Homework 5: Erdős-Pósa property

Algorithms on Directed Graphs, Winter 2018/9

Due: 30.11.2018 by 16:00

In this lecture, we studied the second part of the paper [1]. We prove that every connected planar graph has Erdős-Pósa property. We talked about part 8 of the above paper. They actually did more than what we did in the class. They showed that every planar graph has Erdős-Pósa property.

Exercise 1 (Chordal Graphs)**.** A graph is chordal if it does not contain an induced cycle of length at least 4. That is every cycle of length at least 4 does have a chord. As computer scientists we love chordal graphs: many intractable problems in general graphs are polynomial time solvable. In this exercise, we will learn more about them.

1. Name two natural classes of graphs which are chordal.
2. An interval graph is a graph wherein each of its vertices corresponds to an interval on a real line and there is an edge between two vertices if their corresponding intervals intersect (it is also called line segment intersection graph). Prove that interval graphs are chordal.
3. A graph $G = (V, E)$ is a subtree intersection graph if there is a tree T and a family $\mathcal{F} = \{T_1, \ldots, T_k\}$ of subtrees of T so that there is a one to one mapping $\mu\colon V(G) \to \mathcal{F}$ and there is an edge $\{u, v\} \in E(G)$ if and only if $\mu(u) \cap \mu(v) \neq \emptyset$. Prove tree intersection graphs are chordal.
4. Provide a chordal graph which is not an interval graph.
5. **Bonus Exercise:** Prove that every chordal graph is a tree intersection graph.

Exercise 2 (Grid minors)**.** What is the size of largest grid we can find as a minor in a complete graph of order n?

Exercise 3 (Tournaments and Erdős-Pósa property)**.** Prove that there is a function $f\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ so that for every integers $k, m \in \mathbb{N}$ and every tournament T, one of the followings holds:

1. T contains k acyclic tournaments of order m (m is the number of their vertices).

2. There is a hitting set $S \subseteq V(T)$ such that $|S| \leq f(k, m)$ and every acyclic tournament of order at least m, intersects S.

Exercise 4 (More on tree decomposition)**.** Let H be a connected subgraph of G and let $\mathcal{T} = (T, \beta)$ be a tree decomposition of G (arbitrary tree decomposition). Prove the set of bags of T containing vertices of H induces a connected subtree of T.

References

[1] N. Robertson and P. Seymour. Graph minors. v. excluding a planar graph. *Journal of Combinatorial Theory, Series B*, 41(1):92 – 114, 1986.