Time Complexity of Link Reversal Routing

Talk is based on ...

work with Bernadette Charron-Bost (CNRS, LIX),
Jennifer L. Welch (Texas A&M University), and
Josef Widder (TU Wien)

(Sirocco, '11), (ACM Trans. on Algorithms, '15).

Algorithms on Graphs

Graph rewriting algorithms:
$G_0, G_1, G_2, \ldots$

Link reversal algorithms: on link directions

Applications:

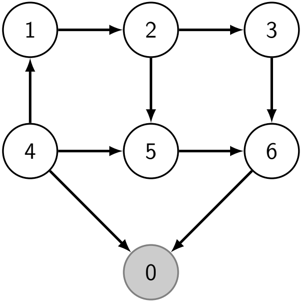
- Routing algorithms
- Behavior of asynchronous circuits
- Behavior of Physarum (?)

The Routing Problem

Requirements:

- Computationally efficient algorithm.
- Adapt to failures/mobility fast.

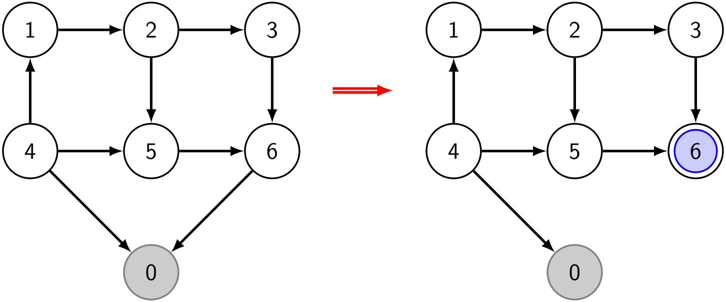
Routing to a destination



acyclic, destination oriented

Acyclic and
destination oriented ⇒
Routing is simple.

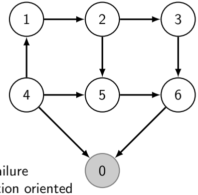
Routing to a destination



acyclic, destination oriented

acyclic

Full Reversal routing (Gafni and Bertsekas, 1981)



before failure
acyclic, destination oriented

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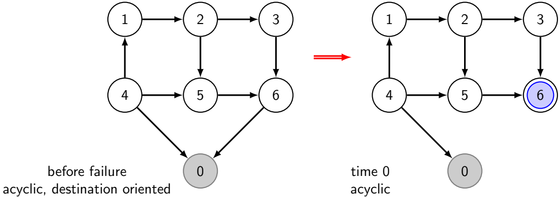

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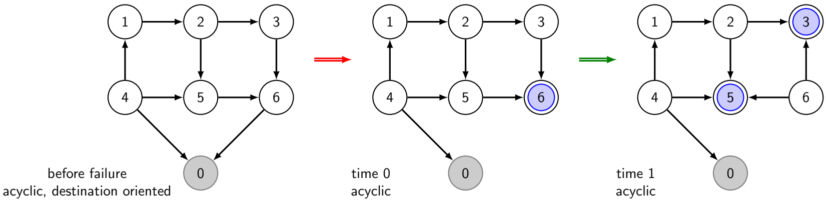

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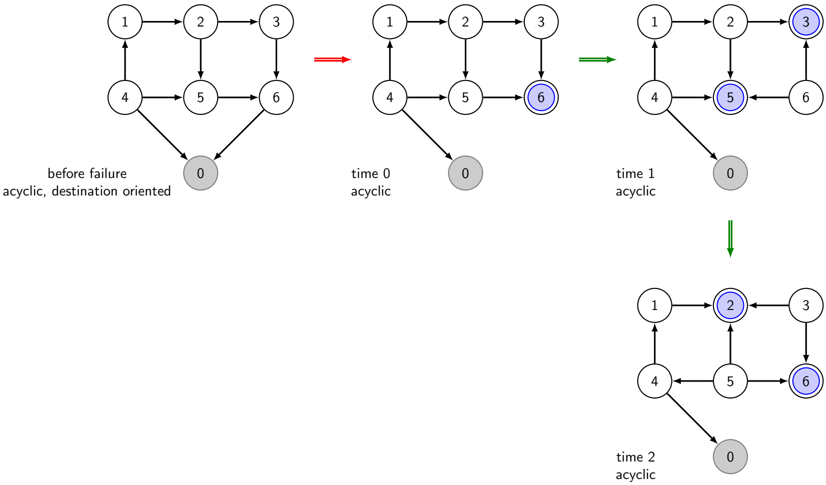

