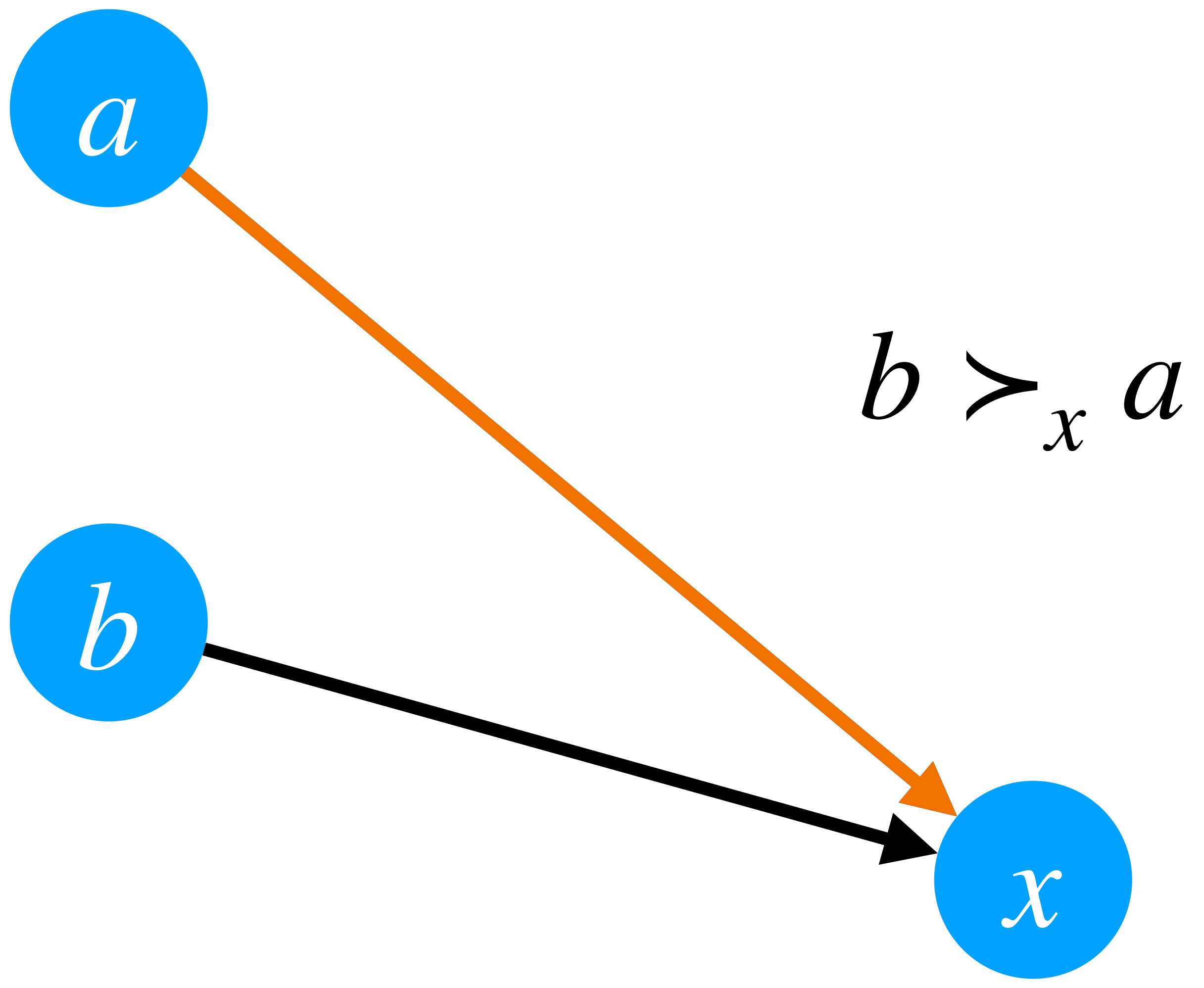

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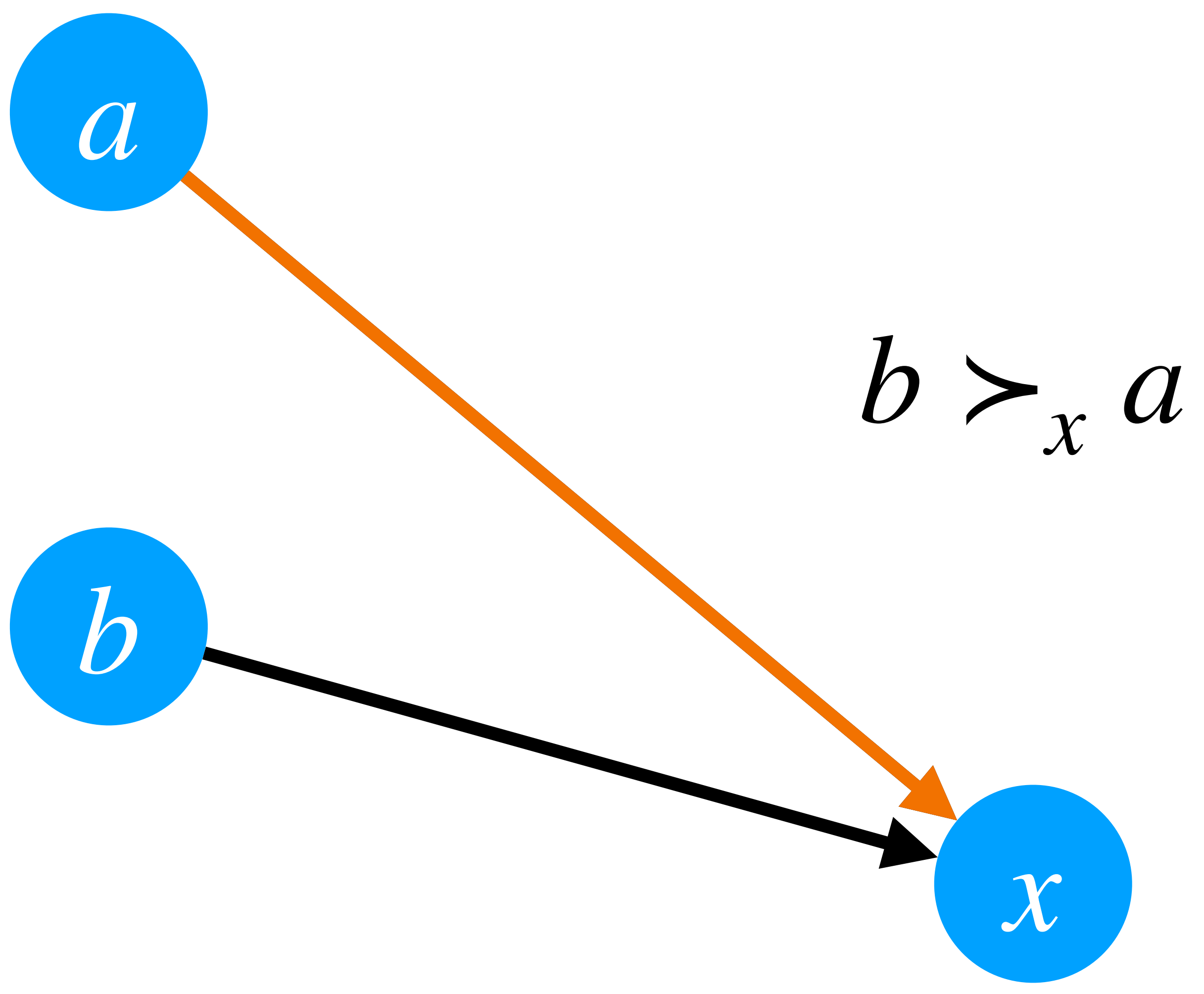

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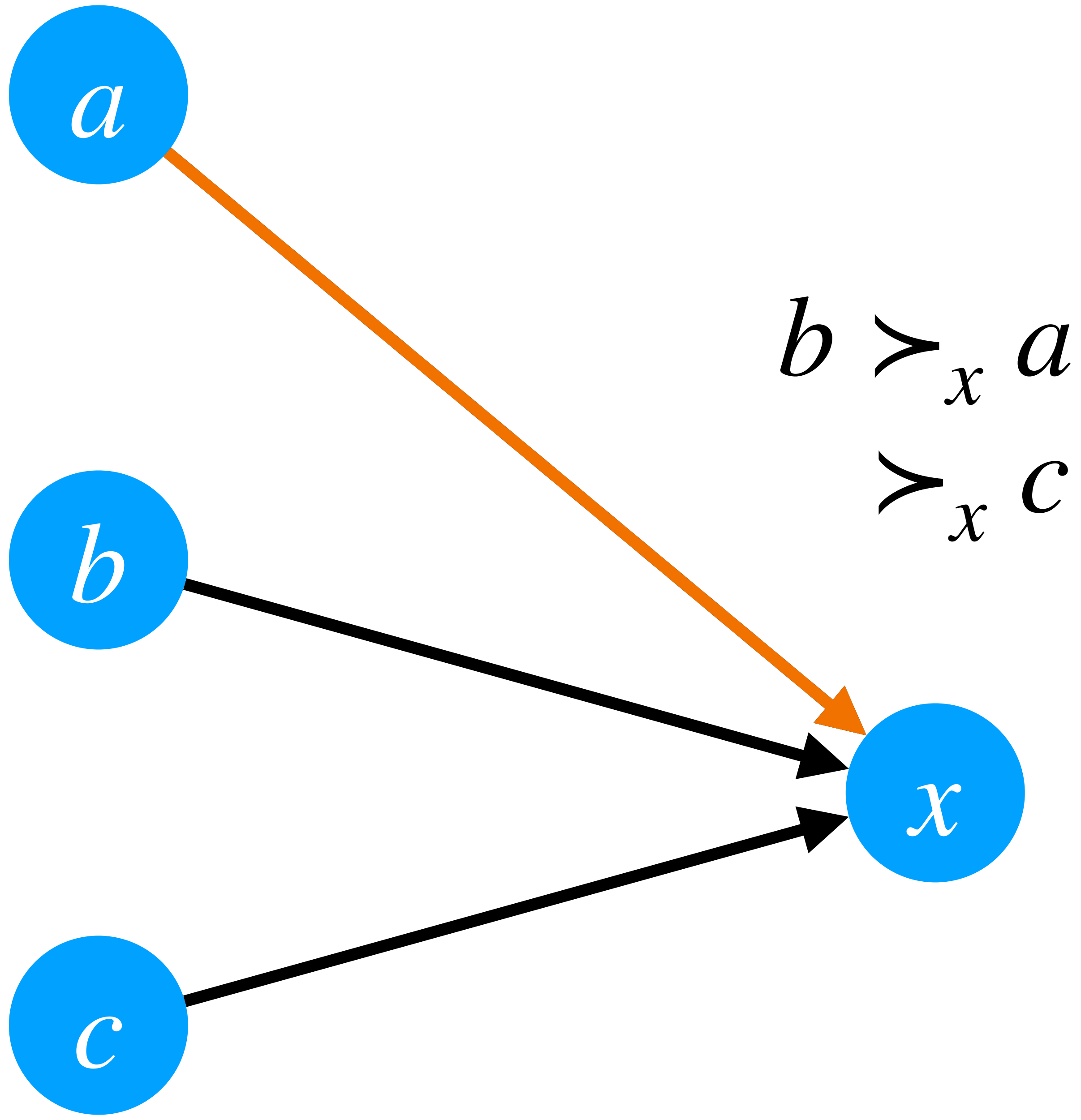

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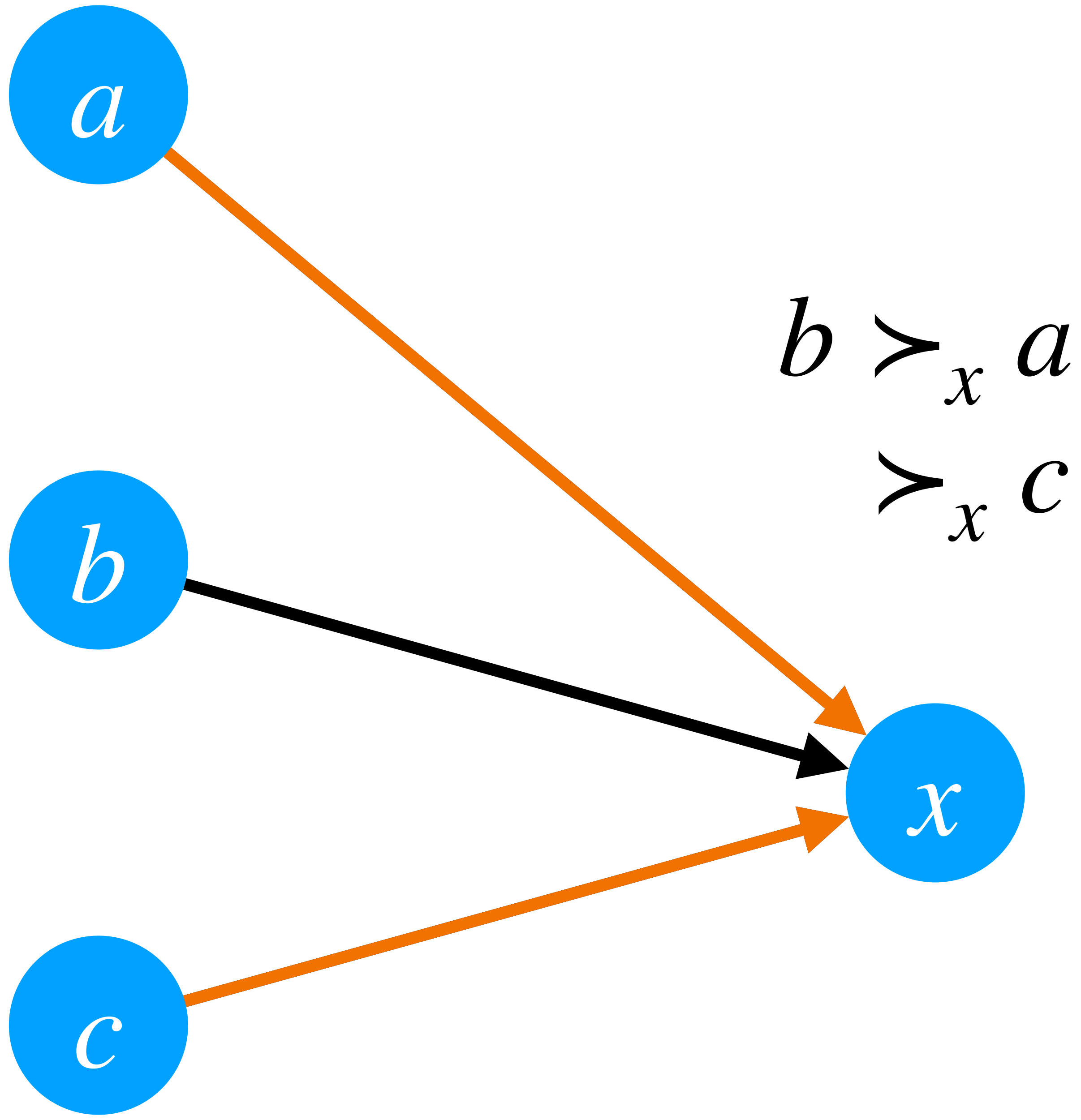

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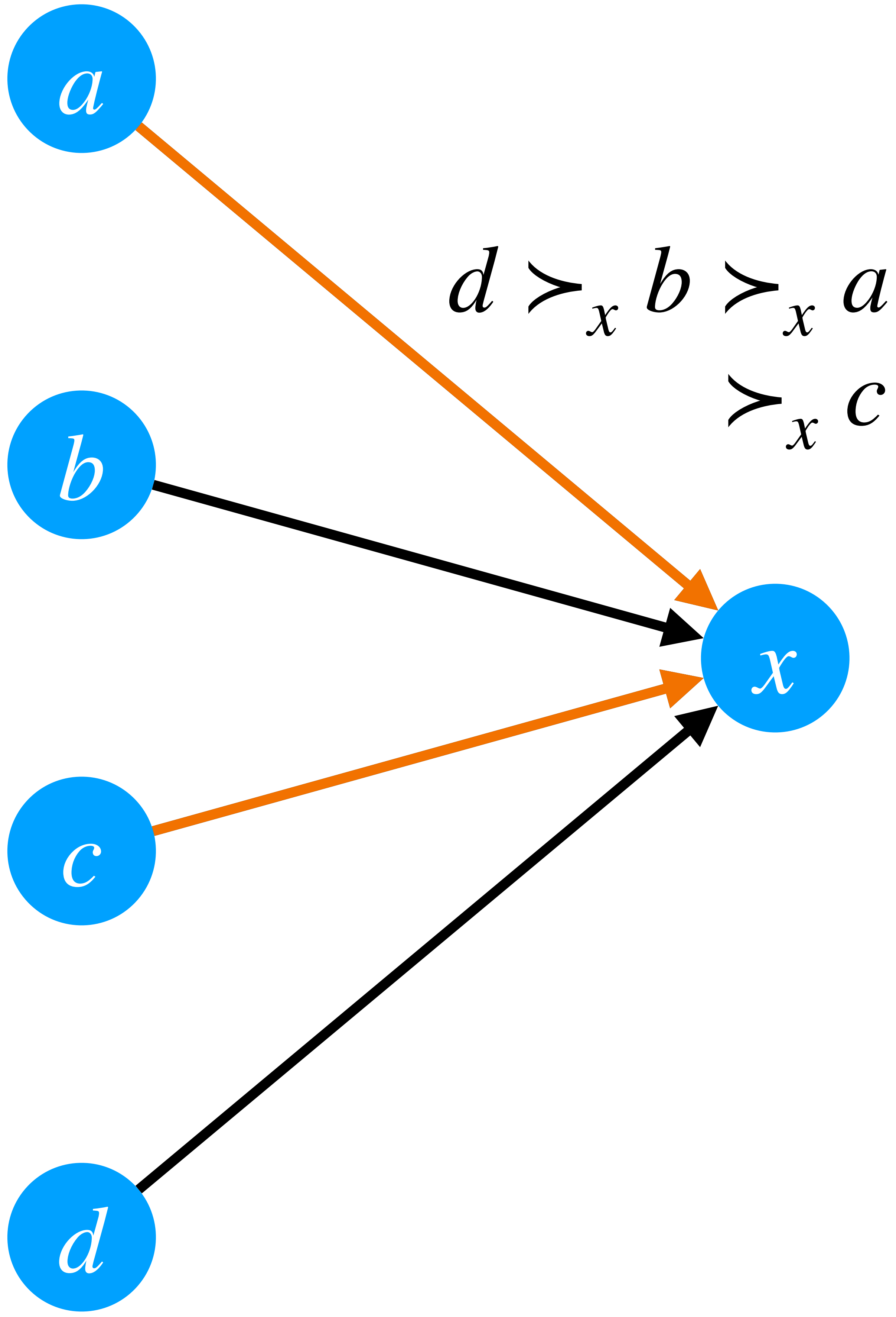

