Prophet Inequalities
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Collection of feasible subsets $F \subseteq 2^E = \{S : S \subseteq E\}$.

Elements arrive one by one in unknown order $\sigma = (\sigma(1), \ldots, \sigma(m))$.

Online selection problem with initial $S = \emptyset$

For $i = 1, \ldots, m$, upon arrival of element $\sigma(i)$:

- Weight $w(\sigma(i))$ is revealed.
- Decide (irrevocably) whether to select or reject $\sigma(i)$, where selecting is only allowed if $S + \sigma(i) \in F$.

Goal: Select subset $S \in F$ maximizing $w(S) = \sum_{e \in S} w(e)$. 

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*With adversarial arrival order*
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**Online procedure for set system** $F = (E, I)$:

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**Example**

Suppose we have a fair die with six sides. Then $g(i) = \frac{1}{6}$ for $i = 1, \ldots, 6$ and $g(i) = 0$ otherwise.
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It then holds that $\mathbb{P}(X \leq z) = \int_{0}^{z} f(x) dx$. 
\[ \mathbb{P}(X \leq z) = \int_{0}^{z} f(x) \, dx \]

The function \( f(x) \) models how the probability mass is spread out.

Example

Consider the uniform distribution over the interval \([a, b]\) with \(0 \leq a < b\).

Then \( f(x) = \frac{1}{b - a} \).

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All the results we discuss today hold for both continuous and discrete distributions, but sometimes need slightly different arguments.
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About the adversary

In general, we assume to have an all-knowing, adaptive adversary

Can choose which element to present in step $i$, based on

Choices of online algorithm in steps 1, $\ldots$, $i - 1$.

Realizations of all elements (including those that have not arrived).

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Example

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Distributions are given by:

- $w_1 \sim X_1 = \begin{cases} 1 & \text{with probability } \epsilon \\ 0 & \text{with probability } 1 - \epsilon \end{cases}$
- $w_2 \sim X_2 = 1 + \delta$ with probability 1

Note that $\mathbb{E}[X_1] = 1$ and $\mathbb{E}[X_2] = 1 + \delta$.

If arrival order would be $(e_1, e_2)$, simply observe realization $w_1$.

- If $w_1 = \frac{1}{\epsilon}$, then select $e_1$ (as $\frac{1}{\epsilon} > 1 + \delta$).
- If $w_1 = 0$, reject $e_1$ and select $e_2$.

Worst-case arrival order is $(e_2, e_1)$.

We don't know realization $w_1$, when deciding on element $e_2$. Nevertheless, it is (intuitively) optimal to select $e_2$.

Why?

$w_2 = 1 + \delta > \mathbb{E}[X_1]$.

In expectation (of $X_1$), we cannot do better if we reject $e_2$.

Performance objective is formalized next.
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  w_1 \sim X_1 &= \begin{cases} 
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(1)

(2)

Note that \( \mathbb{E}[X_1] = \frac{1}{\epsilon} \times \epsilon + 0 \times (1 - \epsilon) = 1 \) and \( \mathbb{E}[X_2] = 1 + \delta \).

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Example
Let $E = \{ e_1, e_2 \}$ of which we may select at most one element. Let $1 > \epsilon, \delta > 0$, and assume that $\frac{1}{\epsilon} > 1 + \delta$. Distributions are given by:

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"Deterministic value $w_2 = 1 + \delta > \mathbb{E}[X_1]$. In expectation (of $X_1$), we cannot do better if we reject $e_2$."
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- Performance objective is formalized next.
Performance of online algorithm

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- Prophet gets to see all realizations \( w_i \sim X_i \) after they are sampled.
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$$\text{OPT}(w_1, \ldots, w_m) := w(S^*) = \max_{S \in \mathcal{F}} \sum_{e \in S} w_e.$$
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Expected weight of (deterministic) algorithm $A$ is

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\text{ALG} = E(y_1, \ldots, y_m) \sim X_1 \times \cdots \times X_m\left[\min_{\sigma} \sigma w(A(\sigma, y_1, \ldots, y_m))\right]
\]

With $w(A(\sigma, y_1, \ldots, y_m))$ weight of set outputted by $A$.

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\]

- With \( w(\mathcal{A}(\sigma, y_1, \ldots, y_m)) \) weight of set outputted by \( \mathcal{A} \).
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Example (cont’d)

\[ E = \{ e_1, e_2 \} \text{ with following distributions.} \]
Example (cont’d)

$E = \{ e_1, e_2 \}$ with following distributions. Let $1 > \epsilon, \delta > 0$, and assume that $\frac{1}{\epsilon} > 1 + \delta$. 

Let $w_1 \sim X_1 = \{ 1 \epsilon \} \text{ with probability } \epsilon$ and $0 \text{ with probability } 1 - \epsilon$ \hfill (3)

Let $w_2 \sim X_2 = \{ 1 + \delta \} \text{ with probability } 1$ \hfill (4)

What can prophet get?

$\text{OPT}(w_1, w_2) = \max\{ w_1, w_2 \} = \{ 1 \epsilon \} \text{ with probability } \epsilon$ and $1 + \delta \text{ with probability } 1 - \epsilon$.