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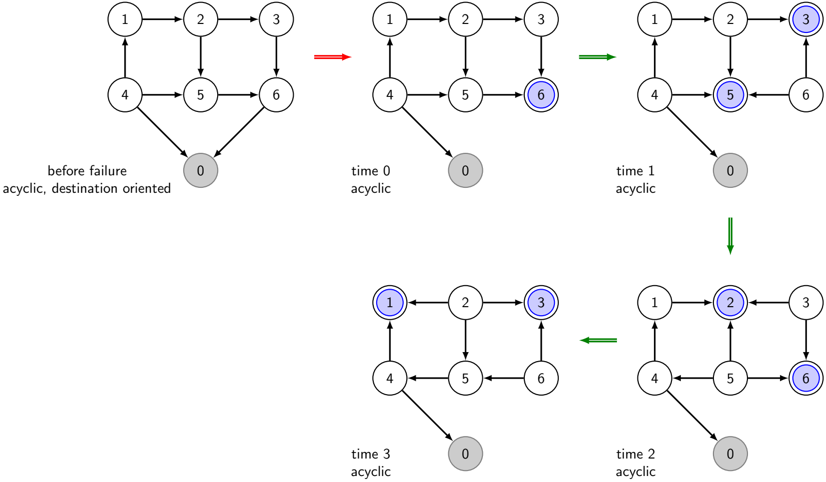

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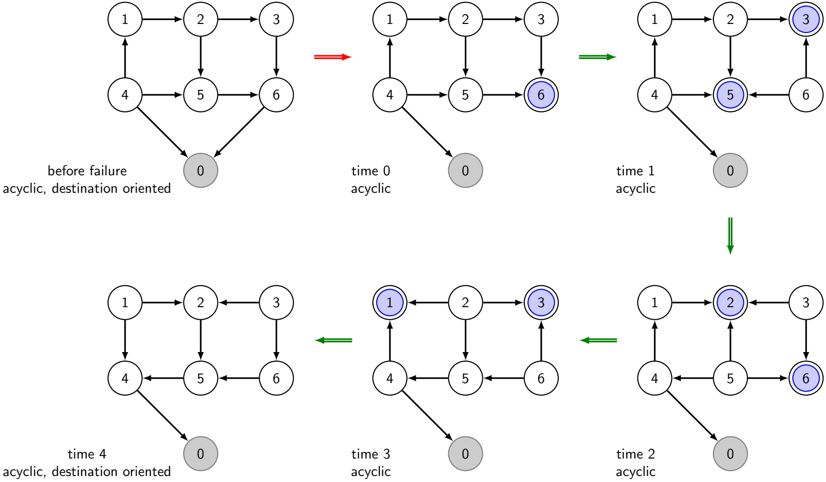
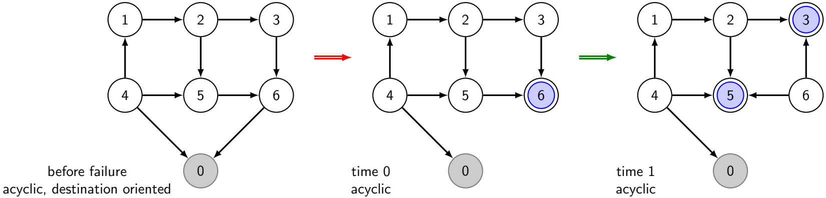


