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Exercises for Randomized and Approximation Algorithms

www.mpi-inf.mpg.de/departments/algorithms-complexity/teaching/winter18/rand-apx-algo/

Exercise Sheet 8: Randomized Rounding

To be handed in by **December 11th, 2018** via e-mail to André Nusser
(CC to Antonios Antoniadis and Marvin Künnemann)

Exercise 1 *(15 Points)* Consider the following problem P: Given a *directed* graph $G = (V, E)$ with weights $w_{i,j} \geq 0$ for each edge $(i, j) \in E$, the task is to partition V into two sets U and $W = V \setminus U$ so as to maximize the total weight of edges going from U to W, i.e., maximize

$$\sum_{\substack{(i,j)\in E \\ i\in U, j\in W}} w_{i,j}.$$

a) *(3 Points)* Give a randomized $\frac{1}{4}$-approximation for the problem.

b) *(2 Points)* Consider the following mixed integer linear program:

$$\begin{array}{lrll} \text{maximize} & \displaystyle\sum_{(i,j)\in E} w_{i,j} z_{i,j} & & \\ \text{s.t.} & z_{i,j} & \leq x_i, & \text{for } (i,j) \in E, \\ & z_{i,j} & \leq 1 - x_j, & \text{for } (i,j) \in E, \\ & x_i & \in \{0,1\}, & \text{for } i \in V, \\ & z_{i,j} & \in [0,1], & \text{for } (i,j) \in E, \end{array}$$

Show that it solves P.

c) *(10 Points)* Consider a randomized rounding algorithm that solves the above mixed integer linear program and independently puts each vertex i into U with probability $1/4 + x_i/2$. Prove that this gives a randomized $1/2$-approximation algorithm for P.

Exercise 2 *(15 Points)* Recall that in the lecture, we analyzed a randomized rounding algorithm for the minimum-capacity multicommodity flow problem that gives a 2-approximation with probability $1 - O(1/n^2)$ whenever the linear programming relaxation has an optimal value of at least $12 \ln n$.

a) *(5 Points)* Give an LP relaxation with *polynomially* many variables and constraints that is equivalent to the one given in the lecture. Describe how to use this LP to make the algorithm given in the lecture run in polynomial time.

b) *(5 Points)* Prove that randomized rounding gives a $O(\frac{\log n}{\log \log n})$-approximation with probability $1-O(1/n^2)$ in the general case. What can you say about the expected approximation ratio?

c) *(5 Points)* Prove that the upper bound is tight in the sense that the expected approximation ratio of randomized rounding is $\Omega(\frac{\log n}{\log \log n})$ in the worst case.
Note: You may need to assume the following helpful fact. Consider n balls and n bins, and put each ball independently into a bin chosen uniformly at random. We say that a bin has load *ℓ if ℓ balls are put into this bin. Then, the expected maximum load of the bins is $\Omega(\frac{\log n}{\log \log n})$.*

Exercise 3 *(10 Points)* Recall the randomized rounding algorithm for MAX SAT given in the lecture. Show that independently setting each Boolean variable x_i to 1 with probability $\frac{1}{2}y_i^* + \frac{1}{4}$ also yields a $\frac{3}{4}$-approximation. (Recall that y^* denotes an optimal solution of the LP relaxation discussed in the lecture.)