Exercise 7: Lost in Complexity

Task 1: Why is everything so Hard?!?

In this exercise, we always consider connected, simple, weighted graphs $G = (V, E, W)$, restrict message size to $\mathcal{O}(\log n)$ bits, and assess worst-case round complexity as a function of the (hop) diameter D and n.

a) Show that finding any approximation to the (weighted) distance between a given pair of nodes $s, t \in V$ takes $\Omega(\sqrt{n}/\log^2 n + D)$ rounds.

   Hint: Use the same technique and graph as in the lecture, just change the weights.

b) Show that finding a Steiner tree requires $\Omega(\sqrt{n}/\log^2 n + D)$ rounds, regardless of the size of the subset T of nodes that needs to be connected to each other (unless it is 1).

   Hint: Attach some irrelevant nodes to the construction from a) for $|T| \leq n/2$ and to the MST construction for $|T| > n/2$.

c) For $s, t \in V$, an s-t cut is a subset $s \in S \subseteq V \setminus \{t\}$. The weight of the cut is the sum of weights of all edges $\{v, w\} \in E \cap (S \times (V \setminus S))$ crossing the cut. Show that finding any approximation to the weight of a minimum s-t cut takes $\Omega(\sqrt{n}/\log^2 n + D)$ rounds.

d) Conclude that finding an approximate maximum flow or even approximating the value of such a flow requires $\Omega(\sqrt{n}/\log^2 n + D)$ rounds.

Task 2: Harder, Better, Slower

Consider graphs $G = (V, E)$ and message size $\mathcal{O}(\log n)$. In this exercise, we show that determining the diameter D of a graph more accurately than factor $3/2$ requires $\Omega(n/\log n)$ rounds or large messages.

a) For a set disjointness instance (x, y) and input size N, construct a graph with $\mathcal{O}(\sqrt{N})$ nodes that has diameter 2 if x and y encode disjoint sets, and diameter 3 otherwise. The graph must have a cut with $\mathcal{O}(\sqrt{N})$ edges between the parts encoding x and y, respectively.

   Hint: Start from $2k$ nodes $l_1, \ldots, l_k, r_1, \ldots, r_k$ where the edge $\{l_i, r_j\}$ is included if and only if $i = j$. Enumerate the bits of x in some convenient way as $x_{i,j}$ so that i and j stay in $\mathcal{O}(\sqrt{N})$; likewise for y. Then add edges so that for all $1 \leq i, j \leq k$ there is some path of length 2 from l_i to r_j (you may assume $i \neq j$) if $x_{i,j} = 0$ or $y_{i,j} = 0$, but not if $x_{i,j} = y_{i,j} = 1$.
   Finally, add two nodes l and r and some edges to make sure that the diameter is 2 if x and y encode disjoint sets and 3 otherwise.

b) Assume for a moment a distributed algorithm exists that uses B-bit messages to compute (or approximate) the diameter of such a graph in T rounds. Show that Alice and Bob can simulate each execution with a total communication complexity of $\mathcal{O}(\sqrt{N}BT)$.

   Hint: Don't think "It can't be that easy!".

c) Conclude that $T \in \Omega(\sqrt{N}/B)$ in the worst case, no matter what algorithm is used. Specifically, conclude that if $B \in \mathcal{O}(\log n)$, it requires $\Omega(n/\log n)$ rounds to determine the diameter of a graph more accurately than up to factor of $3/2$.

Task 3*: Be more Constructive!

a) Check up on the prime number theorem!

b) Show that for any $k \in \mathbb{N}$ and any constant $C \in \mathbb{N}$, the number of primes in the range $[2^k, 2^{k+C}]$ is in $2^{\Theta(k+C)}/k$.

c) Prove that for an N-bit number, the number of different $\Theta(\log N)$-bit primes that divides it is bounded by $\Theta(N/\log N)$. Use this to find suitable choices of k and C such that the number of primes in the range $[2^k, 2^{k+C}]$ is polynomial in N and the probability that, for a fixed N-bit number, a uniformly random prime from this range divides it is at most $N^{-\Theta(1)}$.

d) Check up on the AKS primality test!

e) Infer that there is a protocol solving equality with error probability $N^{-\Theta(1)}$ that uses private randomness, communicates $\mathcal{O}(\log N)$ bits, and requires only polynomial computations, both for construction and execution!

f) Check up on your ability to explain this to others in the exercise session!