Full Reversal routing - alternative execution



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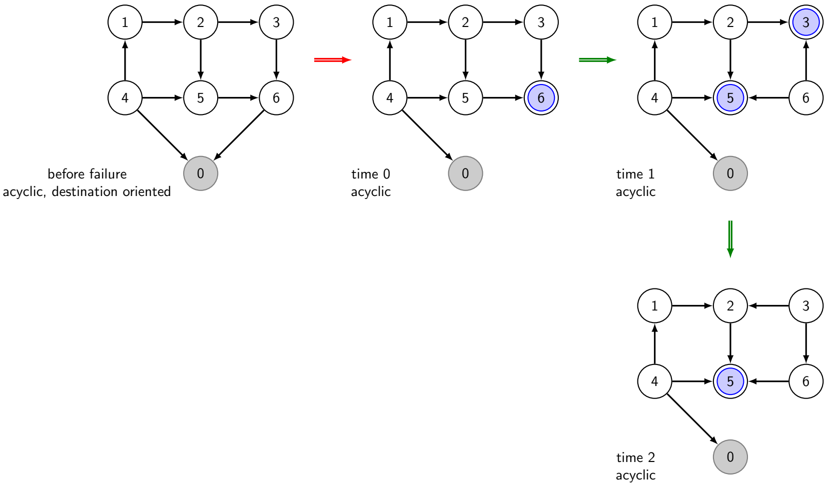

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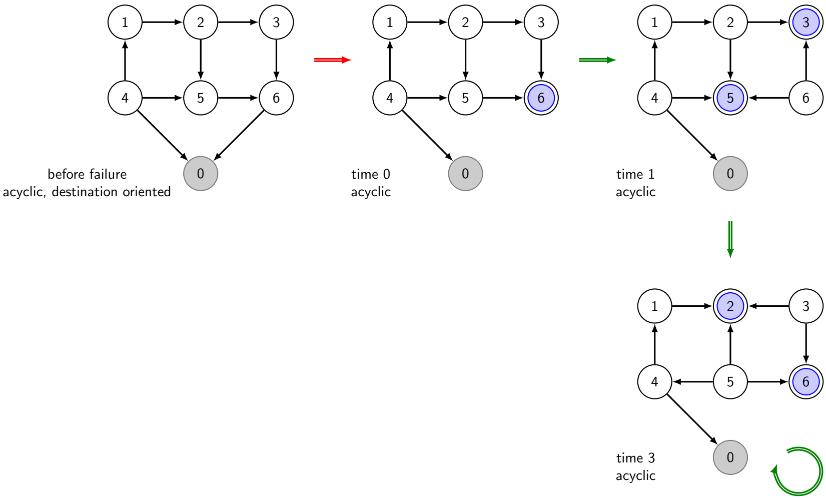

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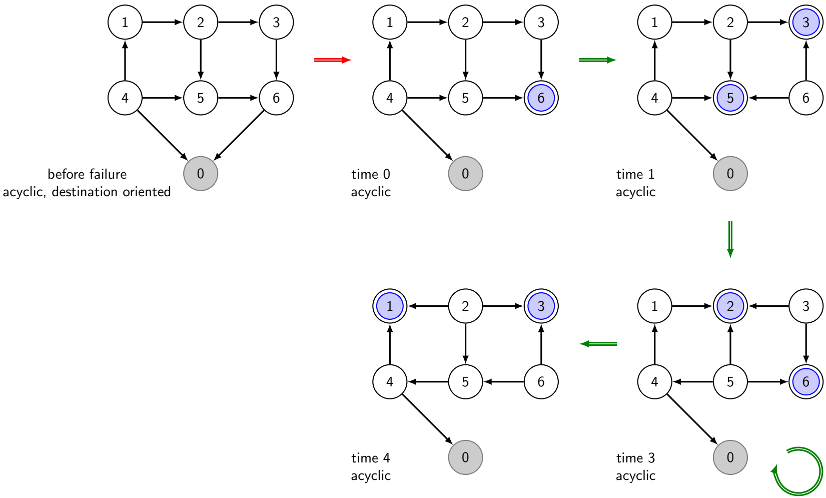

