1 Disjoint Paths and $k$-linkage

In this note we consider two related problems on directed graphs: disjoint paths, and $k$-linkage. In the following sections, we describe three reductions with implications for the computational complexity of disjoint paths and $k$-linkage.

Let $G = (V(G), E(G))$ be a directed graph, and let $A, B \subseteq V(G)$ be disjoint subsets with $|A| = |B| = k$. For concreteness we write $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, \ldots, b_k\}$.

We say that $(G, A, B)$ satisfies the disjoint paths property if there exist $k$ pair-wise vertex disjoint directed paths $P_1, P_2, \ldots, P_k$ such that the first vertex of each $P_i$ is in $A$ and the last vertex of each $P_i$ is in $B$. We say that the paths $P_1, P_2, \ldots, P_k$ form a $k$-linkage if additionally for each $i = 1, 2, \ldots, k$, $P_i$ is a path from $a_i$ to $b_i$. If such a family of paths exist, we say that $(G, A, B)$ satisfies the $k$-linkage property.

2 Reducing Disjoint Paths to Max Flow/Min Cut

We describe a reduction from disjoint paths to the max flow/min cut problem. Let $(G, A, B)$ be an instance of the disjoint paths problem. We construct the graph $G'$ as follows. The vertex set $V(G')$ contains vertices as follows:

- For each $a_i \in A$ and $b_i \in B$, $i \in [k]$, we have $a_i, b_i \in V(G')$.
- For each $v \in V(G) \setminus (A \cup B)$, there are two vertices $v', v'' \in V(G')$.
- There are two additional vertices $s, t \in V(G')$.

The edge set of $G'$ and edge capacities are constructed as follows:

- For each $a_i \in A, b_i \in B$, we have $(s, a_i), (b_i, t) \in E(G')$ with $c((s, a_i)) = c((b_i, t)) = 1$. 
• For each \( v \in V(G) \setminus (A \cup B) \), \((v', v'') \in E(G')\) with \( c((v', v'')) = 1 \)

• For each \((u, v) \in E(G)\) we have
  - \((u'', v') \in E(G')\) if \( u, v \notin A \cup B \)
  - \((u', v) \in E(G')\) if \( u \in A, v \notin A \cup B \)
  - \((u'', v) \in E(G')\) if \( u \notin A \cup B, v \in B \)
  - \((u, v) \in E(G')\) if \( u \in A, v \in B \).

The capacities of all edges above are \( \infty \).

There are no other edges in \( E(G') \). In particular, \( G' \) does not contain any edges of the form \((u, a)\) for \( a \in A, u \neq s \), nor does it contain edges of the form \((b, w)\) for \( w \neq t \).

**Exercise 1.** Prove that the graph \( G' \) constructed above has max flow/min cut value of \( k \) if and only if \((G, A, B)\) satisfies the disjoint paths property.

**Exercise 2.** Use the construction above and the max flow/min cut theorem to prove Menger’s theorem: the size of the minimum vertex cut\(^1\) separating \( A \) and \( B \) is equal to the number of vertex-disjoint paths between \( A \) and \( B \).

### 3 Reducing 3-SAT to \( k \)-linkage

Let \( x_1, x_2, \ldots, x_n \) be Boolean variables, and let \( \Phi \) be a 3-CNF formula over \( x_1, x_2, \ldots, x_n \). That is,

\[
\Phi = \phi_1 \land \phi_2 \land \cdots \land \phi_m,
\]

where each \( \phi_j \) is of the form

\[
\phi_j = y_{j1}^1 \lor y_{j2}^2 \lor y_{j3}^3
\]

and each \( y_{j}^i \) is a literal (equal to some \( x_i \) or its negation \( \overline{x_i} \)). The 3-SAT problem is to determine if there exists an assignment of the \( x_i \) to true or false such that \( \Phi \) evaluates to true—in this case we say that \( \Phi \) is **satisfiable**. 3-SAT is one of the classical NP-complete problems.