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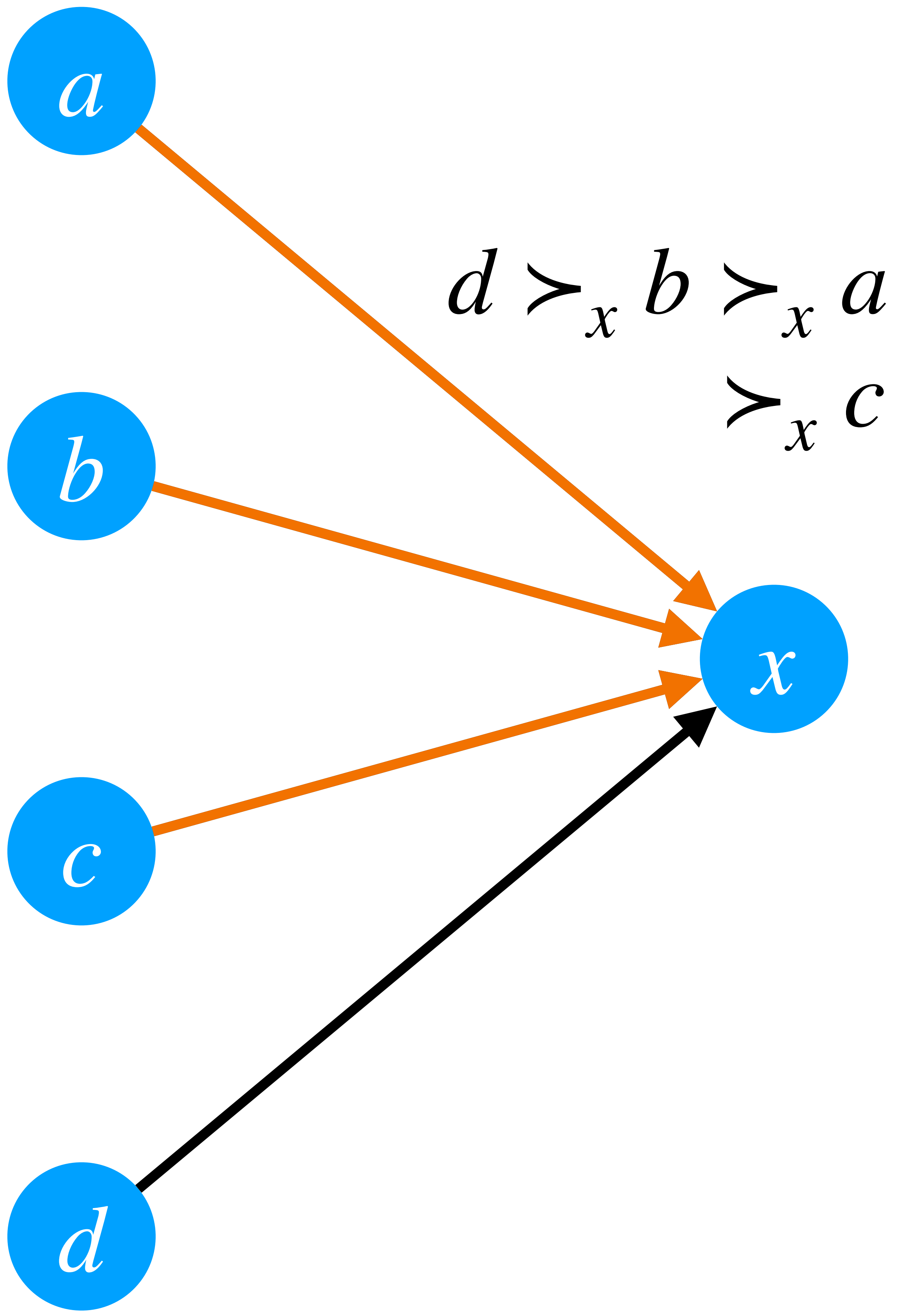

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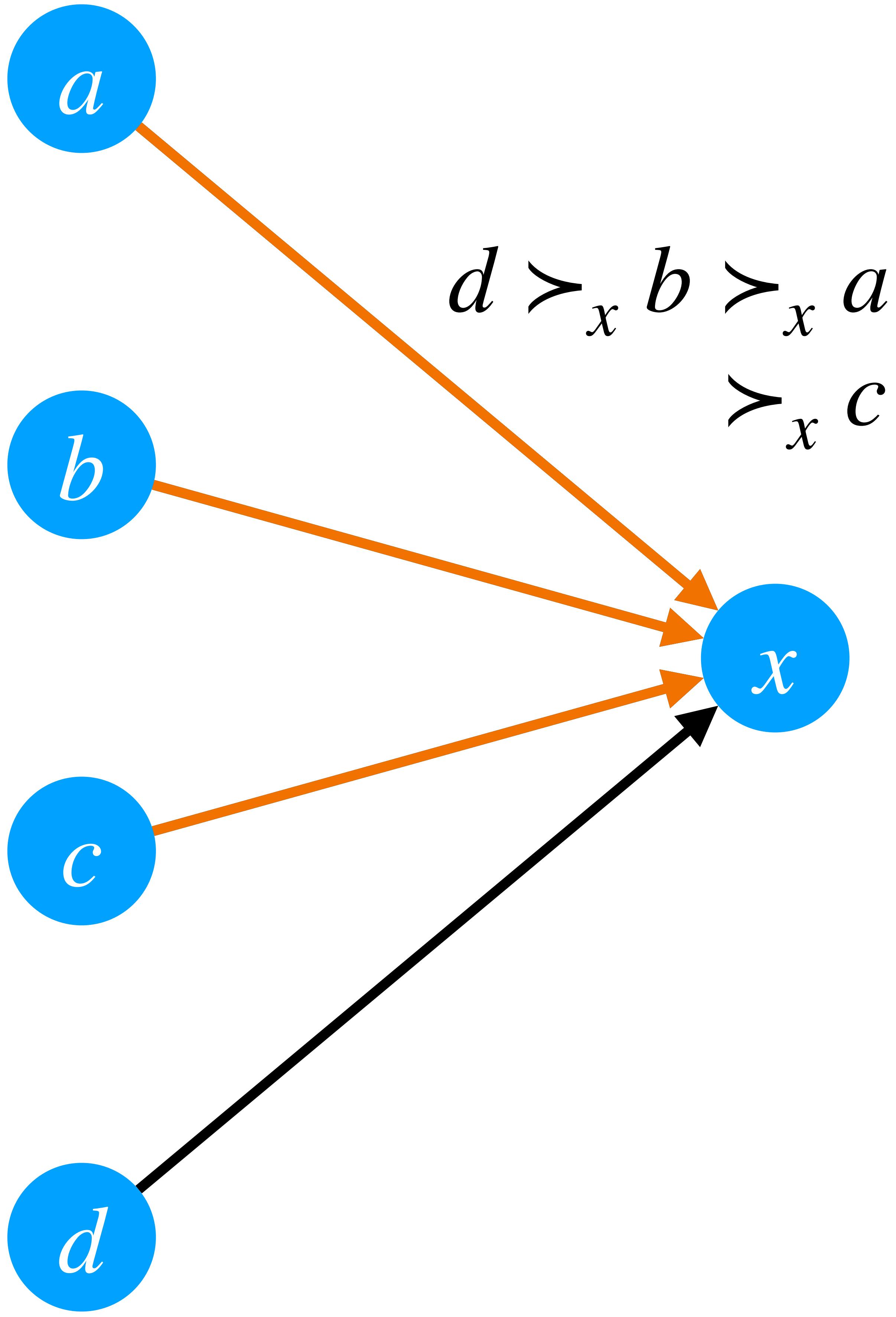

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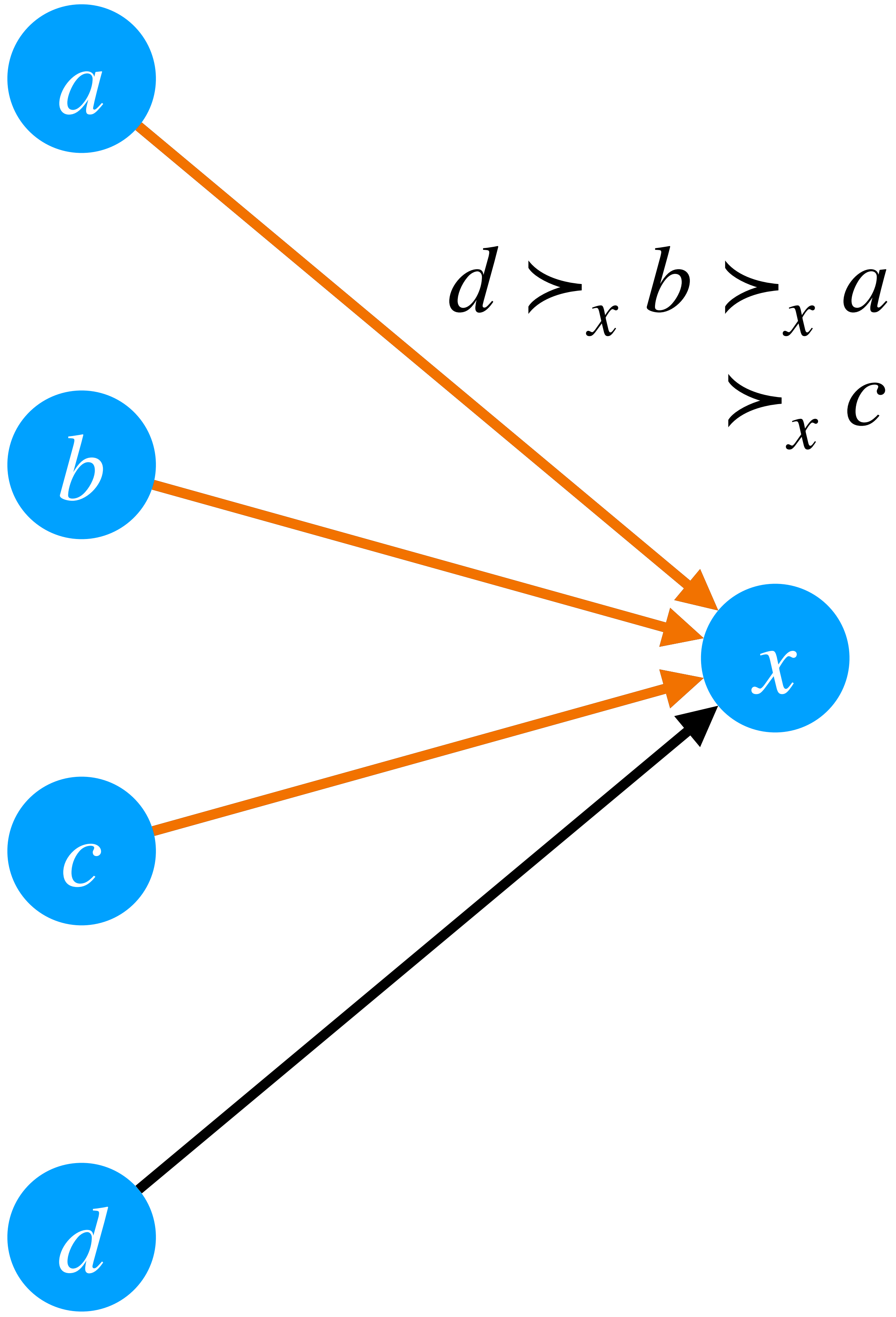

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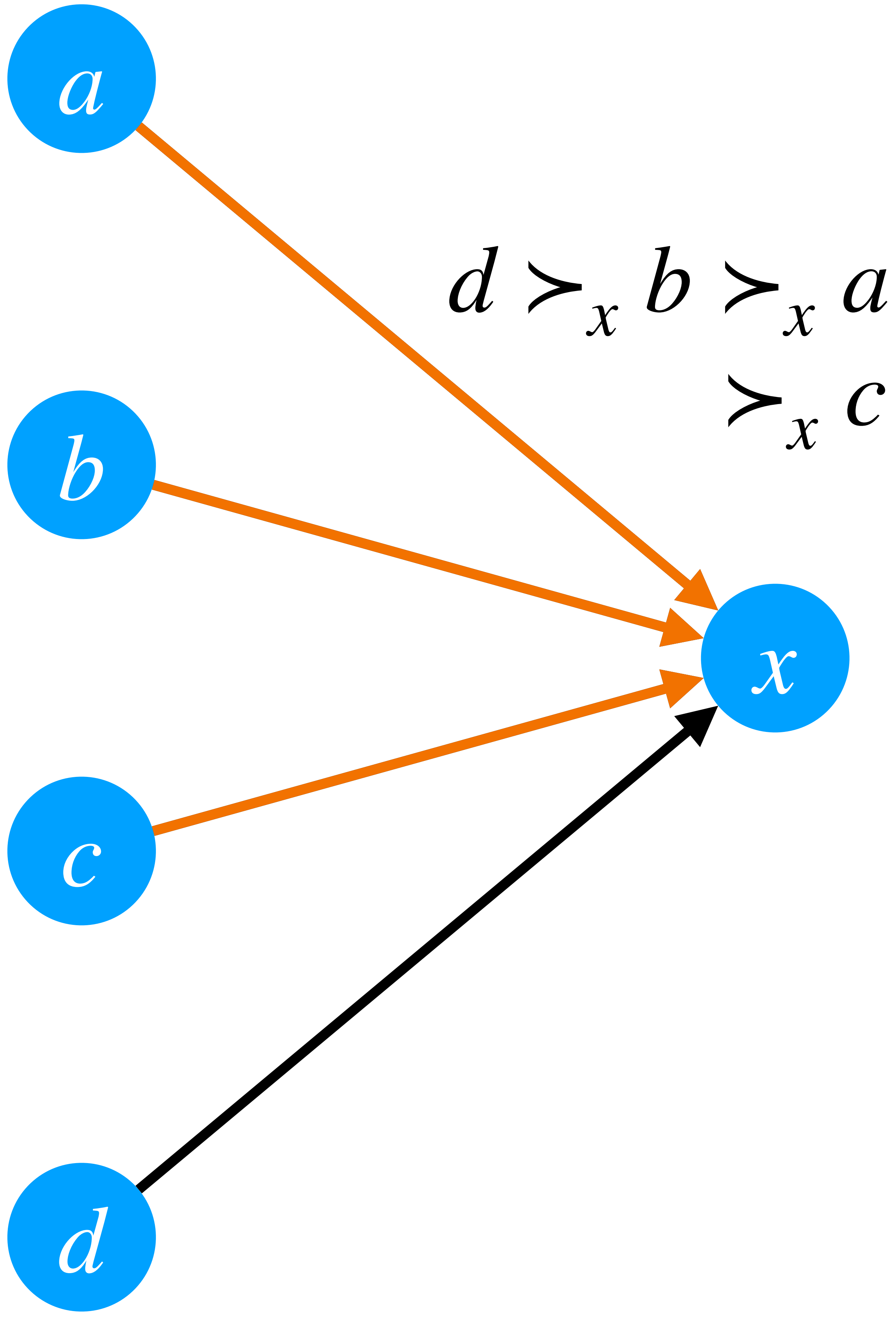
  - at least one new rejection between a pair (a, x)
    occurs in every step except for the last one



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- **Theorem**

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- by contradiction: an applicant gets rejected by their most-preferred attainable company

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In the outcome of applicant-proposing DA:

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- the proof is analogous for applicants (starting from company-proposing DA)