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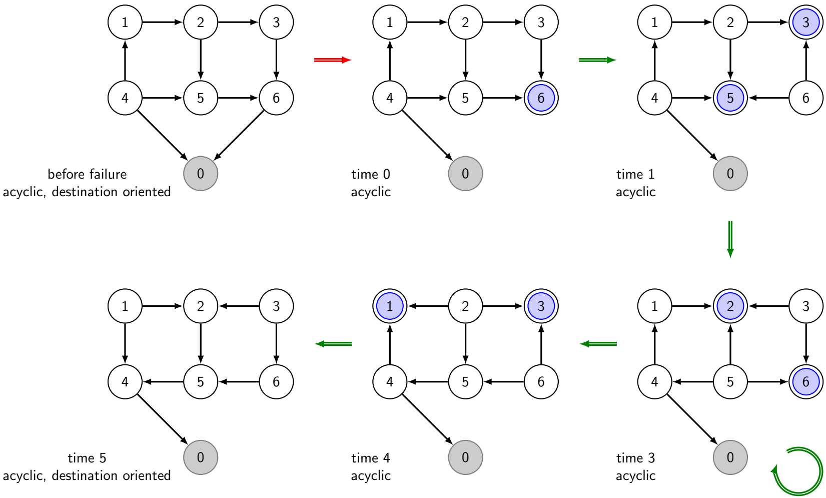

Executions

Execution: $G_0, G_1, G_2, \ldots$

Two extremes:

- Greedy execution: all sinks in G_i make step
- Lazy execution(s): one sink in G_i makes a step

... and many executions in between.

Distributed Algorithm

Local graph rewriting: $\ldots, G_i, G_{i+1}, \ldots$

Local mutex: no two sinks are neighbors

Asynchronous execution possible

Some Facts

Algorithm always terminates destination oriented.

At all steps, graph is acyclic.

Algorithm steps are commutative:

- Final graph is the same.
- Each node performs same number of steps.

Complexity

For an initial graph:

work complexity of node i

- number of steps made by node i in any execution
- first exact expression established in (Busch et al., 2003)
- work complexity for more general algorithms in (Charron-Bost et al., 2009)

time complexity of node i

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Due to concurrency, understanding of work complexity is not sufficient to get exact time complexity.

A dynamical system

System state $\vec{W}(t) = \left(\vec{W}_0(t), \vec{W}_1(t), \ldots, \vec{W}_N(t)\right)$ at time t.

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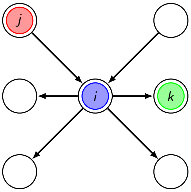
$\vec{W}_i(0) = 0$ for all nodes i

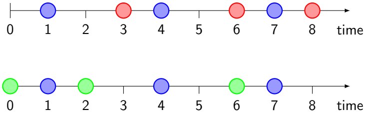
$\vec{W}_0(t) = 0$ for all times t

Looking for a function F such that

$$\vec{W}(t) = F\left(\vec{W}(t-1)\right)$$

The influence of links





Proposition

Between two consecutive steps by a node i, each neighbor of i takes exactly one step.

Proposition

In any FR execution in which i takes a step, before the first step by i, each node $j \in In_i$ takes no step and each node $k \in Out_i$ takes exactly one step.

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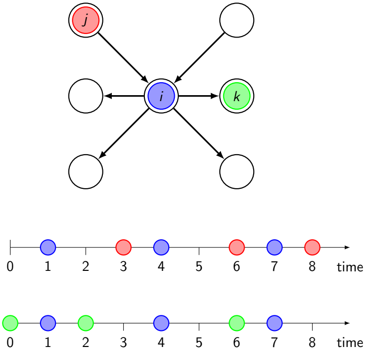

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In any FR execution in which i takes a step, before the first step by i, each node $j \in In_i$ takes no step and each node $k \in Out_i$ takes exactly one step.