Given a 3-CNF formula \( \Phi \) as above, we construct a directed acyclic graph \( G = G(\Phi) \) such that \( G \) satisfies the \( k \)-linkage property if and only if \( \Phi \) is satisfiable. We build the graph \( G \) as follows.

- For each literal \( x_i \), \( V(G) \) contains two vertices \( s_i, t_i \) along with two vertex-disjoint paths from \( s_i \) to \( t_i \). We label the two paths by \( T_i \) and \( F_i \) respectively. Initially \( T_i \) and \( F_i \) each consists of a single edge from \( s_i \) to \( t_i \), but the edges will be sub-divided as the construction proceeds.

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\(^1\)A **vertex cut** separating \( A \) and \( B \) is a set \( X \subseteq V(G) \) such that removing the vertices in \( X \) from \( G \) disconnects \( A \setminus X \) and \( B \setminus X \). In particular, the cuts \( X = A \) and \( X = B \) separate \( A \) and \( B \), so that a min cut has size at most \( k \).
• For each clause $\phi_j$ in $\Phi$, $V(G)$ contains two vertices $u_j$ and $w_j$ along with three vertex-disjoint paths from $u_j$ to $w_j$. The paths correspond to the three literals in $\phi_j$. For each literal $y$ in $\phi_j$:

- if $y = x_i$ appears as a positive literal in $\phi_j$, subdivide the path $F_i$ by adding a new vertex $v$ to this path (between $s_i$ and $t_i$),
- if $y = -x_i$ appears as a negative literal in $\phi_j$, subdivide the path $T_i$ by adding a new vertex $v$ to this path.

Then add the edges $(u_j, v)$ and $(v, w_j)$ to $E(G)$ so that they form a path from $u_j$ to $w_j$.

**Exercise 3.** For the graph $G$ constructed as above, prove that there exists a $k$-linkage from $A = \{s_1, s_2, \ldots, s_n\} \cup \{u_1, u_2, \ldots, u_m\}$ to $B = \{t_1, t_2, \ldots, t_n\} \cup \{w_1, w_2, \ldots, w_m\}$ ($k = n + m$) if and only if $\Phi$ is satisfiable.

### 4 Reducing $k$-linkage for DAGs to Connectivity

Let $G$ be a directed acyclic graph (DAG), and let $A, B \subseteq V(G)$ with $A \cap B = \emptyset$ and $|A| = |B| = k$. Without loss of generality, assume that there are no edges of the form $(u, a)$ for $a \in A$, nor any edges of the form $(b, w)$ for $b \in B$.

We construct a graph $G'$ from $G$ in the following manner. The vertex set $V(G')$ is the the set of $k$-tuples of pair-wise distinct vertices in $G$:

$$V(G') = \{(v_1, v_2, \ldots, v_k) \mid v_1, v_2, \ldots, v_k \in V(G), i \neq j \implies v_i \neq v_j\}.$$  

The edge set $E(G')$ is defined as follows. Since $G$ is a DAG, for every $v = (v_1, v_2, \ldots, v_k) \in V(G')$, there is some index $r$ such that $v_r$ is not reachable from any of the other $v_i$ ($i \neq r$). That is, there is no directed path from any $v_i$ to $v_r$. For this index $r$ (choosing one arbitrarily if there is more than one), and each edge $(v_r, w) \in E(G)$ with $w \neq v_1, v_2, \ldots, v_n$, we add the edge $(v, v_r(w)) \in E(G')$ where $v_r(w)$ is $v$ with $v_r$ replaced by $w$:

$$v_r(w) = (v_1, v_2, \ldots, v_{r-1}, w, v_{r+1}, \ldots, v_k).$$

We will first show that if $G'$ contains a directed path $P'$ from $a = (a_1, a_2, \ldots, a_k)$ to $b = (b_1, b_2, \ldots, b_k)$, then $G$ contains a $k$-linkage from $A$ to $B$. To this end, suppose $P'$ consists of vertices

