Homework 1

Algorithms on Directed Graphs, Winter 2018/9

Due: 11.1.2018 by 16:00

Throughout this sheet, we will take $G = (V, E)$ to be a directed path of length $n + 1$. That is, $V = [n + 1]$, and $E = \{(i, i + 1) \,|\, i \in [n]\}$. The load of buffer i during round t is denoted by $L^t(i)$. For a subset $I \subseteq [n]$ we write $L^t(I) = \sum_{i \in I} L^t(t)$. An ***adversary*** A is a sequence $a_1, a_2, \ldots$ where each $a_t \in [n]$ is the location of the packet injection in round t. All packets must be delivered to node $n + 1$.

Exercise 1. Consider the greedy forwarding algorithm: for each time t and $i \in [n]$, buffer i forwards a packet if and only if $L^t(i) > 0$.

(a) Prove that for any adversary A and for each round t, the total number of packets in the network satisfies $L^t([n]) \leq n$.

(b) Show that there exists an adversary A (i.e. an injection pattern) and buffer i for which $L^t(i) = n$.

Hint: For part (a) argue by induction on i that the number of packets in $[i]$ satisfies $L^t([i]) \leq i$.

Exercise 2. Consider the following centralized for algorithm: in each round t all non-empty buffers $i \geq a_t$ simultaneously forward (recall that a_t is the location of A's injection in round t). Prove that for all $i \in [n]$ and $t \in \mathbf{N}$ we have $L^t(i) \leq 2$.

Recall the Odd-Even Downhill (OED) forwarding protocol, defined as follows: For each $t \in \mathbf{N}$ and $i \in [n]$, buffer i forwards in round t if and only if one of the following conditions hold:

- $L^t(i) > L^t(i + 1)$
- $L^t(i) = L^t(i + 1)$ and $L^t(i)$ is odd.

Exercise 3. Prove that for any adversary A, OED forwarding maintains $L^t([i]) \leq i$ for all $i \in [n]$ and all rounds t. In particular, the total number of packets stored in all buffers of the network is at most n.

Exercise 4. Suppose each buffer in the network is modeled as a last-in, first out (LIFO) queue (or stack). The ***height*** of a packet in a buffer is one plus the number of packets below it in the buffer. (Thus, if a buffer i with load

$L^t(i)$ forwards a packet, the unique packet at height $h = L^t(i)$ is the one which is forwarded.) Suppose that in some round t a packet P is at height $h(t)$ in some buffer i. Prove that for all $s \geq t$ the height $h(s)$ of P in round s satisfies

$$h(s) \leq \begin{cases} h(t) & \text{if } h(t) \text{ is even} \\ h(t)+1 & \text{if } h(t) \text{ is odd.} \end{cases}$$

Hint: Note that for LIFO queues, the height of a packet P only changes when the packet is forwarded.