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Exercises for Fine-Grained Complexity Theory

www.mpi-inf.mpg.de/departments/algorithms-complexity/teaching/winter17/fine-complexity/

Exercise Sheet 3 Due: **Tuesday, December 5, 2017**

Total points : 40

You are allowed to collaborate on the exercise sheets, but you have to write down a solution on your own, ***using your own words.*** *Please indicate the names of your collaborators for each exercise you solve. Further, cite all external sources that you use (books, websites, research papers, etc.).*

You need to collect at least 50% of all points on exercise sheets.

Exercise 1 (*10 points*) In the lecture we generalized **3SUM** to the following problem:

k–**SUM:** Given k sets $A_1, A_2, \ldots, A_k$ of n integers, determine whether there are $a_1 \in A_1$, $a_2 \in A_2, \ldots, a_k \in A_k$ such that $a_1 + a_2 + \ldots + a_k = 0$.

a) *(5 points)* Demonstrate an algorithm solving k-**SUM** that runs in time $O(n^{k/2} \cdot \log n)$ for *even* k, and runs in time $O(n^{(k+1)/2})$ for *odd* k.

b) *(5 points)* Describe how to generalize the $O(n^2 \cdot \text{poly} \log \log n / \sqrt{\log n})$ time algorithm for **3SUM** from the lecture to k–**SUM** for any odd k. Can you obtain a similar improvement for even k?

Exercise 2 (*5 points*)

Consider the following geometric problem:

Segment Visibility: Given a set S of n line segments and two distinguished line segments a and b. Determine whether there are points p on a, q on b such that the line through the points p and q does not intersect any line segment in S (i.e. we want to check whether a is "visible" from b).

Show that if the **Segment Visibility** problem can be solved in time $O(n^{2-\varepsilon})$ for some $\varepsilon > 0$, then **3SUM** can be solved in time $O(n^{2-\varepsilon'})$ for some $\varepsilon' > 0$.

Exercise 3 (*10 points*)

Consider the following problem that can be solved in time $O(n^2 \cdot \log n)$:

$\mathbf{X+Y}$ problem: Given two sets X and Y of n integers, determine whether the multi-set $X + Y := \{a + b \mid a \in X,\, b \in Y\}$ contains n^2 distinct integers.

Show that if the $\mathbf{X+Y}$ problem can be solved in time $O(n^{2-\varepsilon})$ for some $\varepsilon > 0$, then **3SUM** can be solved in time $O(n^{2-\varepsilon'})$ for some $\varepsilon' > 0$.

Exercise 4 (*15 points*) Consider the following problem:

Zero Weight 3–Star: Given a weighted 4–partite graph $G = (V_1 \cup V_2 \cup V_3 \cup V_4, E)$, where $|V_1| = |V_2| = |V_3| = |V_4| = n$, determine whether there are $v_1 \in V_1, v_2 \in V_2, v_3 \in V_3$, and $v_4 \in V_4$ such that $w(v_1, v_2) + w(v_1, v_3) + w(v_1, v_4) = 0$.

In this exercise, we will show that any $O(n^{2-\varepsilon})$ time algorithm for **3SUM** implies an $O(n^{3-\varepsilon'})$ time algorithm for **Zero Weight 3–Star**, *and vice versa* (under randomized reductions).

a) *(4 points)* Show that if there is an algorithm solving **3SUM** running in time $O(n^{2-\varepsilon})$ for some $\varepsilon > 0$, then there is an algorithm solving **Zero Weight 3–Star** running in time $O(n^{3-\varepsilon'})$ for some $\varepsilon' > 0$.

   Hint: Try to find an algorithm for ***Zero Weight 3–Star*** *running in time $O(n^3)$ first.*

In the remaining exercises we will now proceed to show the surprising direction of the equivalence, namely that **Zero Weight 3–Star** is **3SUM**–hard at cubic time.

b) *(2 points)* Show that if there is an $O(n^{3-\varepsilon})$ time algorithm (for some $\varepsilon > 0$) for **Zero Weight 3–Star**, then there is an $O(n^{3-\varepsilon'})$ time algorithm (for some $\varepsilon' > 0$), that decides whether at least one of n given **3SUM** instances is a "YES" instance.

c) *(2 points)* Show that if there is an $O(q \cdot n^{2-\varepsilon})$ time algorithm (for some $\varepsilon > 0$) deciding whether at least one of q given **3SUM** instances is a "YES" instance, then there is also an $O(q \cdot n^{2-\varepsilon'})$ time algorithm (for some $\varepsilon' > 0$) deciding whether at least one of q given **Convolution–3SUM** instances is a "YES" instance.

d) *(4 points)* Fix $0 < \alpha < 1$ and let $t = t(n) = n^\alpha$. Show that if we can decide in time $O(n^{2-\varepsilon})$ (for some $\varepsilon > 0$) whether at least one of $(n/t)^2$ **Convolution–3SUM** instances of size t is a "YES" instance, then we can also solve a single **Convolution–3SUM** instance of size n in time $O(n^{2-\varepsilon'})$ (for some $\varepsilon' > 0$).

e) *(3 points)* Combine the parts b) to d) with the reduction from **Convolution–3SUM** to **3SUM** to obtain the desired reduction from **3SUM** to **Zero Weight 3–Star**, i.e. show that an $O(n^{3-\varepsilon})$ time algorithm for **Zero Weight 3–Star** (for some $\varepsilon > 0$) would imply a randomized $O(n^{2-\varepsilon'})$ time algorithm for **3SUM** (for some $\varepsilon' > 0$) with error probability $O(n^{-100})$.