$$v_1(=a), v_1, v_t, v_{t+1}(=b).$$

From the definition of $G'$, for each edge $(v_j, v_{j+1})$ along $P'$, there is a unique index $r_j$ such that $v_j$ and $v_{j+1}$ differ only in the entry with index $r_j$. Let $u_j$ and $w_j$ denote the $r_j$-th entry of $v_j$ and $v_{j+1}$, respectively. Further, by the definition of $E(G')$, we must have $(u_j, w_j) \in E(G)$. If $r_j = i$, we call $(u_j, w_j)$ an $i$-edge.

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1 No edge of the form $(u, a)$ or $(b, w)$ can appear in any $k$-linkage, so removing such edges will not change whether or not $G$ satisfies the $k$-linkage property.
Claim. The set of $i$-edges induced by $P'$ forms a path $P_i$ from $a_i$ to $b_i$ in $G$.

To see the claim, consider a fixed $i$, and let consider the $i$-edges in the order that they appear in $P'$. Consider the first $i$-edge $(u_j, w_j)$, and let $(v_j, v_{j+1})$ be the corresponding edge in $P'$. Since this the first $i$-edge, the $i$-th entry of $v_j$ is the same as the $i$-th entry of $v_1 = a$. Therefore, $u_j = a_i$, so the first $i$-edge is an edge out of $a_i$: $(a_i, w_j)$. Similarly, the next $i$-edge is an edge from $w_j$ to some other vertex. Continuing in this way, $i$-edges form a path from $a_i$. Since the final vertex in $P'$ is $v_{t+1} = b$, the final $i$-edge in $P'$ must change the $i$-th entry of the corresponding $v_j$ to $b_i$, so that $P_i$ does indeed terminate at $b_i$.

By the claim, the path $P'$ induces $k$ paths $P_1, P_2, \ldots, P_k$ where each $P_i$ is a path from $a_i$ to $b_i$. We now show that the $P_i$ are pair-wise vertex disjoint, thus forming a $k$-linkage from $A$ to $B$. To this end, suppose towards a contradiction that there exist $i \neq j$ and a vertex $w$ appearing in both $P_i$ and $P_j$. Let $u_i$ denote the vertex before $w$ in $P_i$ and let $u_j$ denote the vertex before $w$ in $P_j$. Thus $(u_i, w)$ is and $i$-edge and $(u_j, w)$ is a $j$-edge. Suppose the edges in $P'$ corresponding to $(u_i, w)$ and $(u_j, w)$ are the $t_i$-th and $t_j$-th edges respectively. Assume without loss of generality that $t_i < t_j$. Let $u'_j$ be the $j$-th entry of $v_{t_i}$—i.e., $u'_j$ is the $j$-th entry of $v$ when the $i$-edge $(u_i, w)$ appears in $P'$.

Observe that $j$-edges induced by the sub-path $v_{t_i}, v_{t_i+1}, \ldots, v_{t_j}$ of $P'$ form a path from $u'_j$ to $u_j$ (and then to $w$). Thus, $w$ is reachable from $u'_j$ in $G$. Therefore (by the definition of $G'$) the $i$-th entry of $v_{t_j}$ cannot have changed from $w$ in this sub-path. But this implies that in $v_{t_j}$, both the $i$-th and $j$-th entries are equal to $w$, contradicting the definition of $G'$! Therefore, the paths $P_i$ and $P_j$ do not intersect, as desired.

Exercise 4. Assume (without loss of generality) that $G$ does not contain edges of the form $(u, a)$ for $a \in A$, nor any edges $(b, w)$ for $b \in B$. Prove that if $G$ contains a $k$ linkage from $A$ to $B$, then there exists a path $P'$ in $G'$ from $(a_1, a_2, \ldots, a_k)$ to $(b_1, b_2, \ldots, b_k)$.