Homework 2

Algorithms on Directed Graphs, Winter 2018/9

Due: 9.11.2018 by 16:00

Note. All graphs are directed graphs unless stated otherwise.

Definition. Let $G = (V, E)$ be a graph with $n = |V|$ the number of vertices. A Hamiltonian cycle in $G$ is a simple (directed) cycle of length $n$. A Hamiltonian path is a simple (directed) path on $n$ vertices.

Exercise 1. A classical result in complexity theory asserts that the problem of deciding if a (directed) graph $G = (V, E)$ contains a Hamiltonian cycle is NP-complete. Use this fact to show that deciding if a (directed) graph contains a Hamiltonian path is also NP-complete.

Hint: Given a digraph $G$, describe an efficient method of constructing a graph $G'$ such that $G'$ contains a Hamiltonian path if and only if $G$ contains a Hamiltonian cycle.

Exercise 2. Let $G = (V, E)$ be a graph and $\chi : V \rightarrow [k]$ a coloring of $V$. We say that path $P = (v_1, v_2, \ldots, v_\ell)$ is colorful if the colors $\chi(v_i)$ are pairwise distinct for $i = 1, 2, \ldots, \ell$. Describe an algorithm that finds a colorful path of length $k$ in $G$ (if such a path exists) that runs in time $2^k n^{O(1)}$.

Hint: Use dynamic programming.

Exercise 3. Let $T$ be a directed tree on $k$ vertices. Use color coding to give a $2^{O(k)} n^{O(1)}$-time (randomized) algorithm that determines whether a given graph $G$ contains a subgraph isomorphic to $T$.

Exercise 4. Suppose $G = (V, E)$ contains a simple cycle $C = v_1, v_2, \ldots, v_\ell$ on $\ell \geq 2k + 1$ vertices, but does not contain any simple cycles on $k, k + 1, \ldots, 2k$ vertices. Suppose further, that $V = L \cup R$ is a partition of $G$’s vertices such that $v_1, v_2, \ldots, v_k \in L$, $v_{k+1}, v_{k+2}, \ldots, v_{2k} \in R$, and $(v_1, v_2, \ldots, v_k)$ is a shortest path from $v_1$ to $v_k$ in $G[L]$ (the induced sugraph of $G$ with vertex set $L$). Let $P$ be any (simple) path from $v_1$ to $v_k$ of length $k$ in $G[L]$. Show that $G[V \setminus (P \setminus \{v_1, v_k\})]$ contains a path from $v_k$ to $v_1$.

Hint: Note that this is straightforward if $P$ does not intersect $C$ except at $v_1$ and $v_k$. To prove the general case, assume (without loss of generality) that $C$ is the shortest cycle satisfying the hypotheses of the statement. Try arguing by contradiction.