Antonios Antoniadis and Marvin Künnemann **Winter 2018/19**

Exercises for Randomized and Approximation Algorithms

www.mpi-inf.mpg.de/departments/algorithms-complexity/teaching/winter18/rand-apx-algo/

Exercise Sheet 1: Greedy Approximation Algorithms

To be handed in by **October 23rd, 2018** via e-mail to André Nusser (CC to Antonios Antoniadis and Marvin Künnemann)

Exercise 1 *(10 Points)* In the *k-suppliers* problem we are given as input a positive integer k and a set of vertices V, along with distances $d_{ij} \geq 0$ between any two vertices i, j. We assume $d_{ii} = 0, d_{ij} = d_{ji}$ for each $i, j \in V$, and that the distances obey the triangle inequality, i.e., for each triple $i, j, l \in V$ there holds $d_{ij} + d_{jl} \geq d_{il}$. Furthermore, the vertices are partitioned into *suppliers* $F \subseteq V$ and *customers* $D = V \setminus F$. The goal is to find k suppliers such that the maximum distances from a supplier to a customer is minimized. In other words, we wish to find, among all sets $S \subseteq F, |S| \leq k$, the set S that minimizes $\max_{j \in D} d(j, S)$.

Give a 3-approximation algorithm for the *k-suppliers* problem (and prove that it is indeed a 3-approximation algorithm).

Exercise 2 *(30 Points)* Consider the unweighted version of the *vertex cover problem* that we saw in the lecture, where all weights are equal to one. More formally, we are given a graph $G = (V, E)$ and the goal is to find a set $C \subseteq V$ of minimum cardinality such that for any edge $(i, j) \in E$ either $i \in C$ or $j \in C$.

We define two different greedy algorithms for the unweighted vertex cover problem, *edge-based greedy* and *vertex-based greedy*:

Algorithm: edge-based greedy

$C := \emptyset$
while $E \neq \emptyset$ **do**
  let (i, j) be an arbitrary edge of E
  $C := C \cup \{i, j\}$
  remove from E every edge incident to either i or j.
return C

Algorithm: vertex-based greedy

$C := \emptyset$
while $E \neq \emptyset$ **do**
  let v be the vertex of maximal degree in V
  $C := C \cup \{v\}$
  remove v from V and every edge e incident to v from E.
return C

(The degree of a vertex v is the number of edges incident to v.)

a) *(5 Points)* Argue that both algorithms produce a feasible solution, i.e., the returned set of vertices C has the property for any edge $(i, j) \in E$ either $i \in C$ or $j \in C$. Also show that both algorithms run in time polynomial in $|V|$ and $|E|$.

b) *(10 Points)* Prove that edge-based greedy has an approximation-ratio of 2.

c) *(15 Points)* Prove that vertex-based greedy cannot have an approximation-ratio better than $\Theta(\log n)$. In order to do this, give an example graph for which the solution returned by vertex-based greedy has a value a factor $\Theta(\log n)$ away from the value of the optimal solution.
(Hint: A bipartite graph, is a graph whose vertices can be partitioned into two disjoint sets L and R such that no two graph vertices within the same set are adjacent.

First prove that that it suffices to construct a bipartite graph with $|L| = k$ that satisfies the following properties:

- $|R| = \Theta(k \log k)$*),*
- *vertex-based greedy ouptuts $C = R$ while the optimal solution outputs $C^* = L$.*

Then describe a bipartite graph that satisfies the two aforementioned properties.)