Homework 7/8

Algorithms on Directed Graphs, Winter 2018/9

Due: 21.12.2018 by 16:00

Exercise 1. Suppose $m, n \in \mathbf{N}$ with $m \leq n$, and $p \in [0,1]$. Prove that

$$\sum_{k=m}^{n} \binom{n}{k} p^k (1-p)^{n-k} \leq \binom{n}{m} p^m.$$

Exercise 2. Recall that a ***tournament*** $T = (V, E)$ is an orientation of the complete graph with vertex set V. That is, for every $u, v \in V$ with $u \neq v$, we have either $(u, v) \in E$ or $(v, u) \in E$, but not both. Given a tournament T with $n = |V|$, a ***ranking*** on T is a one-to-one function $R : V \to [n]$. The interpretation is that the (unique) vertex v_1 with $R(v_1) = 1$ is ranked first, v_2 with $R(v_2) = 2$ is ranked second, and so on. For an edge $e = (v_i, v_j) \in E$, the ranking R is ***consistent*** with e if $R(v_i) < R(v_j)$. Otherwise, R is ***inconsistent*** with e.[1] Use the probabilistic method to prove the following facts:

(a) For every tournament T, there exists a ranking R that is consistent with at least half (i.e., $|E|/2$) of the edges.

(b) For every $\varepsilon > 0$ there exists $n = n(\varepsilon)$ and a tournament T on n vertices such that for every ranking R on T, R is inconsistent with at least a $1/2 - \varepsilon$ fraction of edges in E.

Hint: For part (b), you may find the bound $n! \leq n^{n+1/2}e^{1-n}$ useful. You can also use the following Chernoff bound: If $X_1, X_2, \ldots, X_k$ are independent 0-1 random variables with $\Pr(X_i = 1) = \Pr(X_i = 0) = 1/2$ for all i, then for all $\varepsilon \in [0,1]$ we have $\Pr(X < (1-\varepsilon)k/2) \leq e^{-\varepsilon^2 k/4}$ where $X = X_1 + X_2 + \cdots + X_k$.

Exercise 3. Let $G = (V, E)$ be a directed graph with $A, B \subseteq V$ with $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, \ldots, b_k\}$. Recall that a ***$k$-linkage*** from A to B is a family of vertex disjoint paths $P_1, P_2, \ldots, P_k$ where each P_i is a path from a_i to b_i. Suppose that for each $i \in [k]$ there is a family F_i of

[1] We can interpret consistency as follows. If $e = (v_i, v_j) \in E$ means that v_i "beats" v_j in the the tournament, then R being consistent with e means that v_i is ranked higher than v_j.

paths from a_i to b_i such that $|F_i| = m$ and for each $j \neq i$, every path $P \in F_i$ intersects at most ℓ paths in F_j. Show that if $8k\ell/m < 1$, then there exists a k-linkage from A to B.

Hint: Use the Lovasz Local Lemma.

Exercise 4. Let $G = (V, E)$ be a directed graph and $\mathcal{P}$ a family of packets/paths in G. Let S be a schedule for $\mathcal{P}$. A ***T-frame*** is a sequence of T consecutive rounds. The ***frame congestion*** C of a T-frame is the maximum number of packets crossing any one edge in the frame, and the ***relative congestion*** of the frame is defined by $R = C/T$. Prove that for any $\mathcal{P}$ with congestion c and dilation d, there exists a (not necessarily feasible) schedule S such that no packet ever waits in a queue, and the relative congestion of any T-frame for $T \geq \log d$ is at most 1.

Hint: Without loss of generality, assume that $d = c$. Consider the family of schedules where each packet chooses a delay t_P uniformly at random from $[\alpha d]$ uniformly at random, and starting at round t_P packet P is forwarded along its path without ever being delayed. Use the Lovasz Local Lemma to prove that there exists a set of initial delays such that resulting schedule satisfies the desired conclusion.