Exercise 2: Flirting with Synchrony and Asynchrony

Task 1: Growing Balls

In this exercise, we will see how a crucial step of the $\gamma$-synchronizer works; specifically, that a desirable partition of the nodes exists.

Denote by $B(v, r)$ the ball of radius $r$ around $v$, i.e., $B(v, r) = \{u \in V : \text{dist}(u, v) \leq r\}$. Consider the following partitioning algorithm. Note the difference between intercluster edges and intrachuster edges.

**Algorithm 1** Cluster construction. $\rho \geq 2$ is a given parameter.

1. while there are unprocessed nodes do
2. select an arbitrary unprocessed node $v$;
3. $r := 0$;
4. while $|B(v, r + 1)| > \rho|B(v, r)|$ do
5. $r := r + 1$
6. end while
7. makeCluster($B(v, r)$) // all nodes in $B(v, r)$ are now processed
8. remove all cluster nodes from the current graph
9. end while
10. select intercluster edges

a) Show that Algorithm 1 constructs clusters of radius at most $\log_\rho n$.

b) Show that Algorithm 1 produces at most $\rho n$ intercluster edges.

c) For given cluster radius $k \in \{1, \ldots, \lfloor \log n \rfloor\}$, determine an appropriate choice $\rho(k) \geq 2$, proving the precondition of Corollary 2.14!

**Hint:** As a short-hand, we often don’t write out common terms like $n$ that are assumed to be globally known. Specifically, $\rho(k)$ may also depend on $n$, as if we had written $\rho(k, n)$. If in doubt, then we weren’t clear enough, so tell us!

Task 2: Showing Dijkstra, and Bellman & Ford the Ropes

a) Show that if the asynchronous Bellman-Ford algorithm from the lecture is executed synchronously, it sends only $O(|E|)$ messages.

b) Use this to construct an asynchronous BFS tree construction algorithm of time complexity $O(D)$ that uses $O(|E|D)$ messages and terminates. You may assume that $D$ is known here.

c) Can you give an asynchronous Bellman-Ford-based algorithm that sends $O(|E| + nD)$ messages and runs for $O(D^2)$ rounds?

**Hint:** Either answer is feasible, provided it is backed up by appropriate reasoning!
Task 3*: Liaison with Leslie Lamport

a) Look up what the happened-before relation, Lamport clocks, and vector clocks are.

b) Contemplate their relation to synchronizers and what you’ve learned in the lecture.

c) Discuss your findings in the exercise session!