$\Rightarrow$ Links induce strict alternation.

Recurrence

Strict alternation $\Rightarrow$

$\vec{W}_i(t) \leq \vec{W}_j(t-1) + 1$ for $j \in In_i$
$\vec{W}_i(t) \leq \vec{W}_k(t-1) + 0$ for $k \in Out_i$

Theorem

In a greedy FR execution, for any node i other than 0 and any $t \geq 1$,

$$\vec{W}_i(t) \leq \min\left\{\vec{W}_j(t-1) + 1,\ \vec{W}_k(t-1) + 0 \colon j \in In_i,\ k \in Out_i\right\}.$$

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$$\Big\downarrow \text{ min-plus algebra}$$

$$\vec{W}_i(t) = \textstyle\sum_{j\in In_i} \vec{W}_j(t-1)\otimes 1 \ + \ \sum_{k\in Out_i} \vec{W}_k(t-1)\otimes 0.$$

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$\Rightarrow$ discrete *linear* dynamical system in min-plus algebra

Reduction to graph properties

- computing: $\vec{W}(t) = A^t \otimes \vec{0} = (A^{t/2})^2 \otimes \vec{0}$

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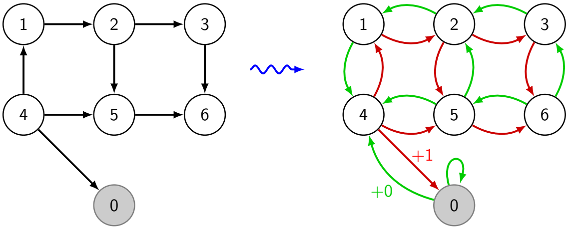

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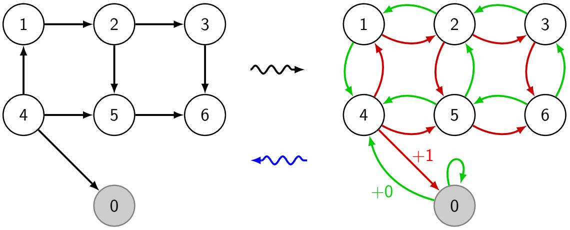
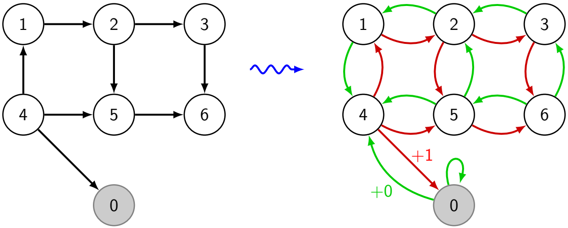


- $W_i(t) = \min\{weight(p) : \; p \text{ is path to } i \text{ of length } t\}$
- $W_i(t) = \min\{r(c) : \; c \text{ is chain to } i \text{ of length } t\}$

Simple work complexity proof

$$w_i = \lim_{t\to\infty} W_i(t) = \lim_{t\to\infty} \min\left\{weight(p) : \ p \text{ is path to } i \text{ of length } t\right\}$$



- A path long enough to reach 0, will loop there for free.
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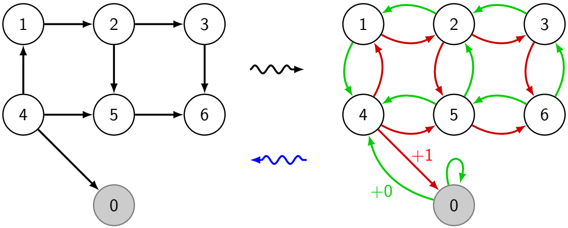

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Can we say something about time complexity?

