Homework 10: Separators and Cuts Part I

Algorithms on Directed Graphs, Winter 2018/9

Due: 19.01.2019 by 16:00

Materials of this lecture are from chapter 8 of the parametrized algorithm book. For definitions please take a look at the book.

Before going into exercises I have to fix one of my mistakes in the lecture: In the proof of Lemma 8.10 I said \( d(R_{\text{max}} \cup R) > d(R) \) and we arrive at a contradiction, but as one of you complained during the lecture this is wrong. The correct argument is that: if \( d(R_{\text{max}} \cup R) \leq d(R) \) with the fact that \( R_{\text{max}} \not\subseteq R \) we have that \( R \) is a proper subset of \( R_{\text{max}} \cup R \). Given that \( d(R_{\text{max}} \cup R) \) is small we get a contradiction with the assumption that \( R \) is an important cut.

**Exercise 1** (submodular functions). Prove the followings.

1. Prove the function \( d_G \) is submodular for every undirected graph \( G \).

2. Let define \( \Delta_G(R) \) for digraph \( G \) to be the set of edges so that their tail is in \( R \) and their head is in \( G \setminus R \). Define \( d_G(R) = |\Delta_G(R)| \). Is \( d_G \) a submodular function?

3. Solve the exercise 8.2 of the book.

**Exercise 2** \( (R_{\text{min}}, R_{\text{max}}) \). Solve the exercises 8.6 and 8.7.

**Exercise 3** (Enumerating important cuts). Complete missing proofs from the lecture.

1. We explained half of the proof of theorem 8.11. Provide a full proof of it. Of course you can read it first, however you have to write your own understanding.

2. The proof of 8.11 was a constructive proof. Turn it to an algorithm with running time \( O(4^k P(n)) \) for some polynomial function \( P \).