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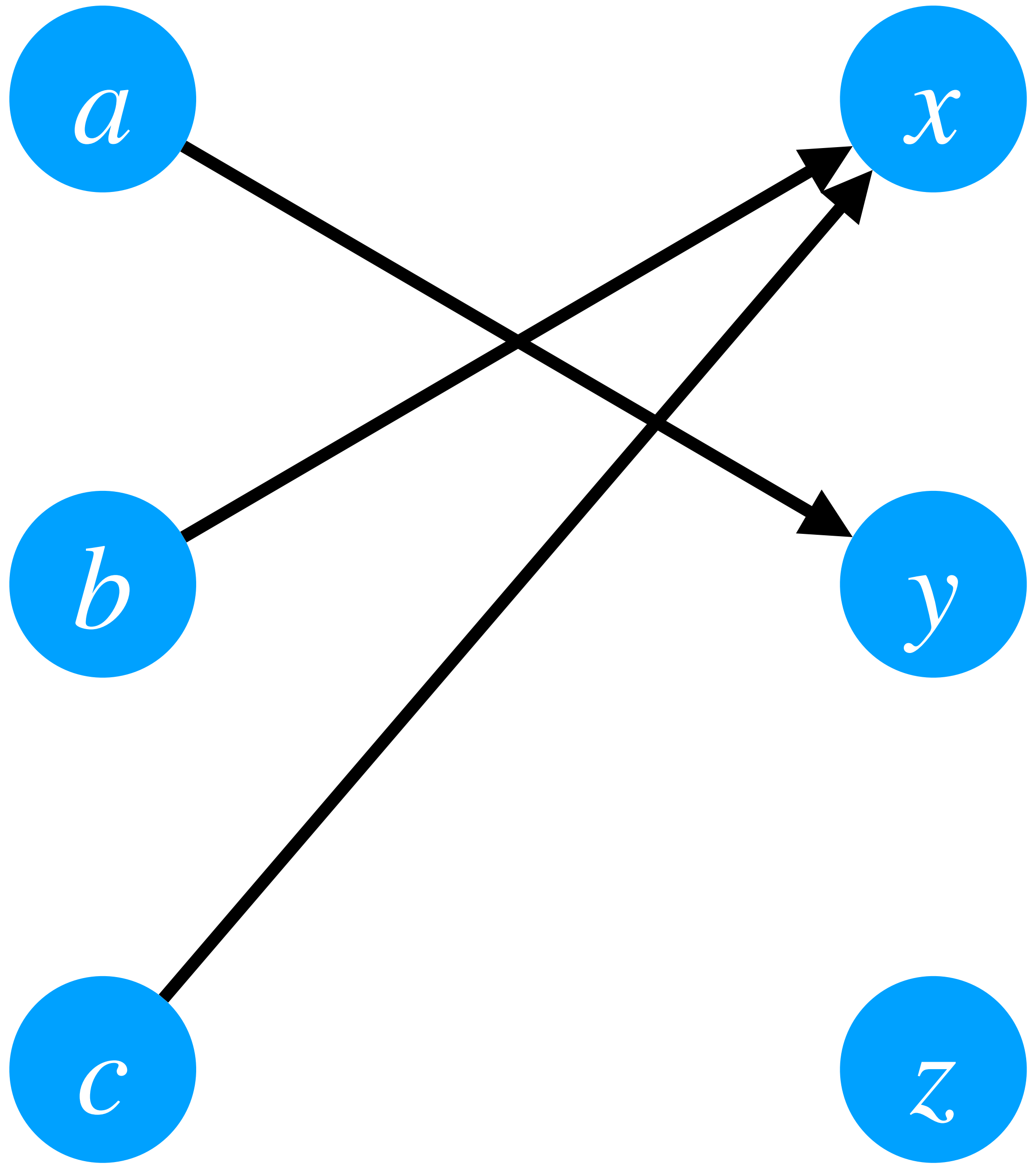

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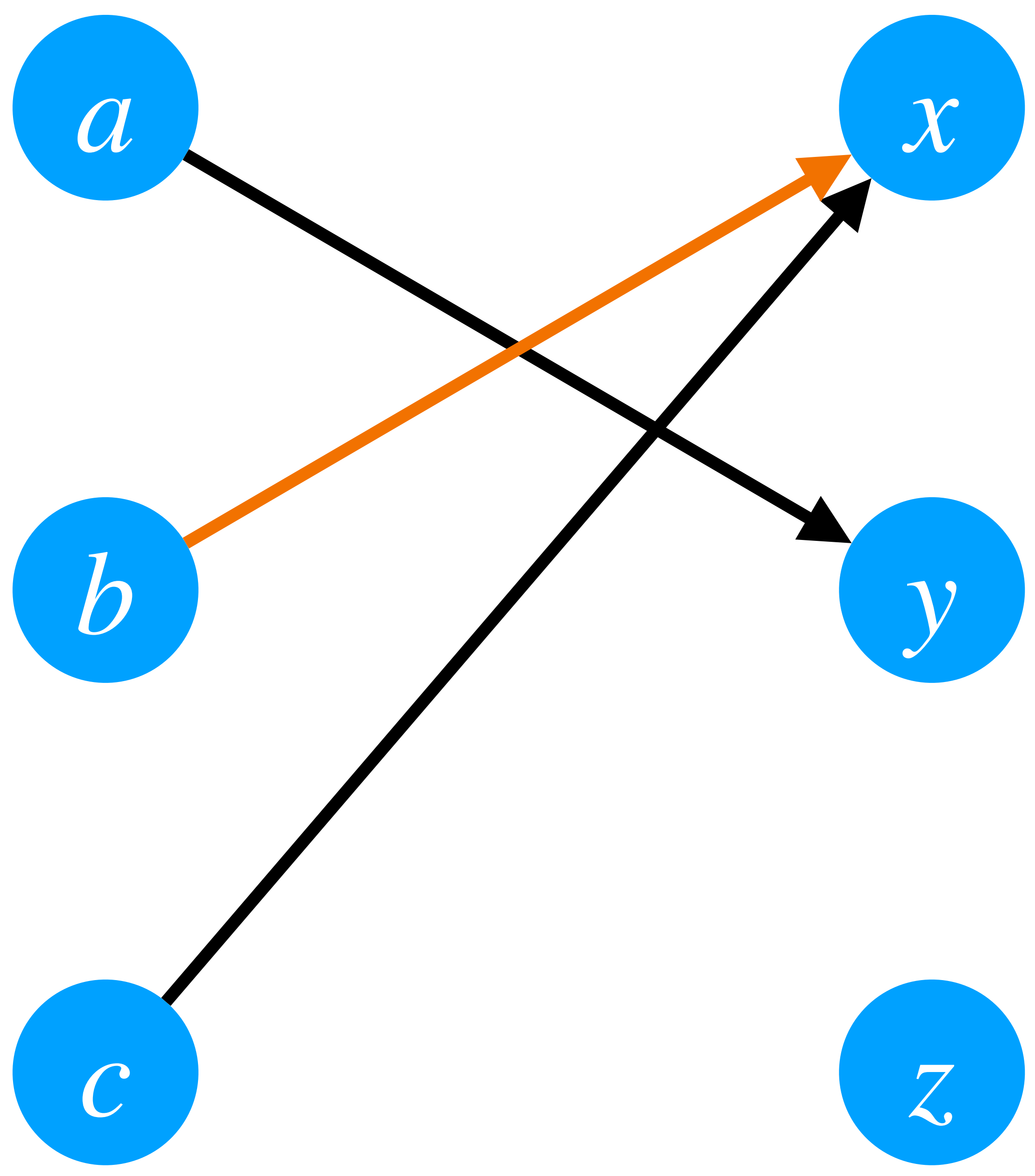

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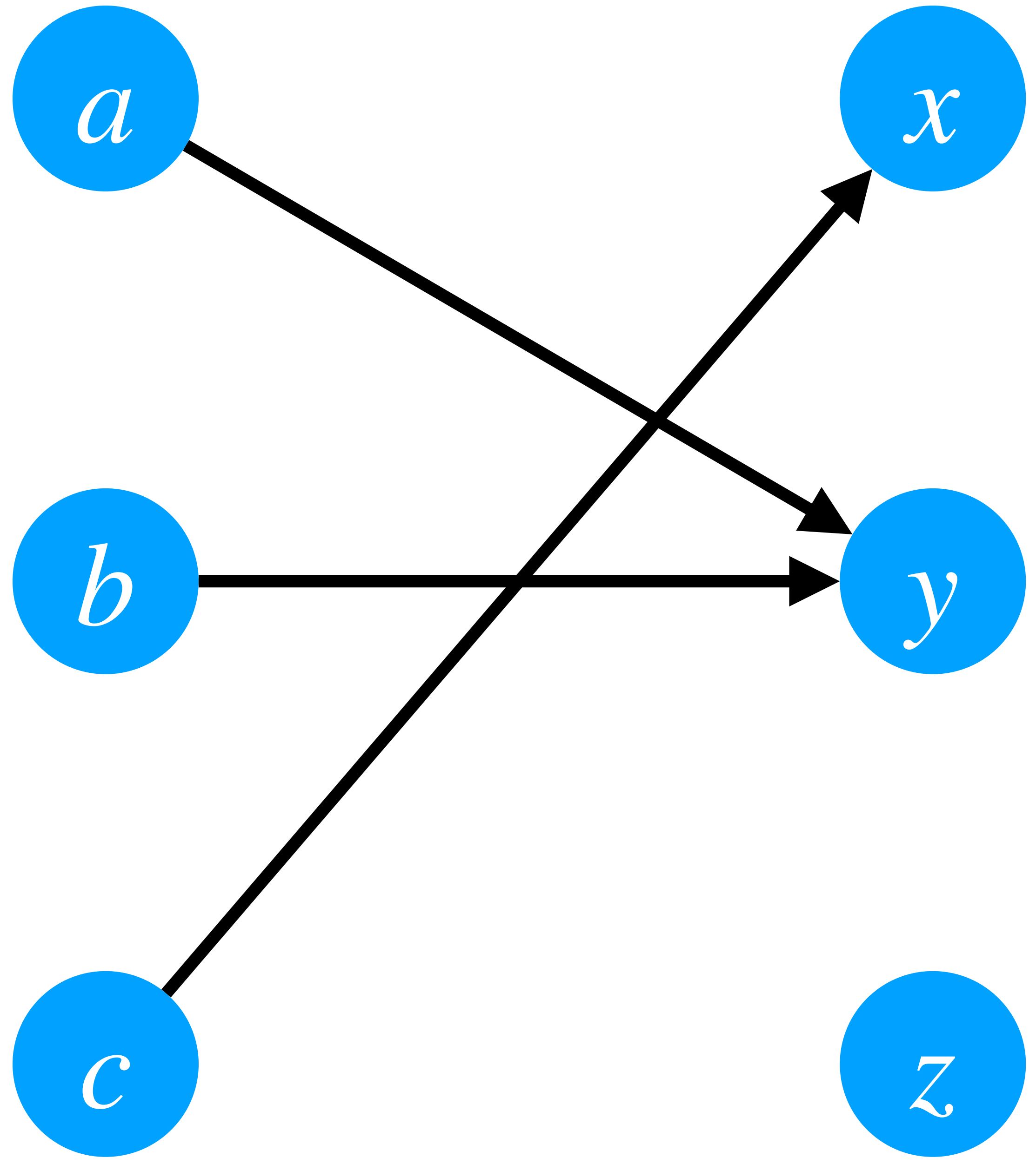

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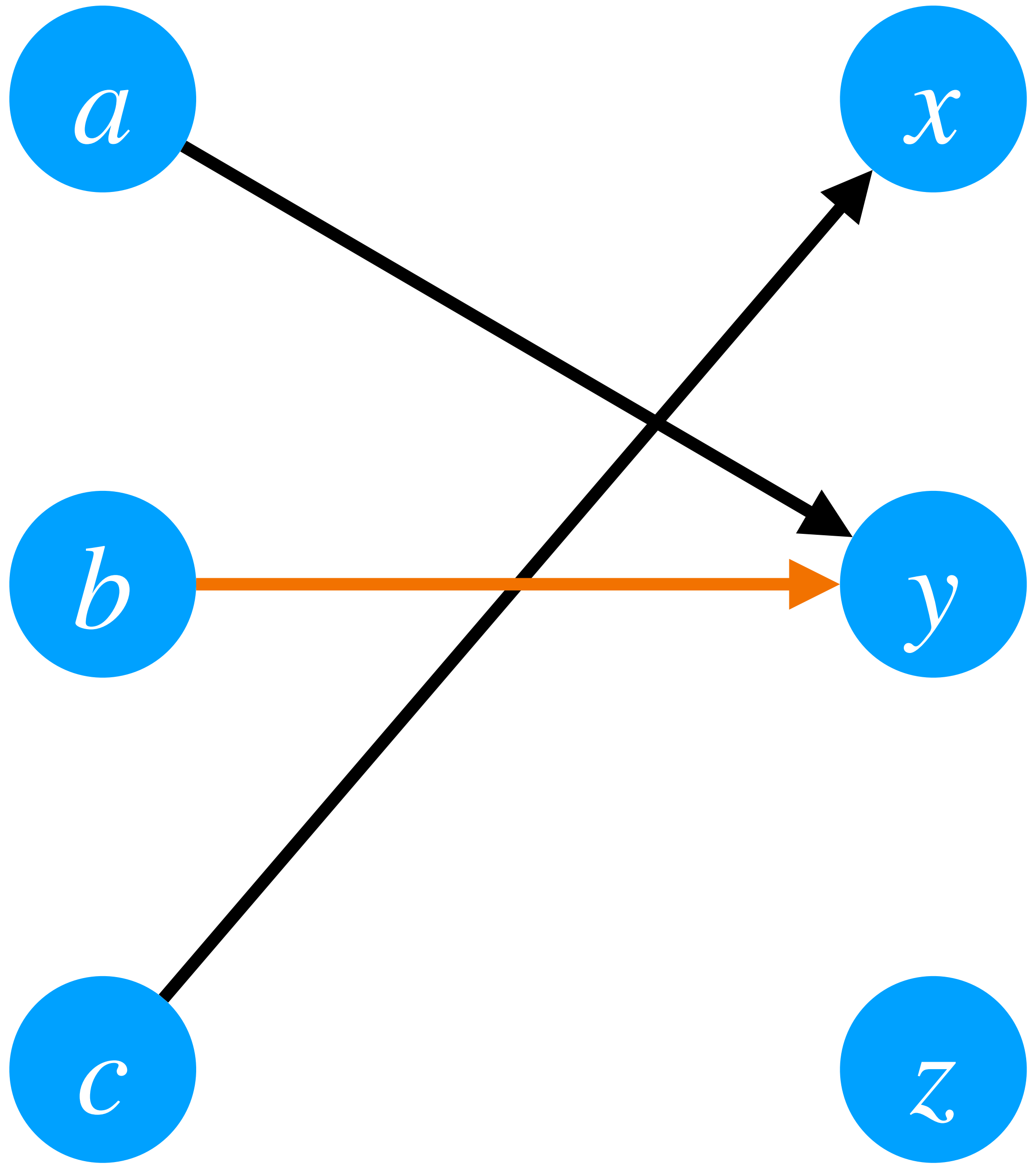

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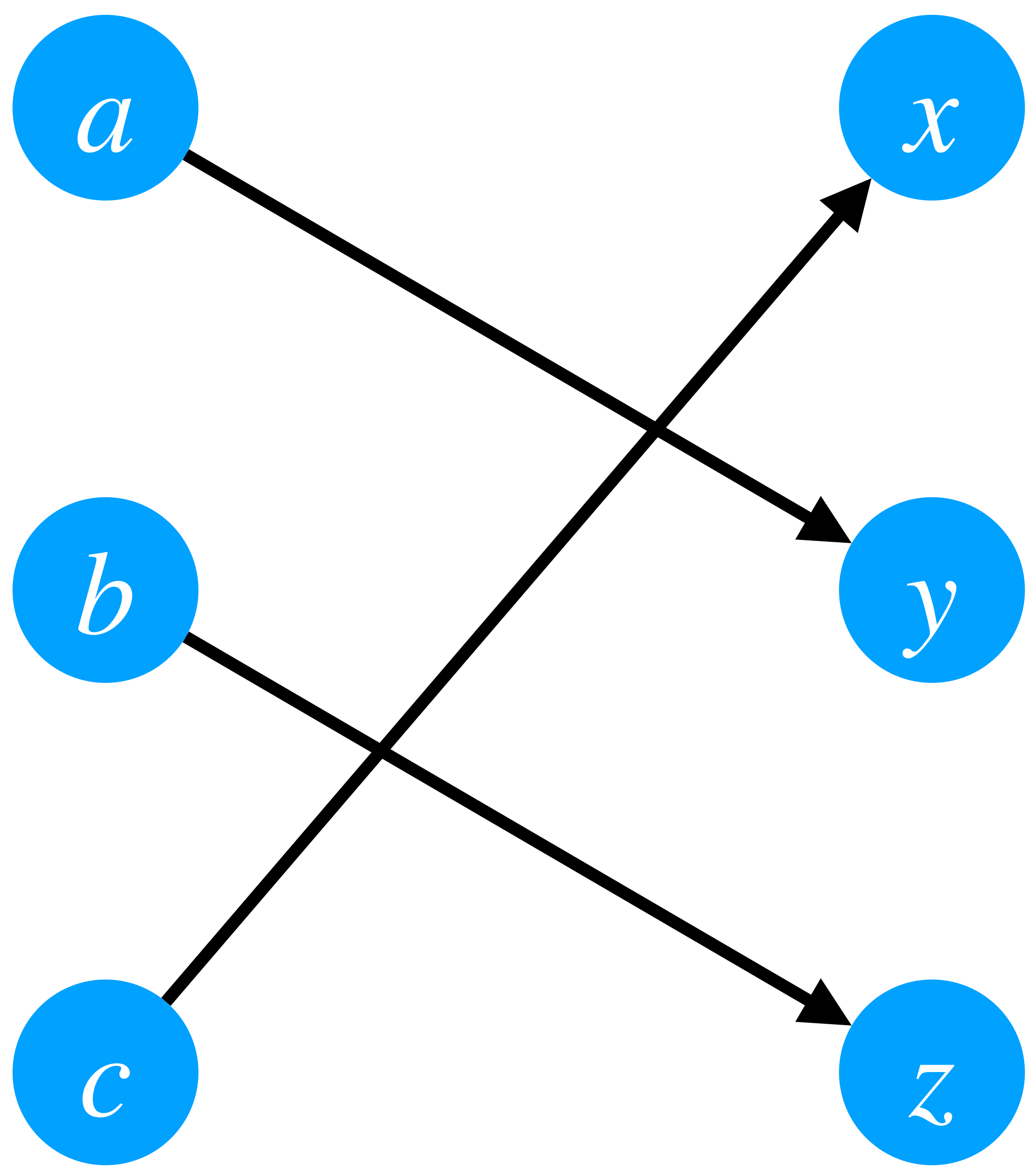