Dual of $W_i(t)$ is $T_i(w)$

- $W_i(T_i(w)) = w$ $\Rightarrow$ $W_i(\theta_i) = w_i$
- $W_i(T_i(w) - 1) = w - 1$ $\Rightarrow$ $W_i(\theta_i - 1) = w_i - 1$

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$\Downarrow$

Theorem

The termination time θ_i of any node i that takes a step is equal to

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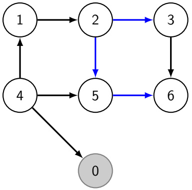
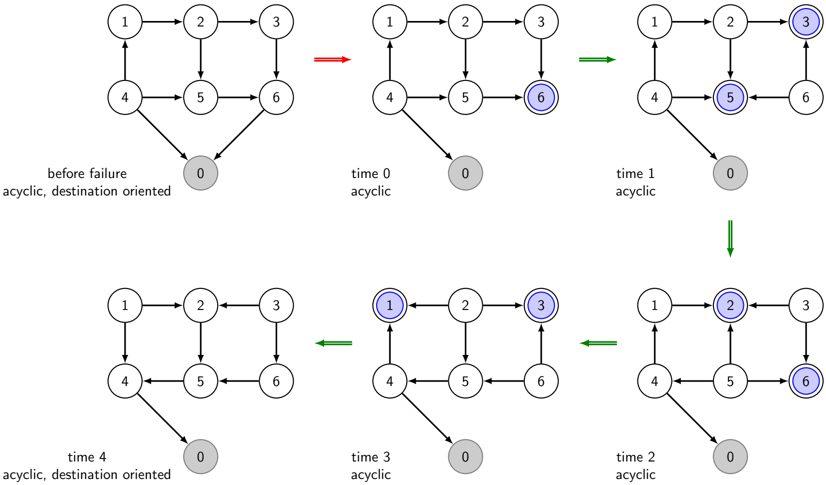


$w_3 = 2$

$\theta_3 = 4$

Full Reversal routing (Gafni and Bertsekas, 1981)



Immediate results

Corollary

There is an N node graph family with FR time complexities scaled from $\Theta(N)$ to $\Theta(N^2)$

Corollary

In any tree with $N+1$ nodes, the FR time complexity is at most equal to $2N-1$.

Corollary

In general FR time complexity is unstable. Adding one link can increase it from $\Theta(N)$ to $\Theta(N^2)$.

FR only for routing?

Distributed algorithm where no two sinks are neighbors.

$\Rightarrow$ Distributed scheduling with local mutex.

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$\Rightarrow$ What if we remove the special node 0?

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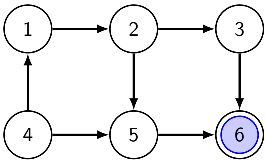
acyclic



Will run forever.

Distributed scheduling

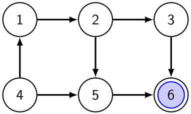
Scheduling frequency $\lim_{t\to\infty} W_i(t)/t$

Distributed scheduling

Again a dynamical system $\vec{W}(t)$

$\vec{W}_i(t) = \min\{r(c) : \ c \text{ is chain to } i \text{ of length } t\}$

acyclic



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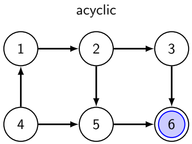
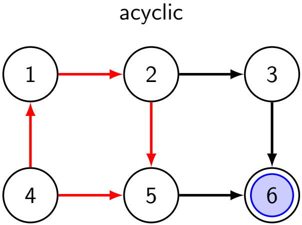


$\Rightarrow$ eventually c will find a cycle γ with minimal $r(\gamma)/\ell(\gamma)$.

Distributed scheduling

Theorem

$\lim_{t\to\infty} W_i(t)/t = \min\{r(\gamma)/\ell(\gamma) : \gamma \textit{ is cycle}\}$



$\lim_{t\to\infty} W_i(t)/t = 1/4$

Beyond Full Reversal

LR $\widehat{=}$ Reverse only some links.

Proof idea:

- Not necessarily linear in N-dimensional system
- But: simulate nodes with one or two nodes (transformed graph).
- Relate chains in transformed graph to chain in original graph.

$\Rightarrow$ analogous results with other potentials Π.

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