Homework 2

Algorithms on Directed Graphs, Winter 2018/9

Due: 9.11.2018 by 16:00

Note. All graphs are *directed graphs* unless stated otherwise.

Definition. Let $G = (V, E)$ be a graph with $n = |V|$ the number of vertices. A ***Hamiltonian cycle*** in G is a simple (directed) cycle of length n. A ***Hamiltonian path*** is a simple (directed) path on n vertices.

Exercise 1. A classical result in complexity theory asserts that the problem of deciding if a (directed) graph $G = (V, E)$ contains a Hamiltonian cycle is NP-complete. Use this fact to show that deciding if a (directed) graph contains a Hamiltonian *path* is also NP-complete.

Hint: Given a digraph G, describe an efficient method of constructing a graph G' such that G' contains a Hamiltonian path if and only if G contains a Hamiltonian cycle.

Exercise 2. Let $G = (V, E)$ be a graph and $\chi : V \to [k]$ a coloring of V. We say that path $P = (v_1, v_2, \ldots, v_\ell)$ is ***colorful*** if the colors $\chi(v_i)$ are pairwise distinct for $i = 1, 2, \ldots, \ell$. Describe an algorithm that finds a colorful path of length k in G (if such a path exists) that runs in time $2^k n^{O(1)}$.

Hint: Use dynamic programming.

Exercise 3. Let T be a directed tree on k vertices. Use color coding to give a $2^{O(k)} n^{O(1)}$-time (randomized) algorithm that determines whether a given graph G contains a subgraph isomorphic to T.

Exercise 4. Suppose $G = (V, E)$ contains a simple cycle $C = v_1, v_2, \ldots, v_\ell$ on $\ell \geq 2k + 1$ vertices, but does not contain any simple cycles on $k, k + 1, \ldots, 2k$ vertices. Suppose further, that $V = L \cup R$ is a partition of G's vertices such that $v_1, v_2, \ldots, v_k \in L$, $v_{k+1}, v_{k+2}, \ldots, v_{2k} \in R$, and $(v_1, v_2, \ldots, v_k)$ is a shortest path from v_1 to v_k in $G[L]$ (the induced sugraph of G with vertex set L). Let P be any (simple) path from v_1 to v_k of length k in $G[L]$. Show that $G[V \setminus (P \setminus \{v_1, v_k\})]$ contains a path from v_k to v_1.

Hint: Note that this is straightforward if P does not intersect C except at v_1 and v_k. To prove the general case, assume (without loss of generality) that C is the shortest *cycle satisfying the hypotheses of the statement. Try arguing by contradiction.*