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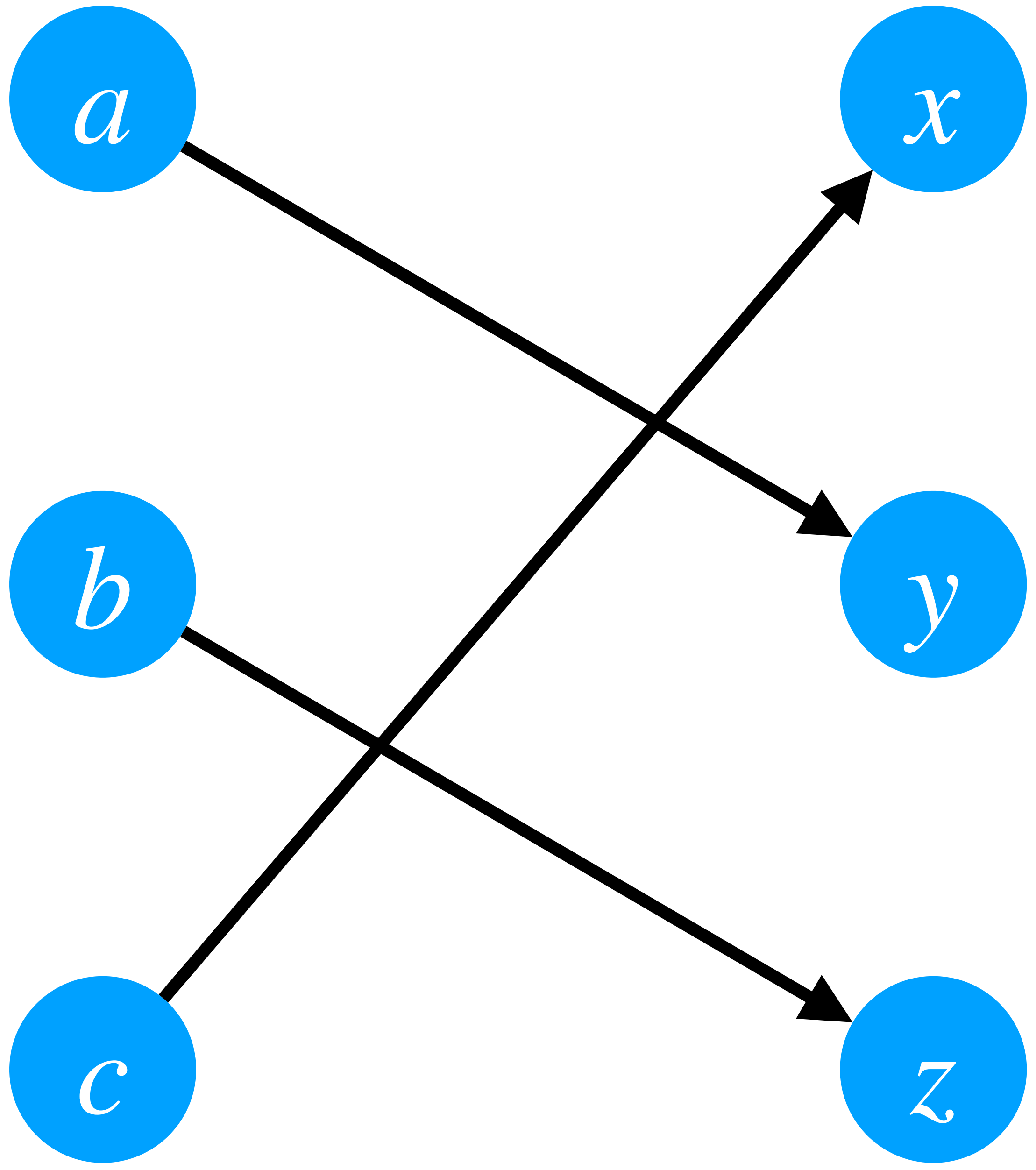

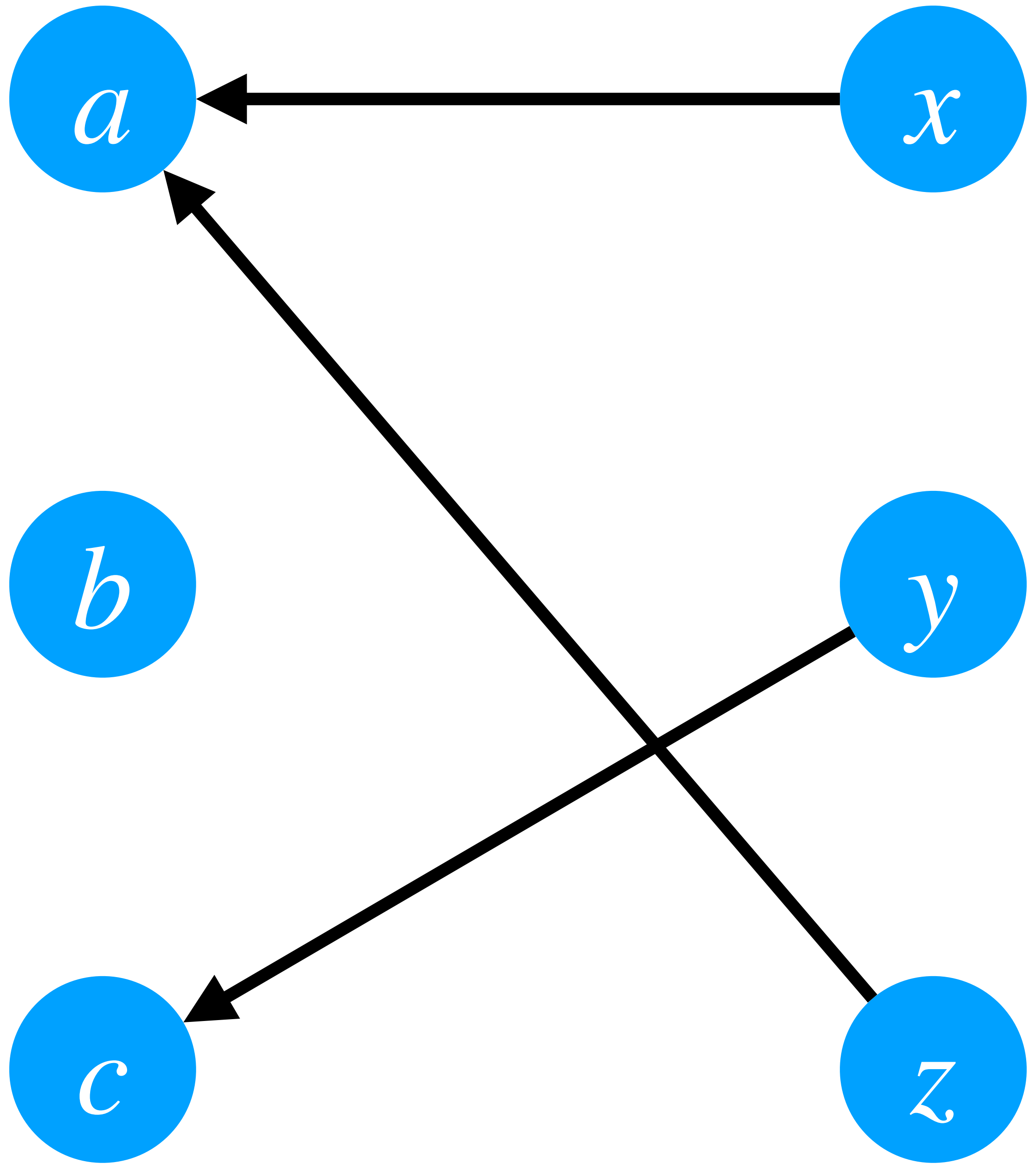

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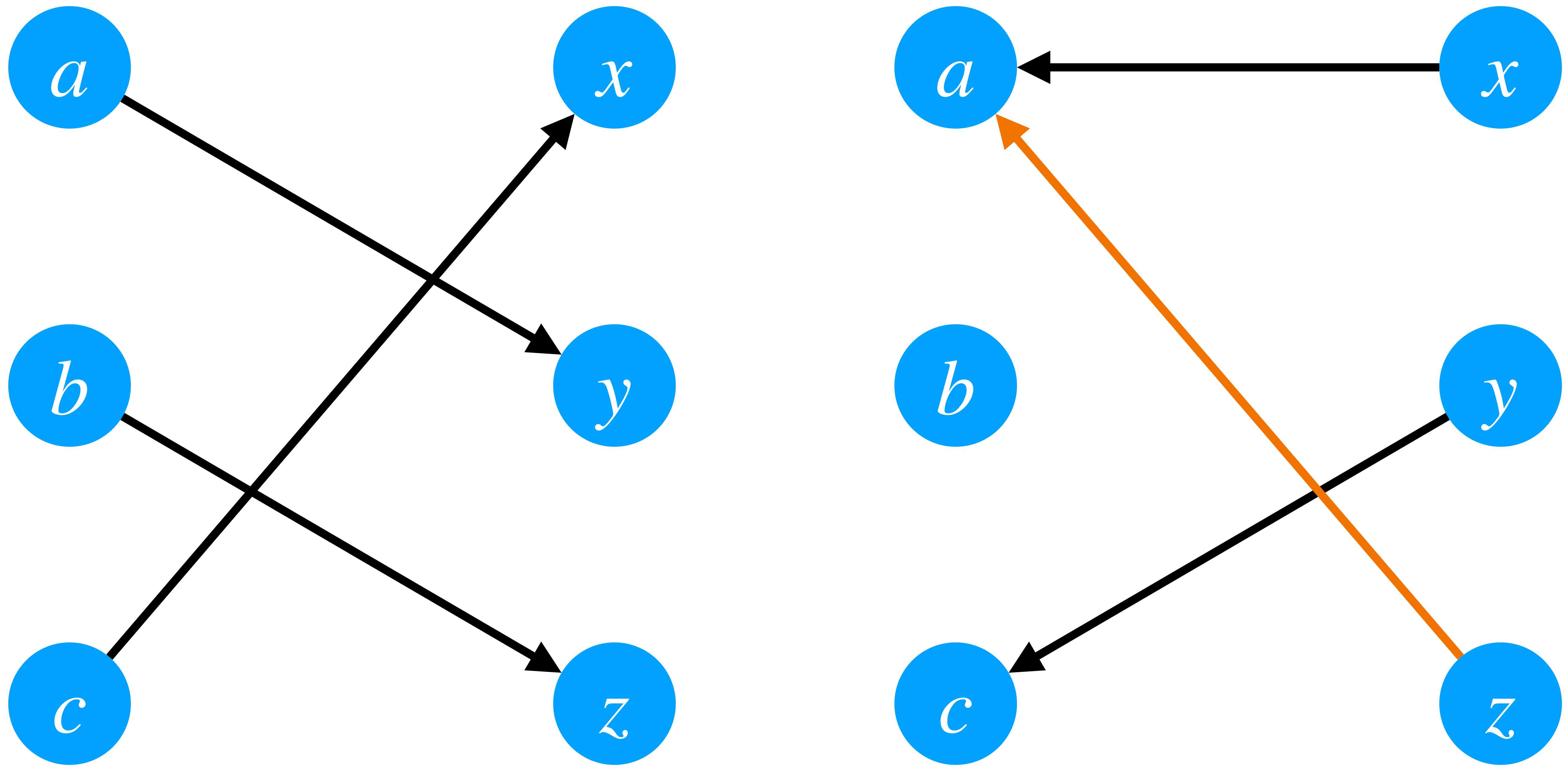

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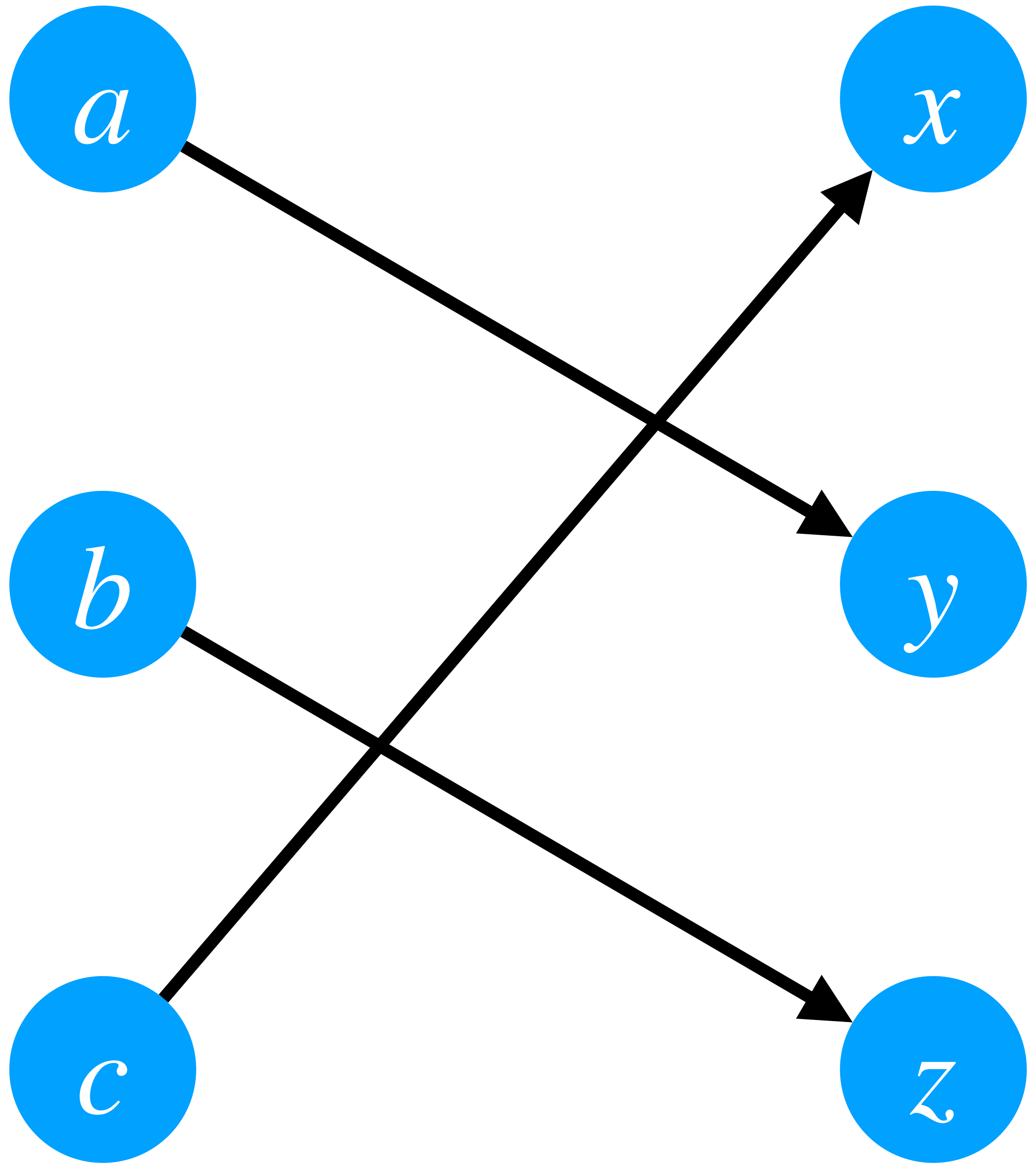

