Homework 12

Algorithms on Directed Graphs, Winter 2018/9

Due: 1.2.2018 by 16:00

Definition. Let $G = (V, E)$ be a digraph. We say that G is ***Eulerian*** if G is (weakly) connected, and for all $v \in V$ we have $|\delta^+(v)| = |\delta^-(v)|$, where $\delta^+(v)$ and $\delta^-(v)$ denote the sets of out- and in-edges of v, respectively. An ***Eulerian circuit*** is an edge-simple cycle C that contains every edge $e \in E$ (exactly once).

Exercise 1. Prove that if G is Eulerian, then it contains an Eulerian circuit.

Exercise 2. Suppose $G = (V, E)$ is Eulerian, and let $A, B \subseteq V$ be disjoint. Let $\Gamma = (V, E')$ be the undirected graph formed by replacing each directed edge $(u, v) \in E$ with the undirected edge $\{u, v\}$ in E'.

(a) Show that if (A, B) is a partition of V (i.e, $V = A \cup B$), then we have

$$|\{(u, v) \in E \mid u \in A, v \in B\}| = |\{(u, v) \in E \mid u \in B, v \in A\}| .$$

That is, the number of edges from A to B is equal to the number of edges from B to A.

(b) Suppose that for all $v \in V$, the in-degree of v satisfies $|\delta^-(v)| \leq \Delta$. Suppose that in Γ, there are $(\Delta + 1)k + 1$ vertex disjoint (undirected) paths from A to B. Prove that in G, there are $k + 1$ directed vertex disjoint paths from A to B. (Do not assume here that A and B form a partition of V—they can be arbitrary disjoint sets.)

Definition. Let $G = (V, E)$ be a digraph, and let $A, B \subseteq V$.

- We say (A, B) is a ***separation*** of order $k = |A \cap B|$ if there is no (directed) edge $(u, v) \in E$ with $u \in A \setminus B$ and $v \in B \setminus A$.
- A subset $X \subset V$ is ***node well-linked*** (NWL) if for all $A, B \subseteq X$ with $|A| = |B|$ there are $k = |A| = |B|$ vertex disjoint paths from A to B in G.
- A subset $X \subset V$ is ***edge well-linked*** (EWL) if for all $A, B \subseteq X$ with $|A| = |B|$ there are $k = |A| = |B|$ edge disjoint paths from A to B in G.

- For $\alpha \in [0,1]$, a subset $X \subseteq V$ is α-NWL if every separation (A,B) has order at least $\alpha \min\{|X \cap A|, |X \cap B|\}$.
- For $\alpha \in [0,1]$, a subset $X \subseteq V$ is α-EWL if for every partition $V = A \cup B$ we have $|\delta^+(A)| \geq \alpha \min\{|X \cap A|, |X \cap B|\}$.

Exercise 3. Let $G = (V,E)$ be a digraph and $X \subseteq V$.

1. Show that if X is 1-NWL then X is NWL.
2. Show that if X is 1-EWL then X is EWL.

Definition. Let $G = (V,E)$ be a digraph and $S \subset V$. The ***edge expansion*** of S is defined to be

$$\alpha_S(G) = \frac{|\delta^+(S)|}{\min\{|S|, |V \setminus S|\}}.$$

The ***edge expansion*** of G is

$$\alpha(G) = \min_{S \subset V} \alpha_S(G).$$

Exercise 4. Let $G = (V,E)$ be a digraph.

(a) Prove that if G has edge expansion $\alpha = \alpha(G)$ then V is α-EWL.

(b) Prove that if G is α-EWL and has maximum in-degree Δ, then G is (α/Δ)-NWL.