Homework 6: Erdős-Pósa property on digraphs

Algorithms on Directed Graphs, Winter 2018/9

Due: 7.12.2018 by 16:00

Recall that an edge is butterfly contractible if it is the only outgoing or the only incoming of one of its endpoints. A graph H is a butterfly minor of a graph G, if H can be obtained by taking a subgraph of G and then applying butterfly contractions. We say a graph H embeds in a graph G if H is a butterfly minor of G.

Most parts of this lecture were based on the following paper.

https://arxiv.org/abs/1603.02504

Exercise 1 (Cylindrical Grids)**.** Prove the followings.

1. Let H be such that:

   $$\begin{cases} V(H) = \{1, \dots, 6\} \\ E(H) = \{(1,2), (2,3), (3,1), (1,4), (4,1), (2,5), (5,2), (3,6), (6,3)\} \end{cases}$$

   Prove that H is not embeddable in any finite cylindrical grid.

2. **Open Problem:** Is there an algorithm which takes a graph H as input and determines whether there is an integer k such that H embeds in G_k (a cylindrical grid of order k). Note that here we are talking about decidability of the problem. We do not care if the algorithm is superexponential with respect to $|H|$.

Exercise 2 (Complete the Proof Part I)**.** Recall the construction of the graph G', H and the function $f\colon \mathbb{N} \to \mathbb{N}$ from the lecture. G' is also depicted in Fig. 4.1. of the referenced paper. Prove that $G - S$ does contain a butterfly model of H for every set $S \subset V(G)$ of size at most $f(2 + |H|)$.

Exercise 3 (Complete the Proof Part II)**.** Show that every butterfly minor model of H in G' does contain a vertex $x \in G_{2f_2+1}$ and a vertex $y \in H'_i$ for some $i \in [k]$. (hint: proof of this and the previous homework is given in the proof of Theorem 4.4 of the referenced paper. There we introduced an equivalent definition of butterfly minor, so-called tree-like minor. That definition makes the proof easier. We already discussed the proof ideas, hence, you should be able to stick on the original definition and prove the theorem.)