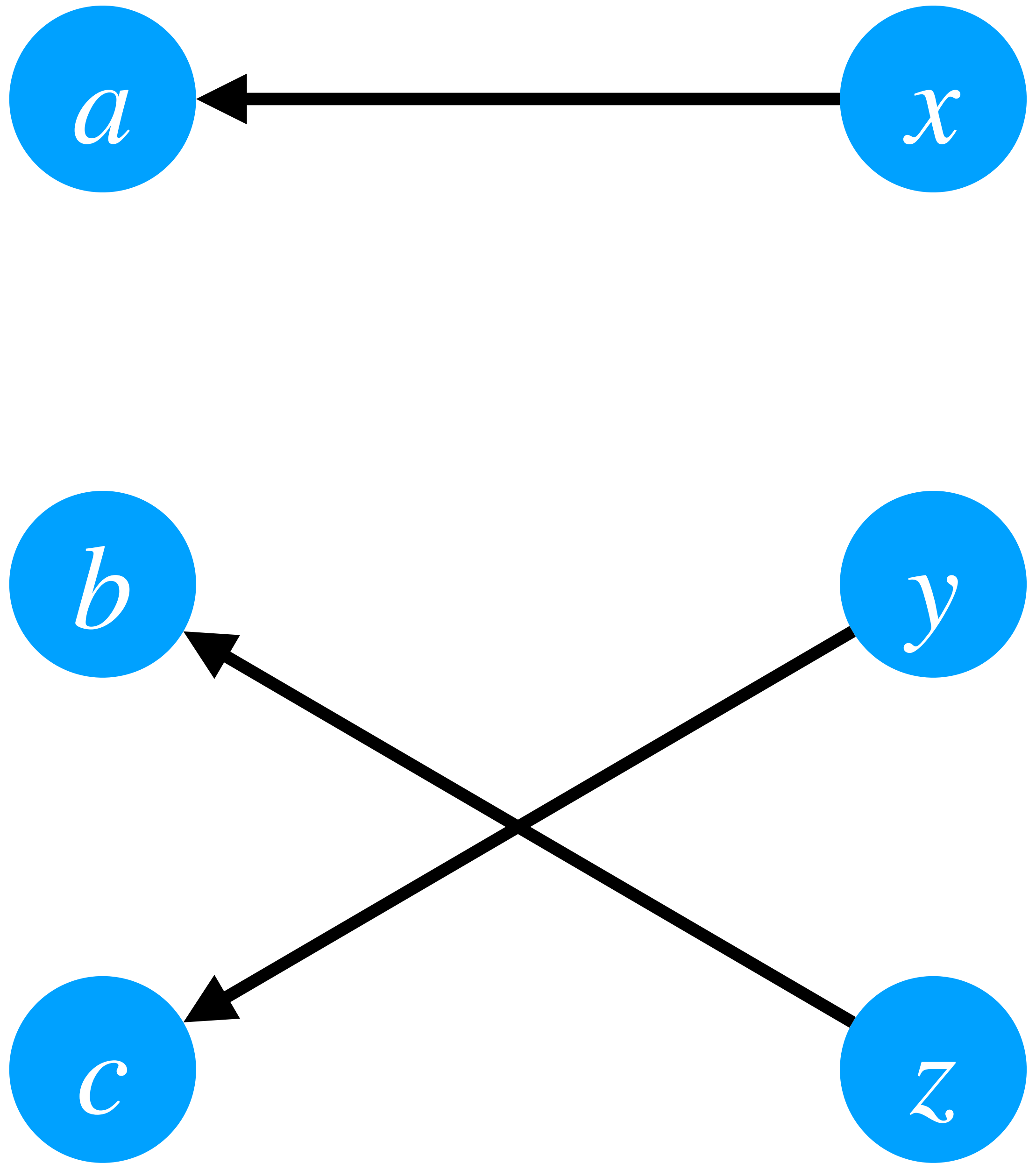

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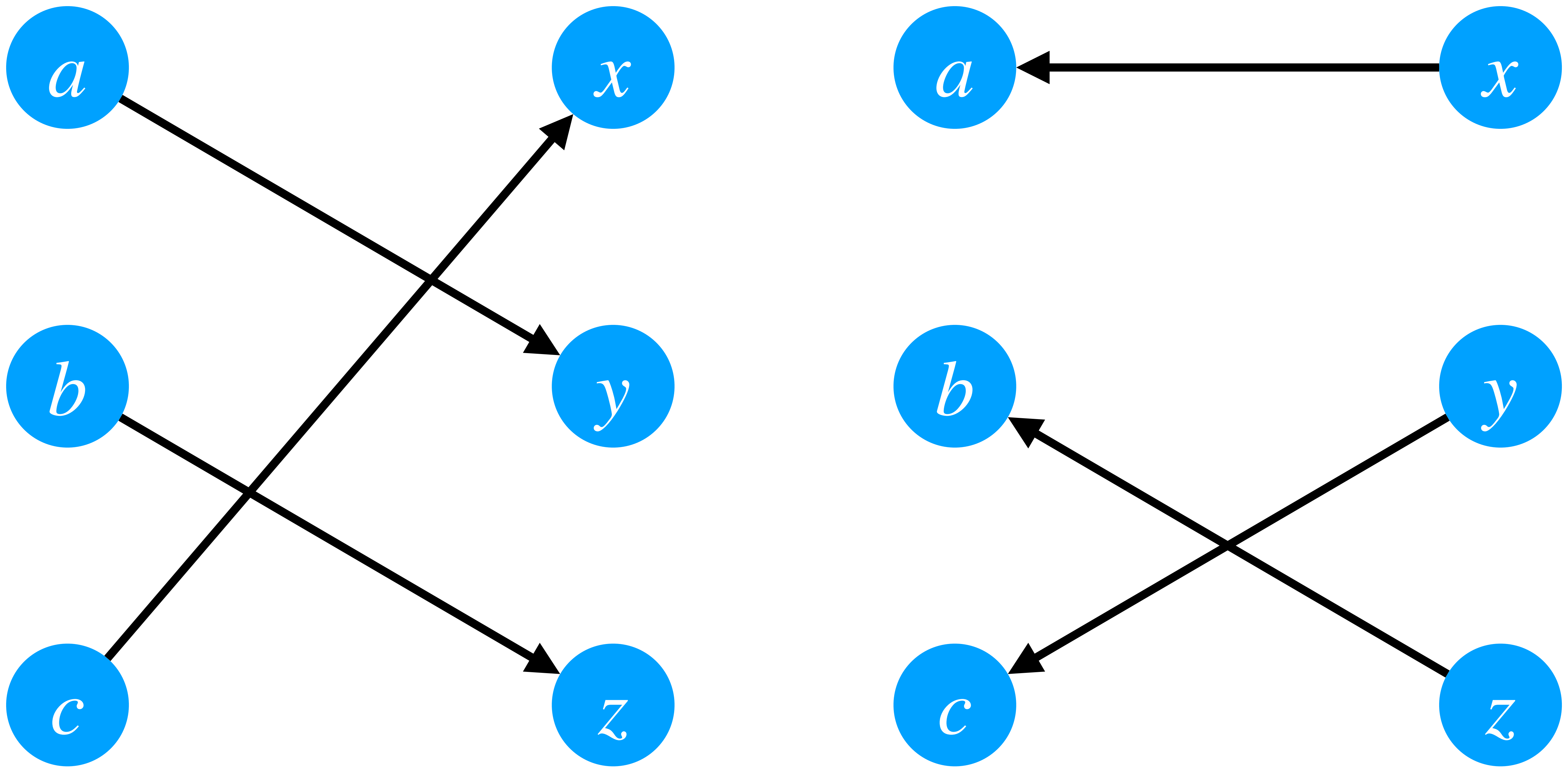

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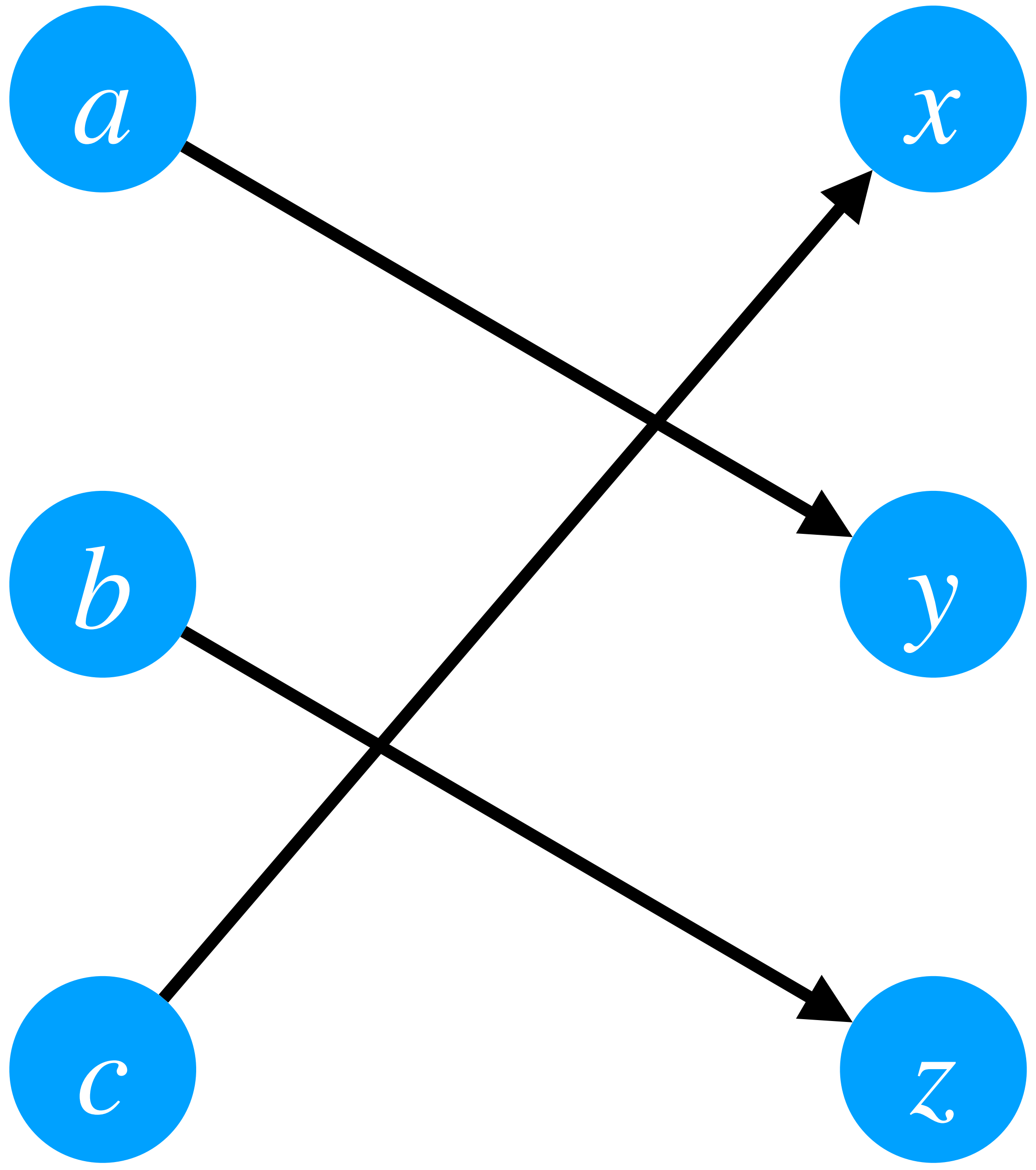

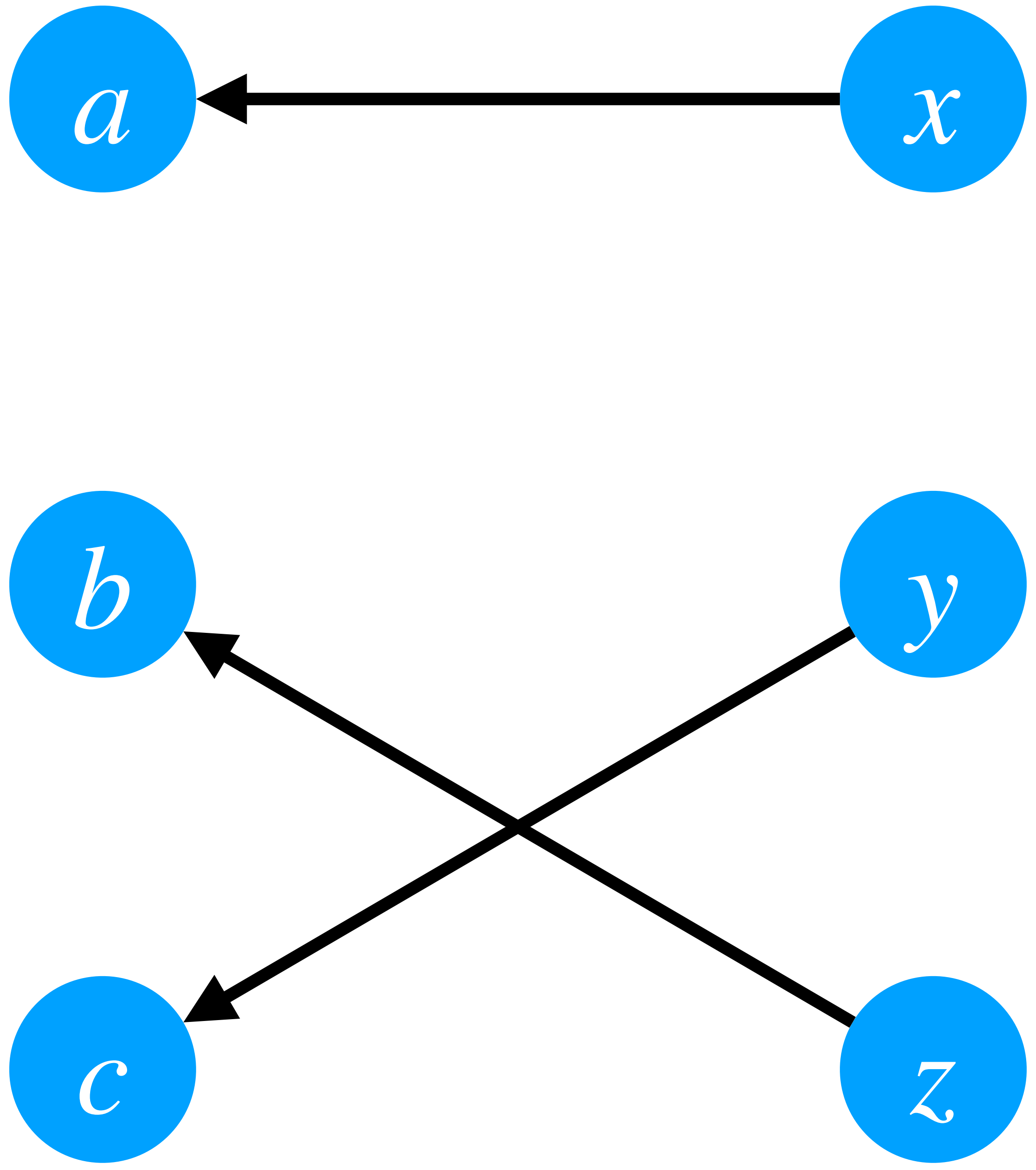

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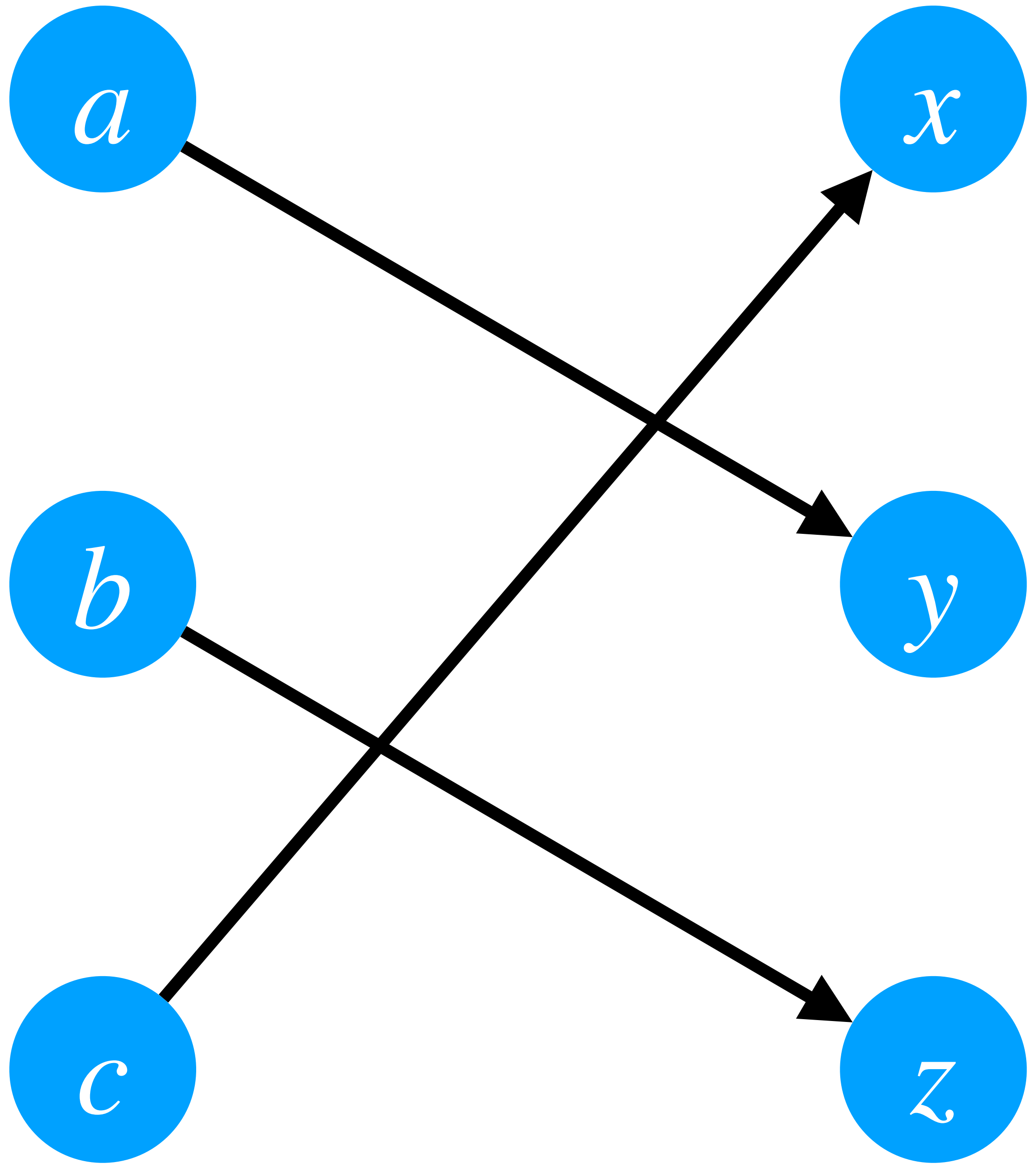

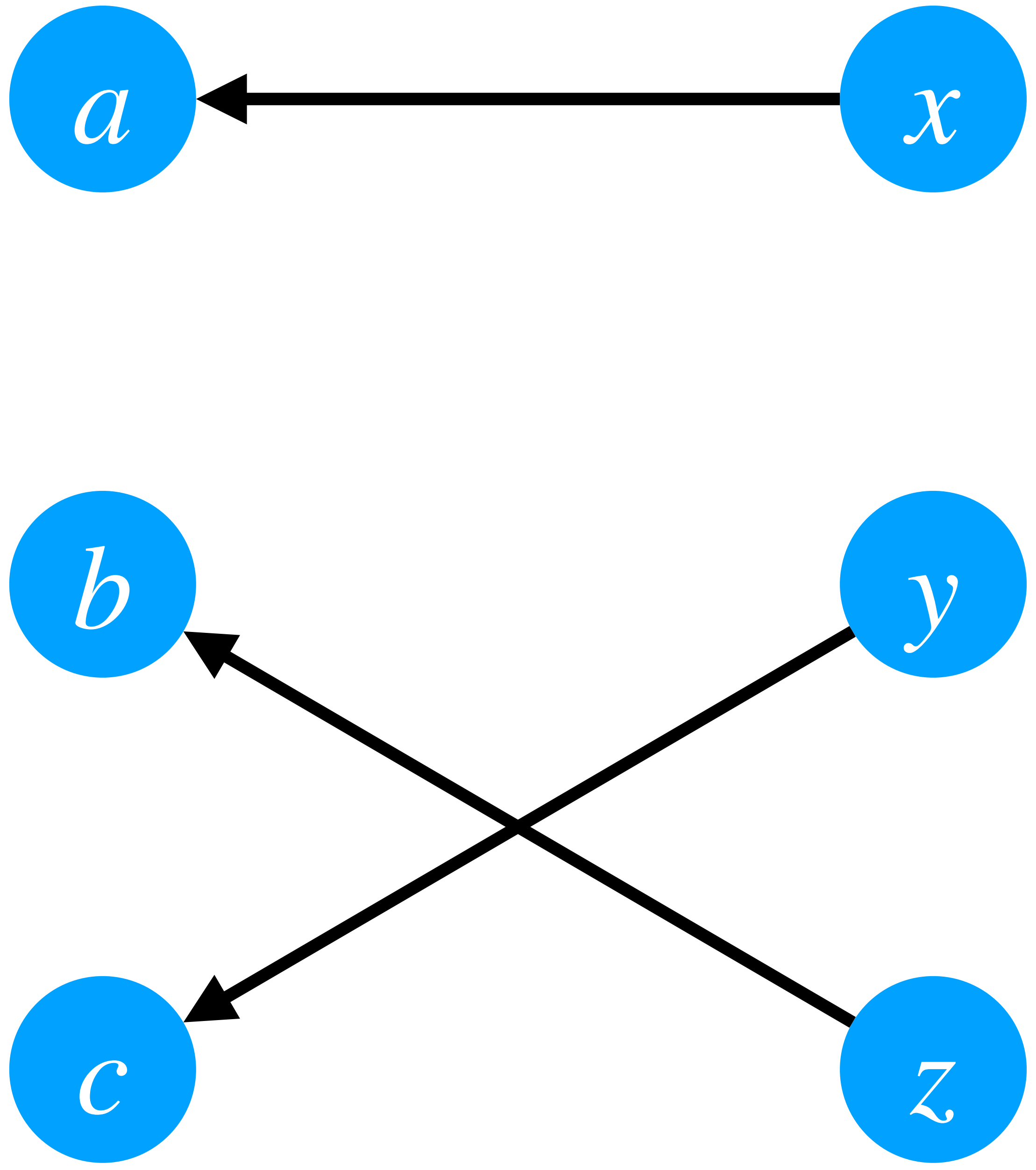

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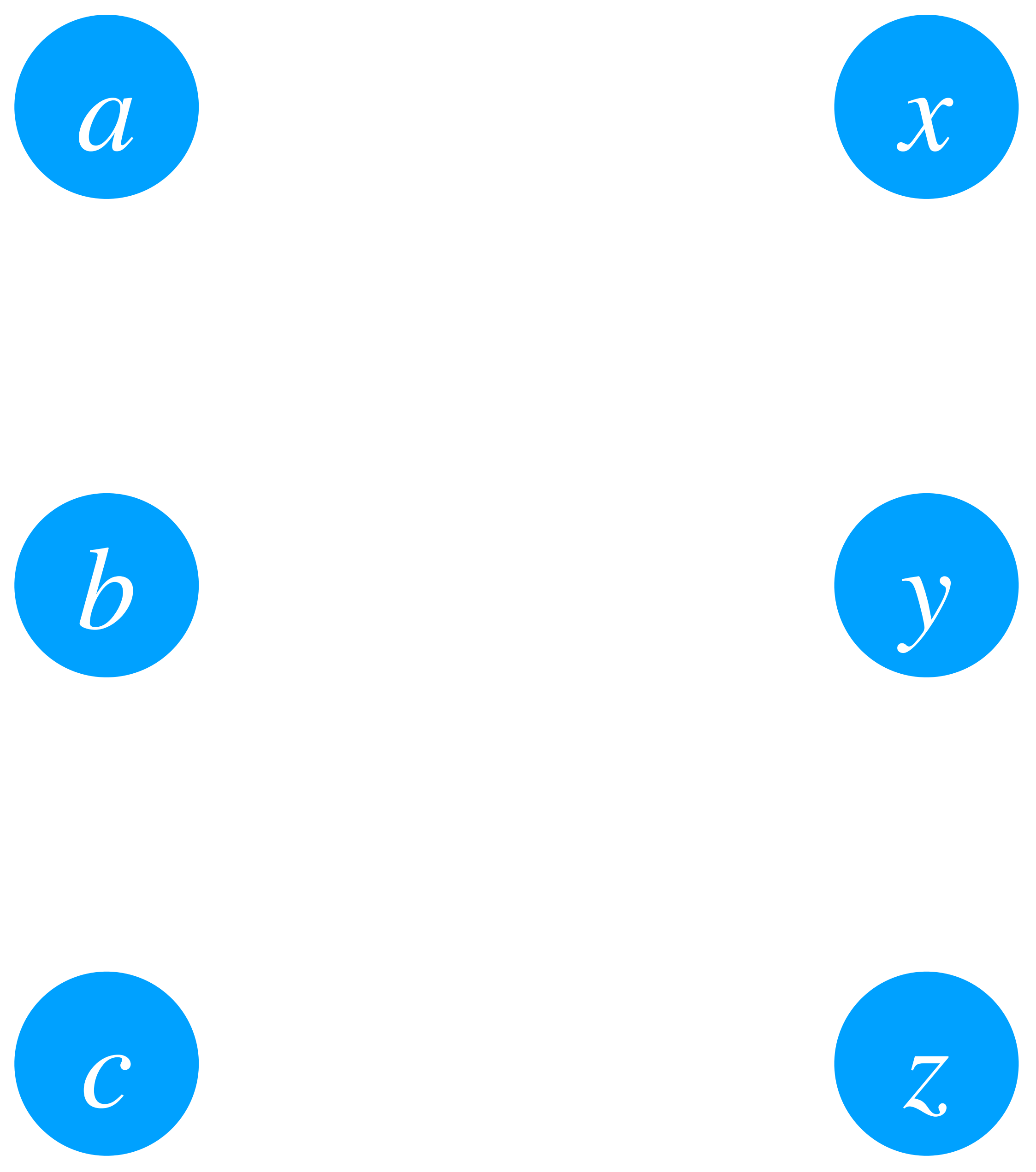

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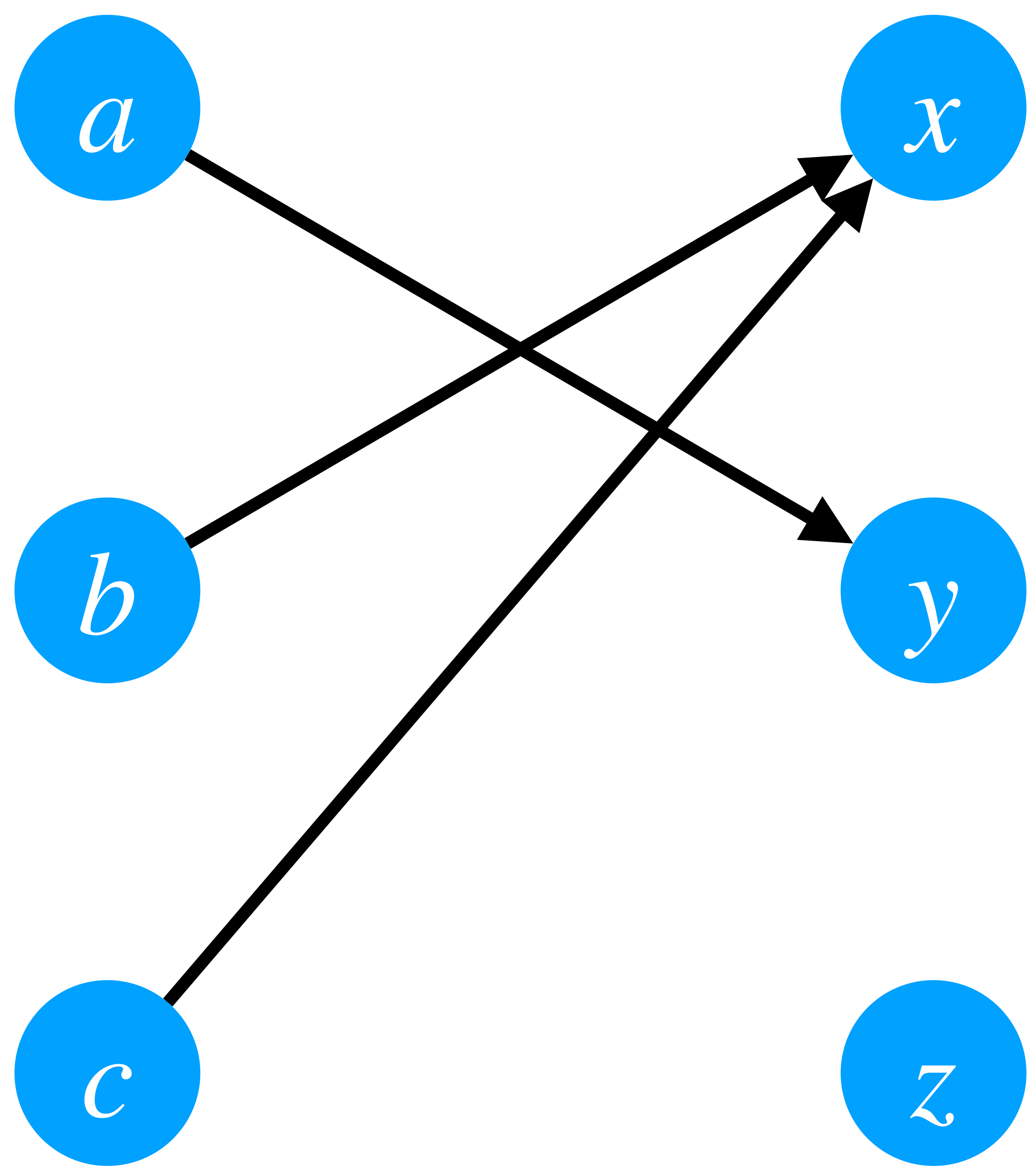

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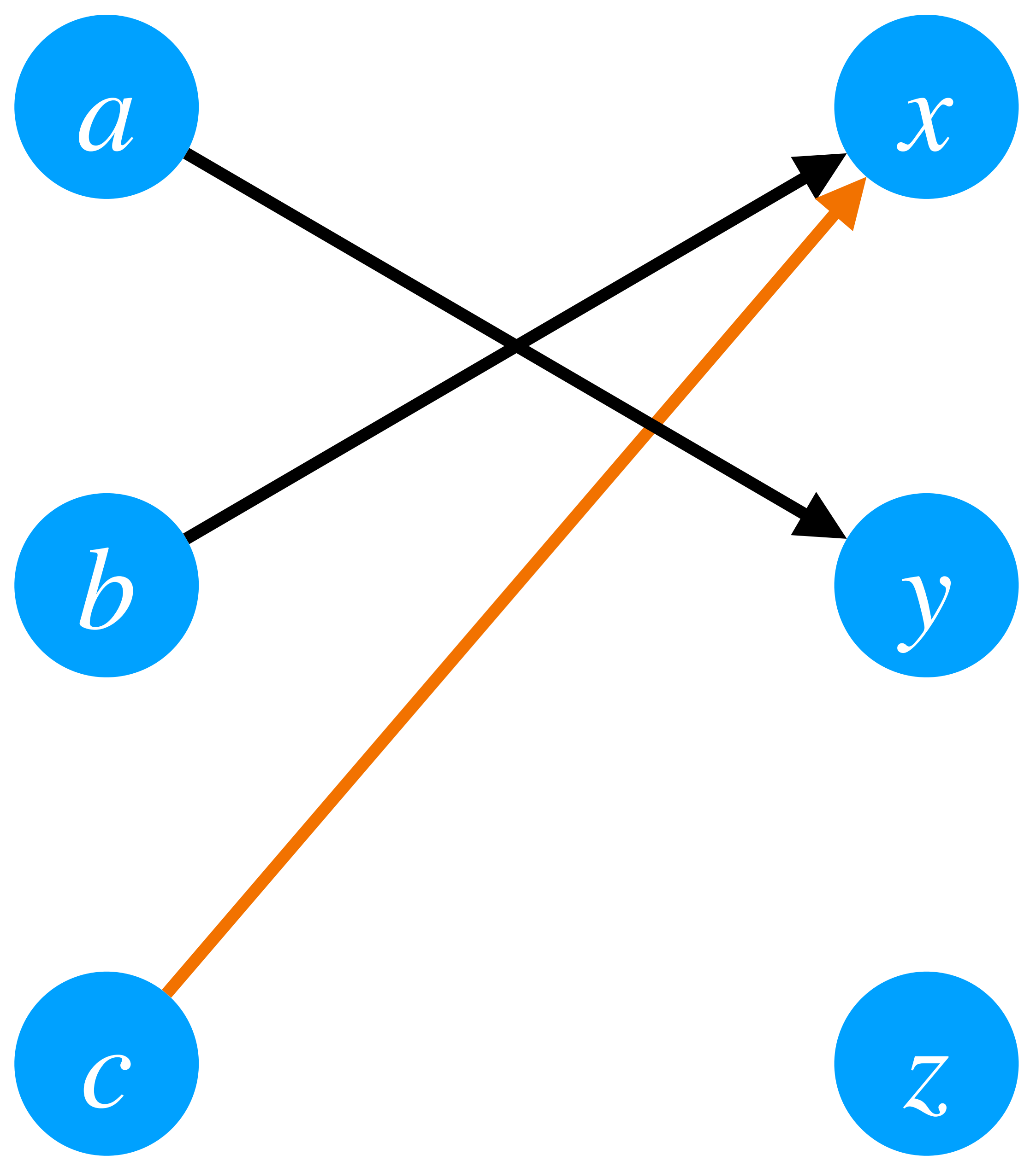

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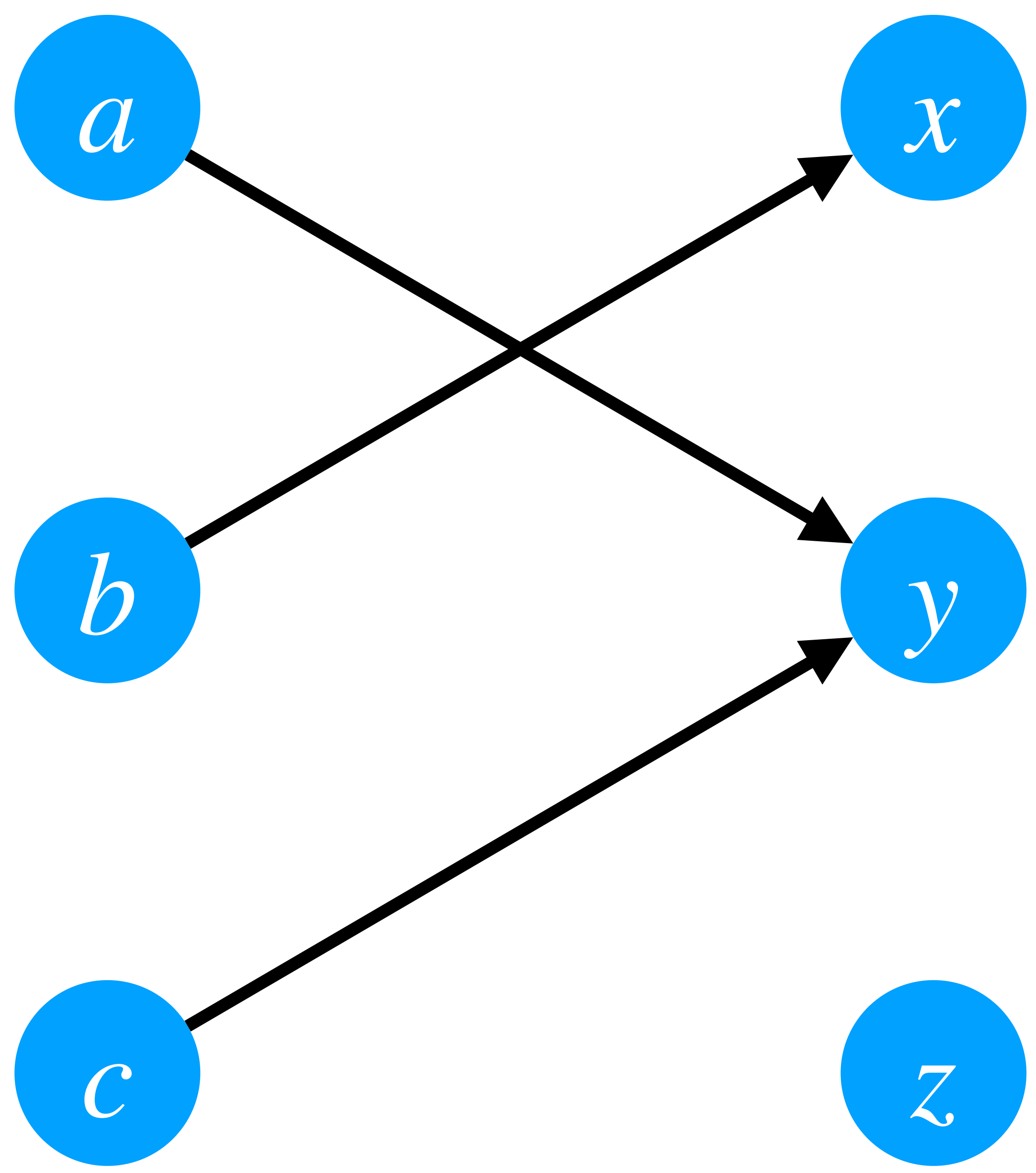

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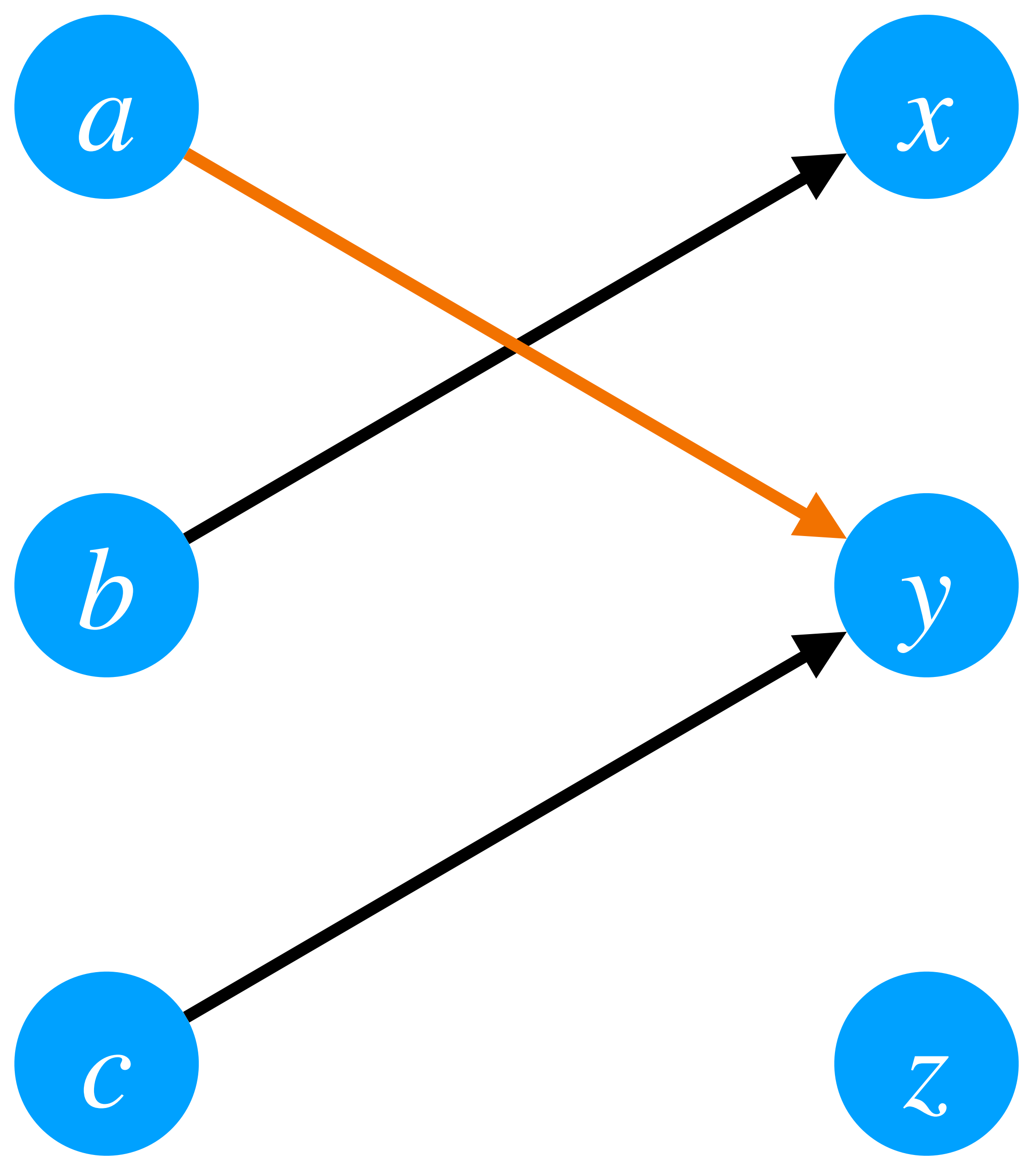

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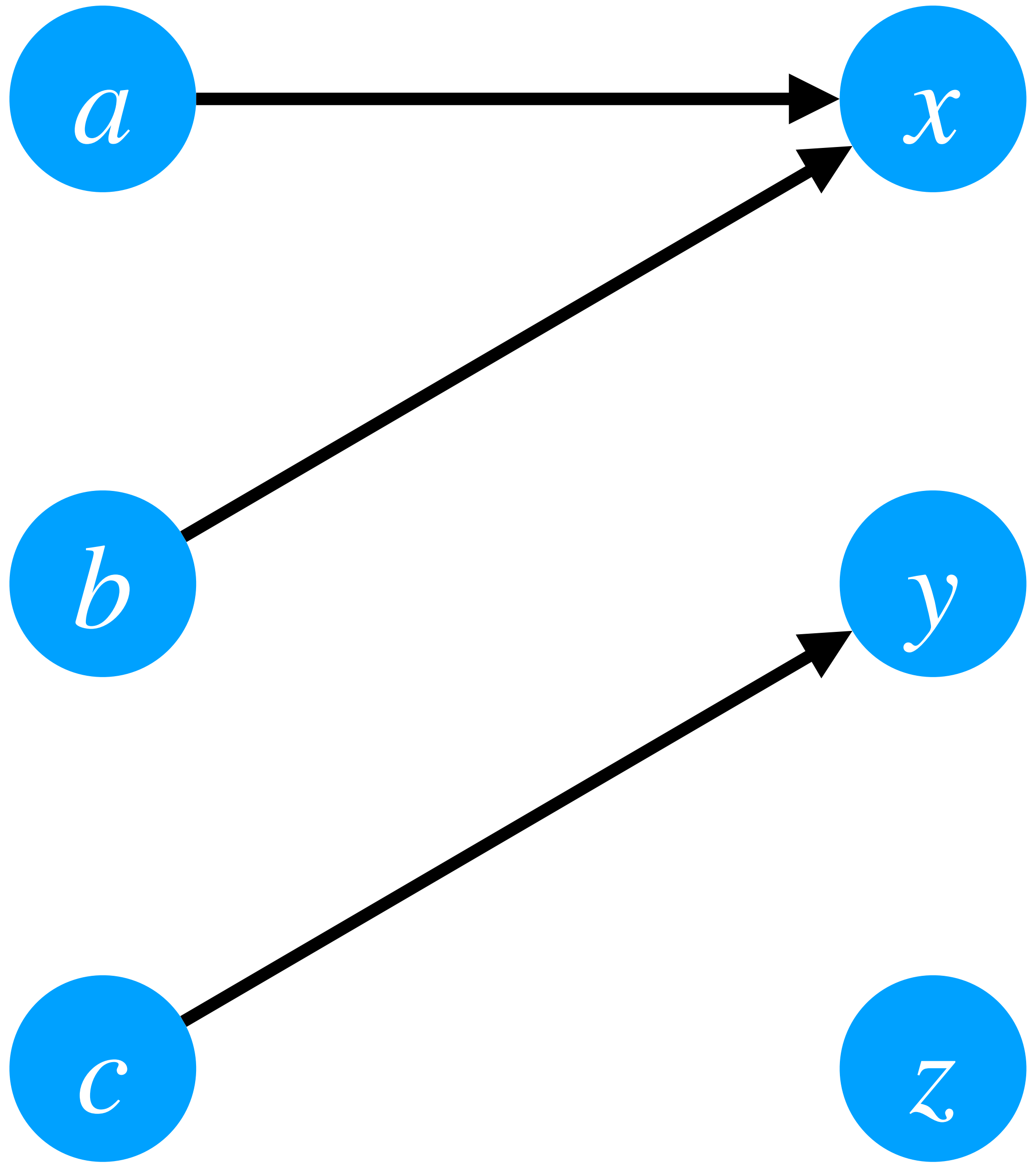

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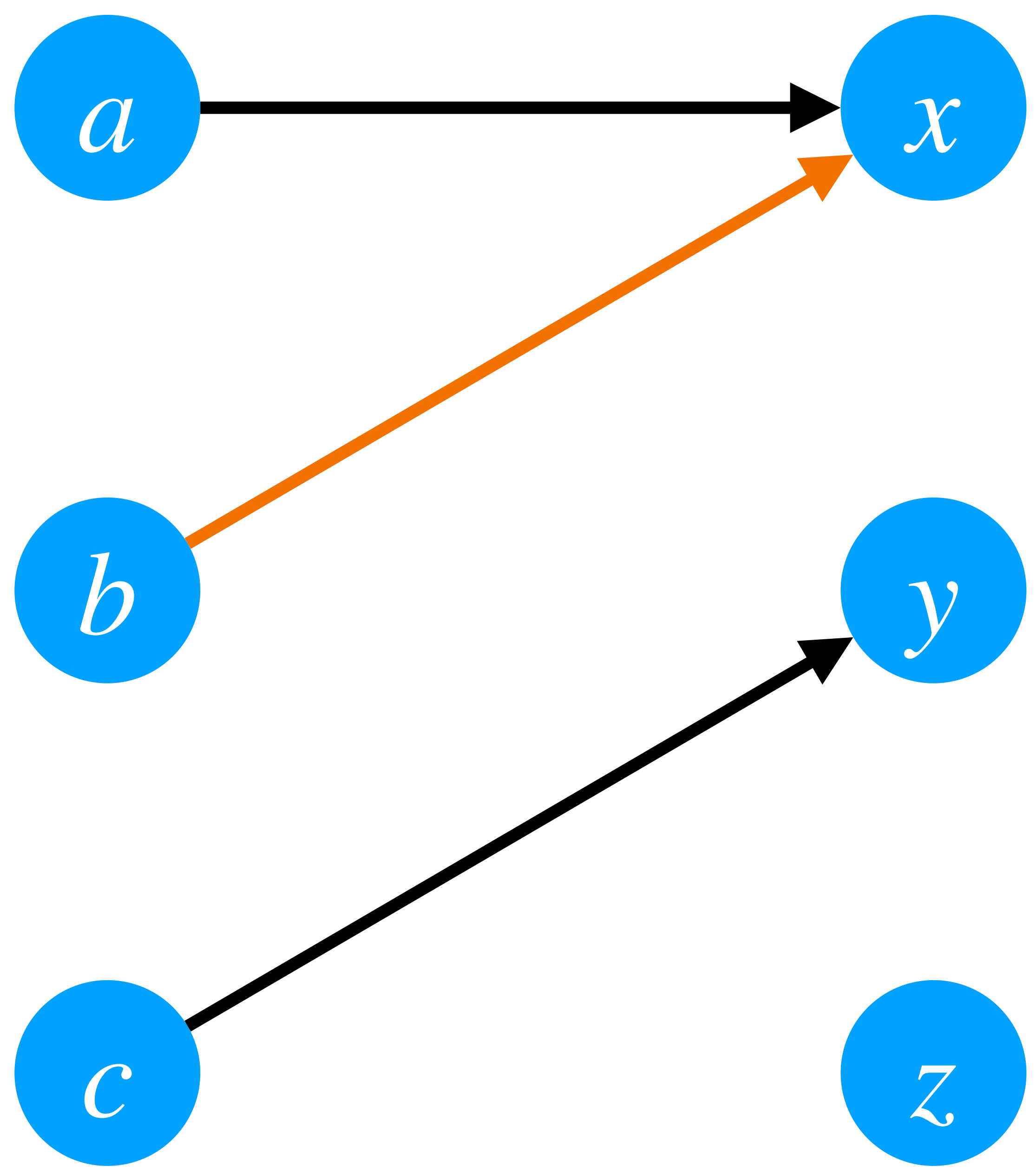

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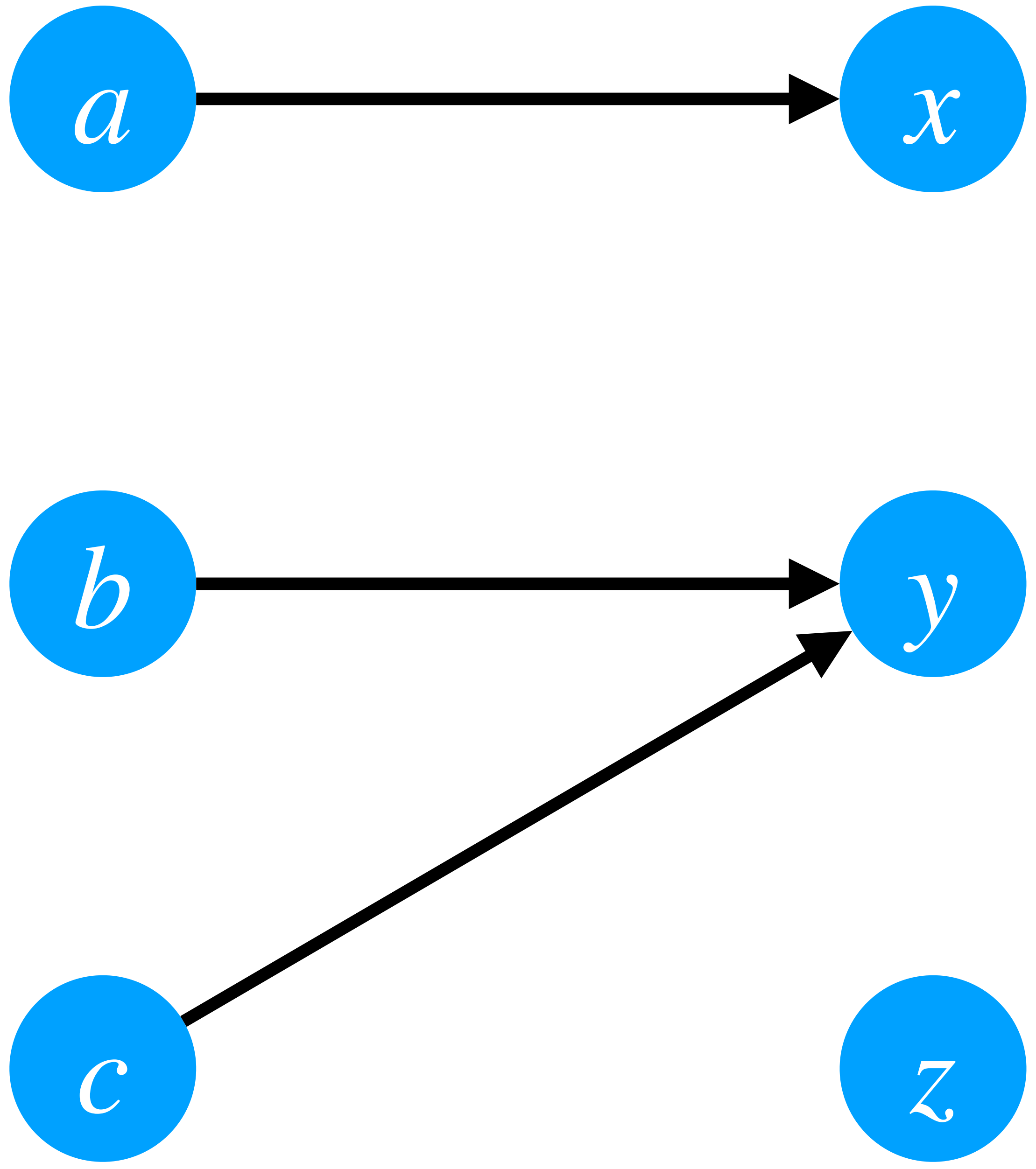

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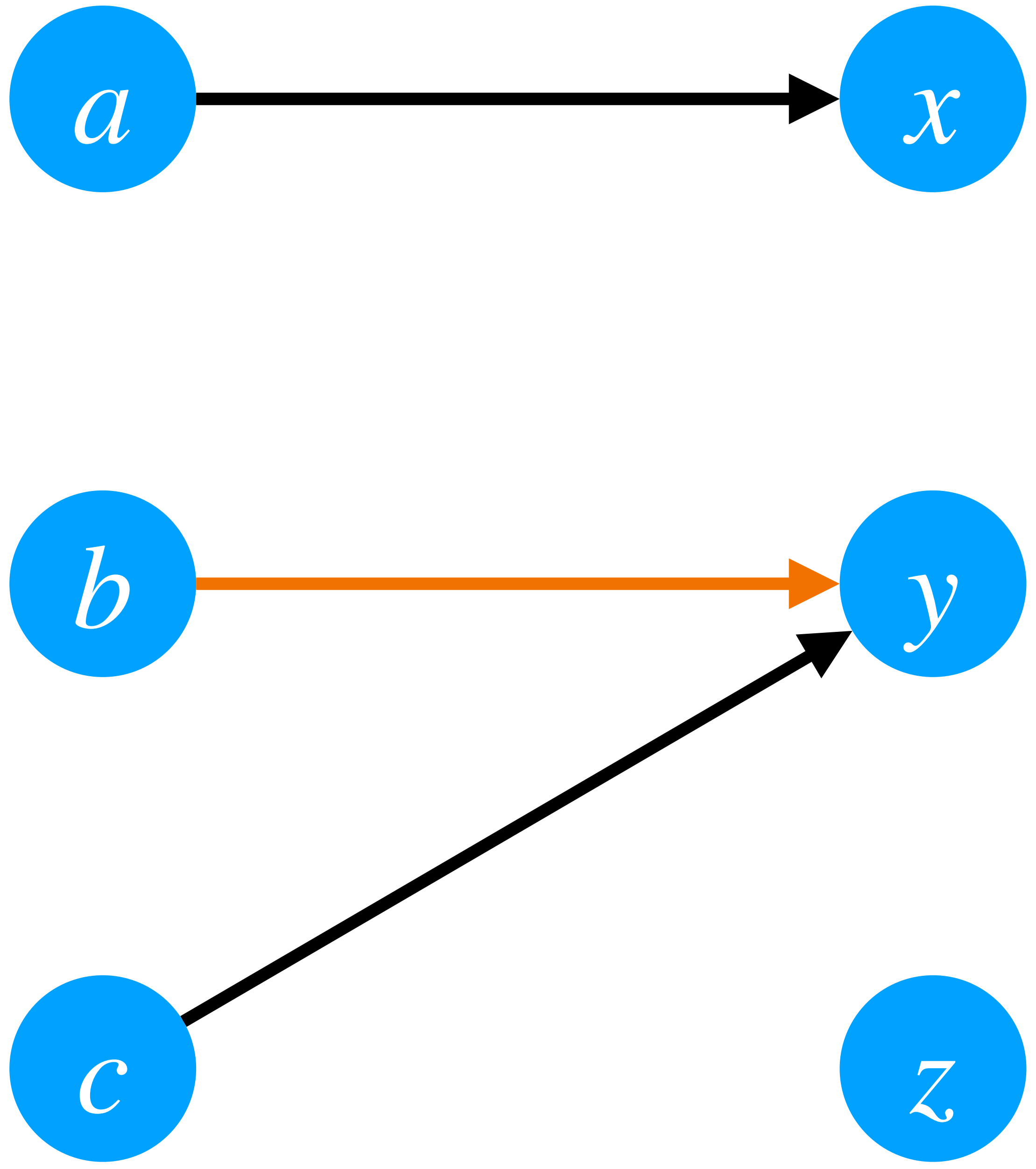

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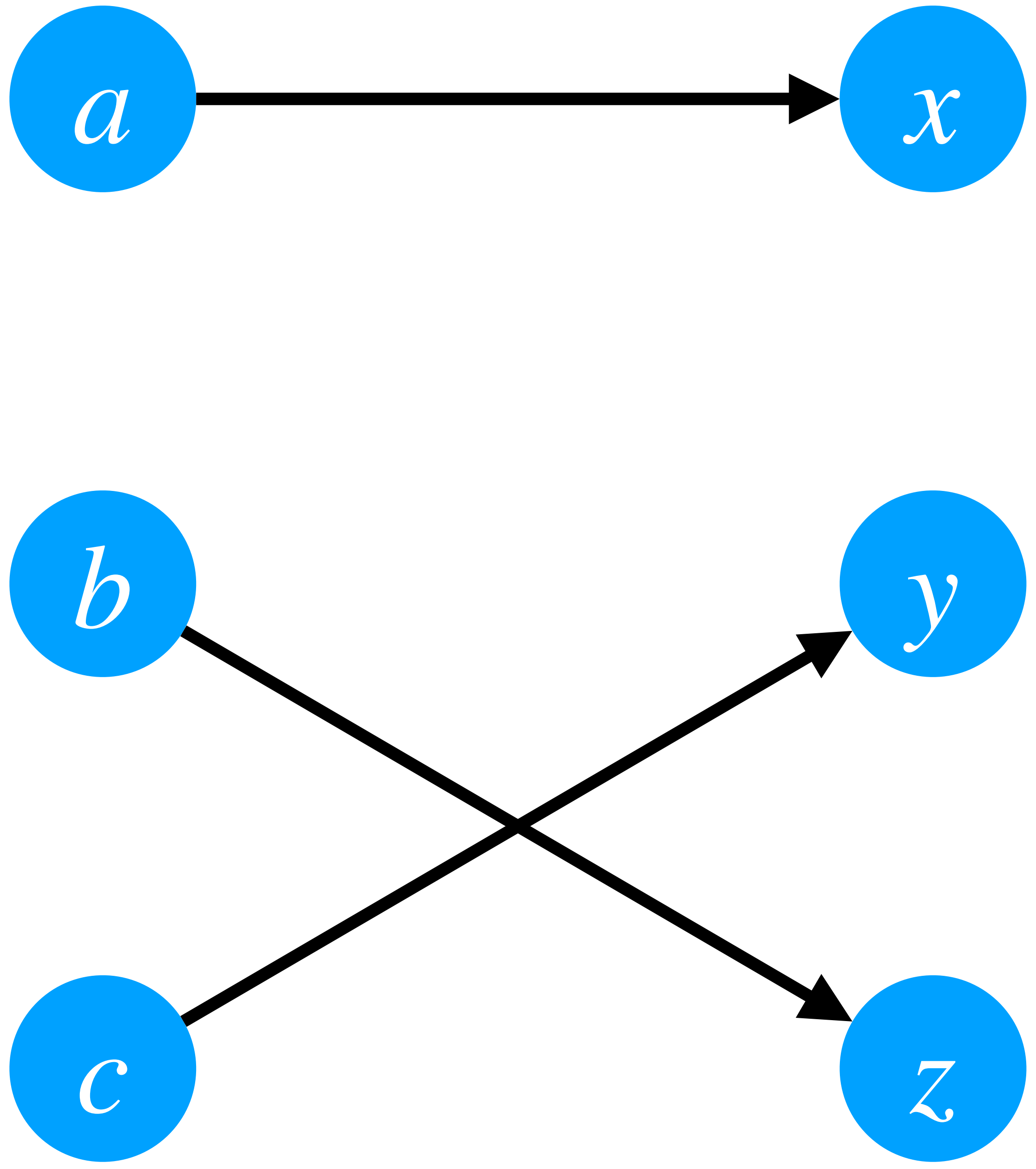

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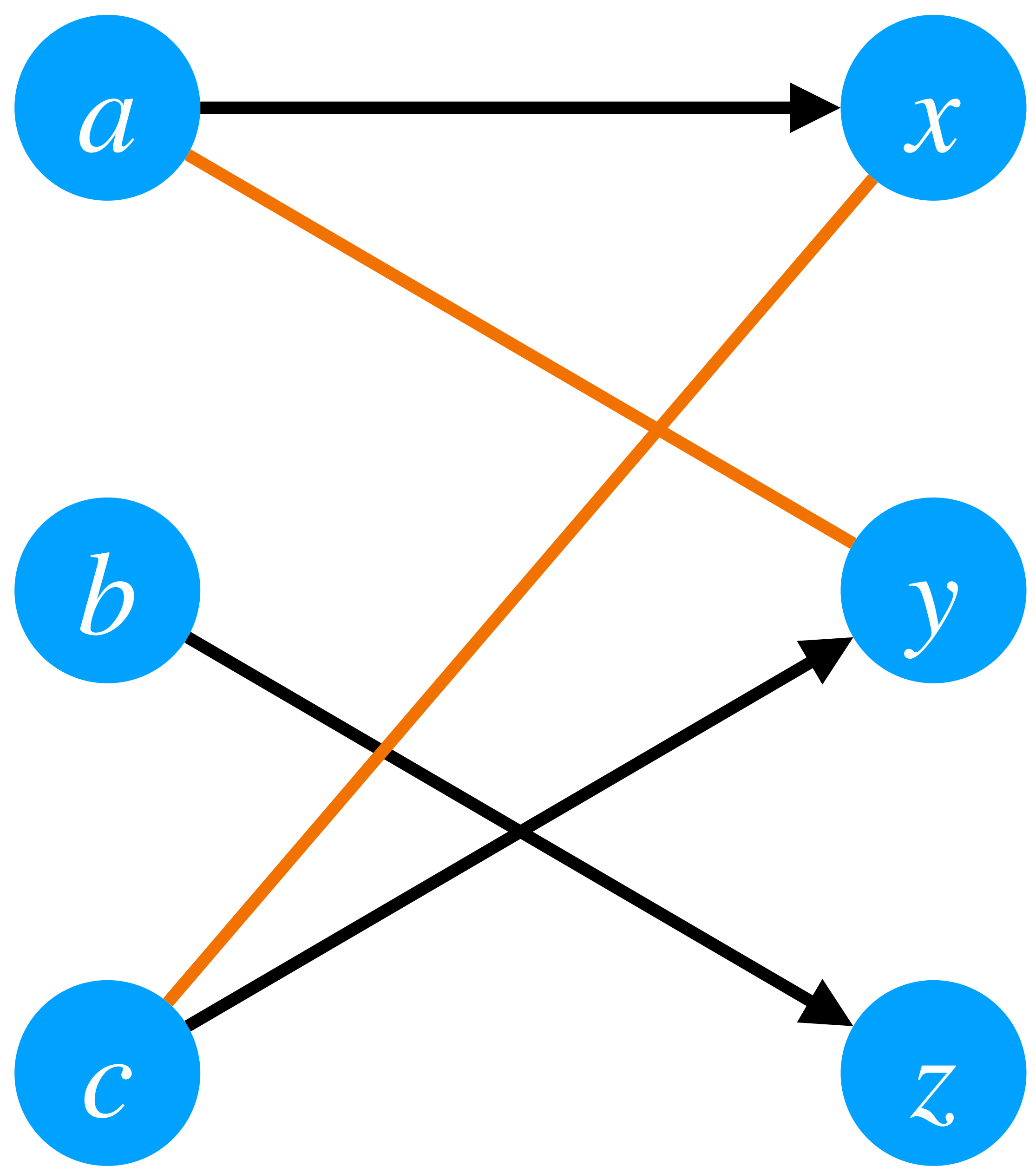

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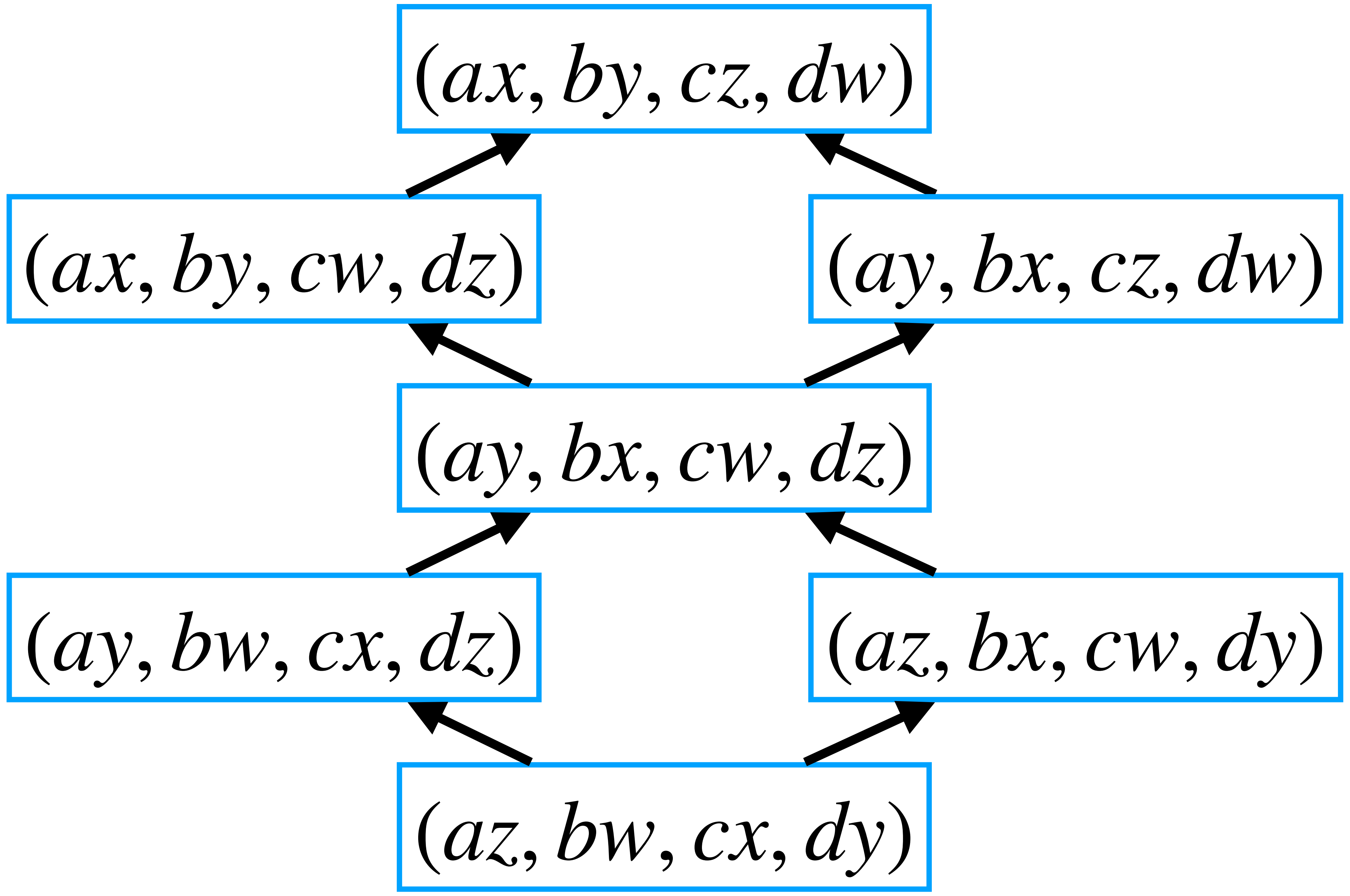

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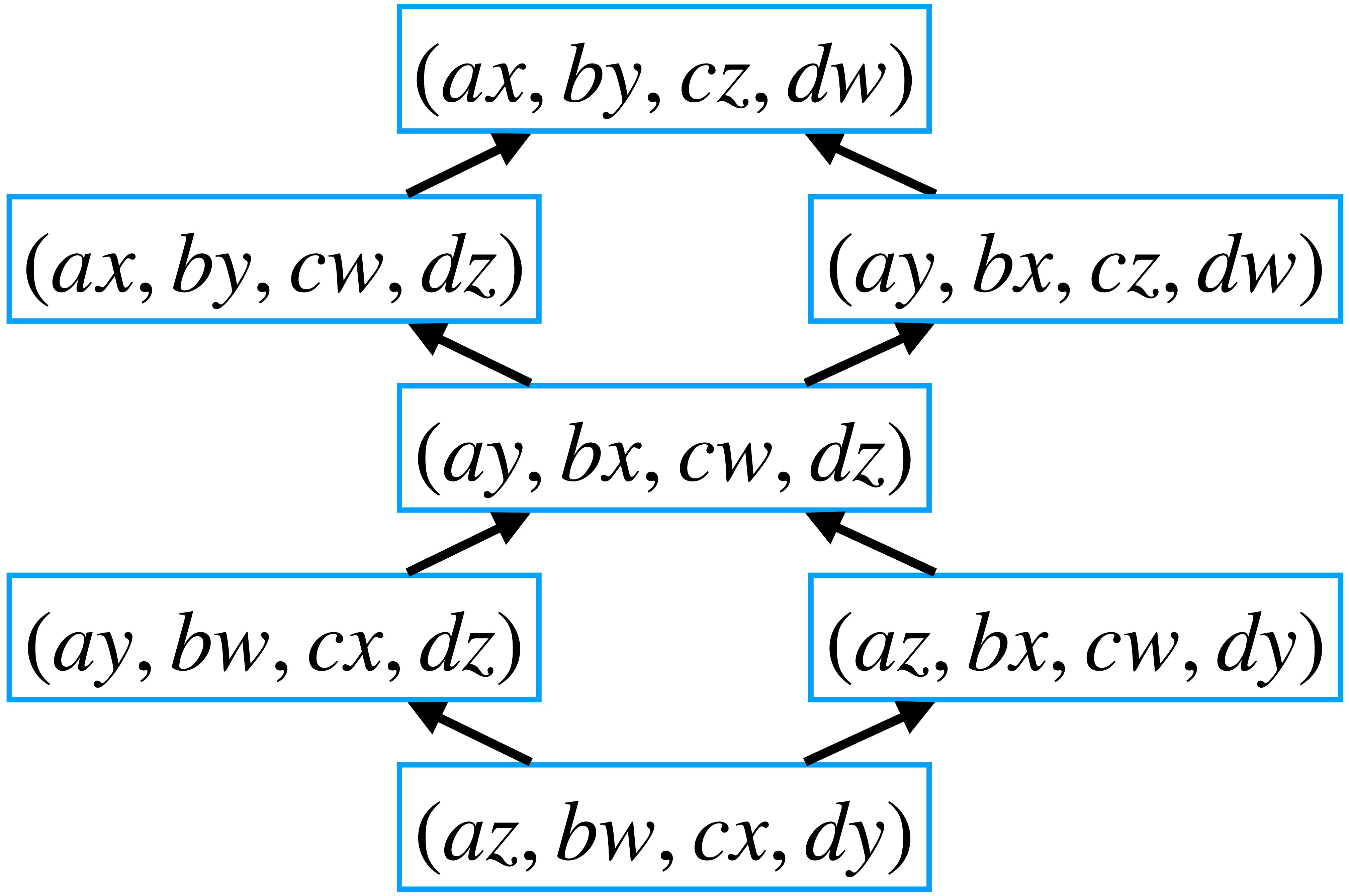

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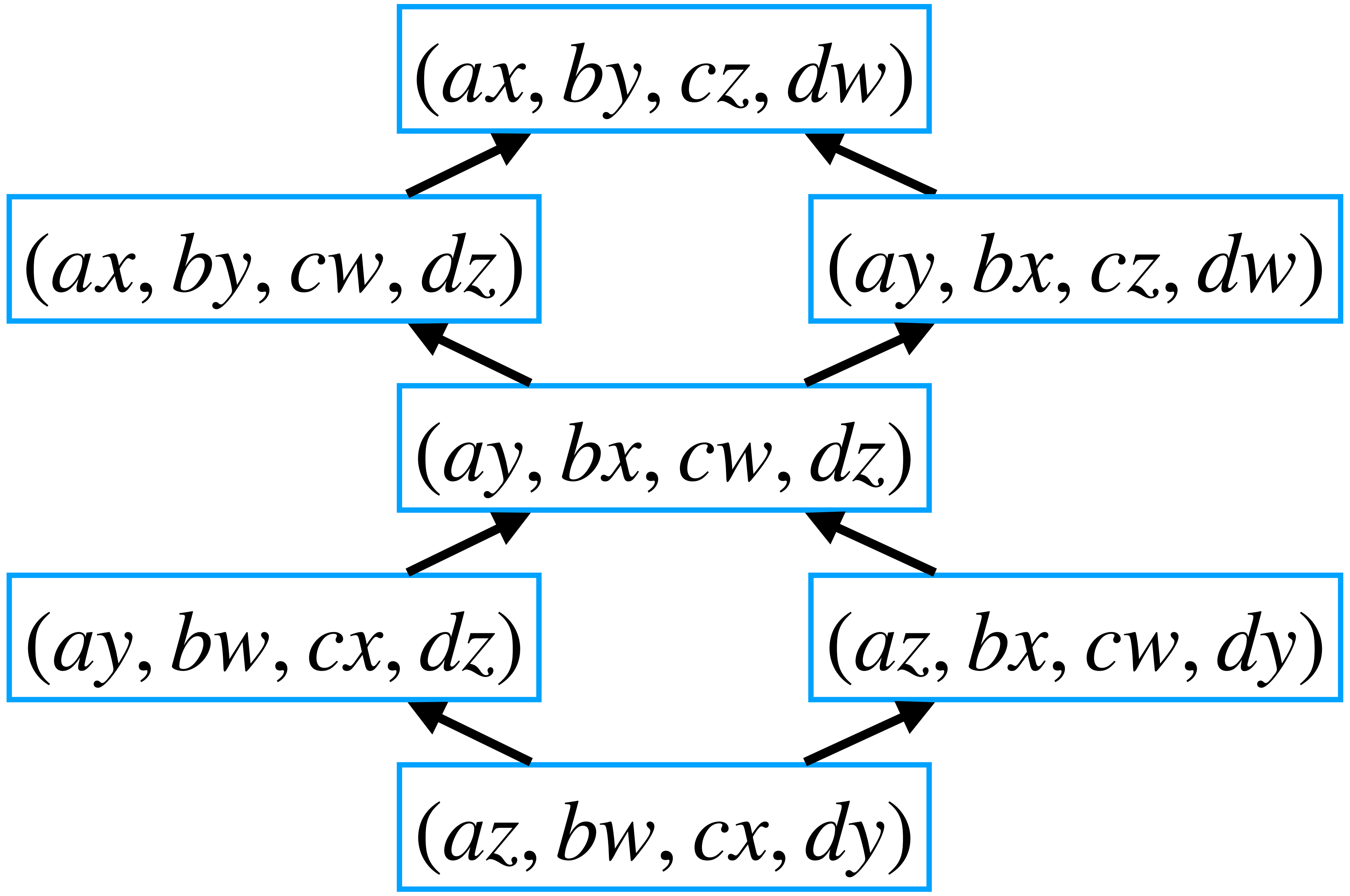

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  - Chilean school admission system gives interesting examples! [Correa et al. '21]

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