Then $E_{\text{OPT}}(y_1, y_2)[\max_{i} y_i] = \epsilon \epsilon + (1 + \delta)(1 - \epsilon) \rightarrow 2$ as $\epsilon, \delta \rightarrow 0$.

Optimal algorithm $A$ is to select $e_2$ (again, think about it).

Worst-case order is $(e_2, e_1)$ with $E_{\text{OPT}}(y_1, y_2)[w(A(\sigma, y_1, y_2))] = 1$.

I.e., optimal algorithm only half as bad as prophet ($\alpha = \frac{1}{2}$).

Also shows that, in general, we cannot hope for $A$ with $\alpha > \frac{1}{2}$, already in setting where we can select at most one (out of two) elements.
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(4)
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Then $\mathbb{E}_{\text{OPT}(y_1, y_2)}[\max_i y_i] = \frac{1}{\epsilon} \times \epsilon + (1 + \delta) \times (1 - \epsilon) \rightarrow 2$ as $\epsilon, \delta \rightarrow 0$. 

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Then \( \mathbb{E}_{\text{OPT}(y_1, y_2)}[\max_i y_i] = \frac{1}{\epsilon} \times \epsilon + (1 + \delta) \times (1 - \epsilon) \to 2 \) as \( \epsilon, \delta \to 0 \).

Optimal algorithm \( A \) is to select \( e_2 \) (again, think about it).

- Worst-case order is \( (e_2, e_1) \) with \( \mathbb{E}_{(y_1, y_2)}[w(A(\sigma, y_1, y_2))] = 1 \).
Example (cont’d)

$E = \{e_1, e_2\}$ with following distributions. Let $1 > \epsilon, \delta > 0$, and assume that $\frac{1}{\epsilon} > 1 + \delta$. Let

\[
w_1 \sim X_1 = \begin{cases} \frac{1}{\epsilon} & \text{with probability } \epsilon \\ 0 & \text{with probability } 1 - \epsilon \end{cases}
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w_2 \sim X_2 = \begin{cases} 1 + \delta & \text{with probability } 1 \end{cases}
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What can prophet get?

$$\text{OPT}(w_1, w_2) = \max\{w_1, w_2\} = \begin{cases} \frac{1}{\epsilon} & \text{with probability } \epsilon \\ 1 + \delta & \text{with probability } 1 - \epsilon \end{cases}. \quad (3)$$

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Optimal algorithm $A$ is to select $e_2$ (again, think about it).

- Worst-case order is $(e_2, e_1)$ with $\mathbb{E}_{(y_1, y_2)}[w(A(\sigma, y_1, y_2))] = 1$.

- I.e., optimal algorithm only half as bad as prophet ($\alpha = \frac{1}{2}$).
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$E = \{ e_1, e_2 \}$ with following distributions. Let $1 > \epsilon, \delta > 0$, and assume that $\frac{1}{\epsilon} > 1 + \delta$. Let

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- I.e., optimal algorithm only half as bad as prophet ($\alpha = \frac{1}{2}$).

- Also shows that, in general, we cannot hope for $A$ with $\alpha > \frac{1}{2}$, already in setting where we can select at most one (out of two) elements.
Selecting single element

Prophet Inequality with $\alpha = \frac{1}{2}$
Krengel, Sucheston and Garling (1978) show there is a prophet inequality with $\alpha = \frac{1}{2}$.
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- Simple algorithm was given by Samuel-Cahn (1984):

1. Set threshold $T$ to be the median of distribution $X_{\text{max}} = \max_i X_i$.
2. Select first element $e_i$ whose realized $w_i \sim X_i$ exceeds threshold.
3. Median of distribution $X$ is value $m$ such that $P(X < m) \leq \frac{1}{2}$ and $P(X > m) \leq \frac{1}{2}$.

For continuous distributions, the median is the "middle value" of the distribution.

Example: Suppose we have uniform distribution over (continuous) interval $[a, b]$.

Then $m = \frac{a + b}{2}$. 
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Example

Suppose we have uniform distribution over (continuous) interval $[a, b]$. Then $m = \frac{a+b}{2}$. 
As an alternative to Samuel-Cahn’s median-based threshold, Kleinberg and Weinberg (2012) gave another threshold-based algorithm.

Theorem (Kleinberg and Weinberg, 2012)
The KW-algorithm selects an element $e^*$ with the property that
$$\mathbb{E}[X_1, \ldots, X_m \mid w(e^*)] \geq \frac{1}{2} \cdot \mathbb{E}[\max_j X_j].$$
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- Extends to case where multiple elements may be selected under matroid constraint.
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KW-algorithm for (unknown) arrival order $\sigma$

Let $X_i$ be the distribution from which element $e_i$'s weight is drawn.

Set threshold $T = \mathbb{E}[\max_j X_j]^2$.

For $i = 1, \ldots, m$: If $w(\sigma(i)) \geq T$, select $\sigma(i)$ and STOP.

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**Proof:** Let \( \tau \in \{1, \ldots, m\} \) be (random) step in which element is select, and let \( X_\tau \) be the (random) weight of the selected element,
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**Proof:** Let \( \tau \in \{1, \ldots, m\} \) be (random) step in which element is select, and let \( X_\tau \) be the (random) weight of the selected element, i.e., it holds that

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- Assume w.l.o.g. that \( \sigma = (e_1,\ldots,e_m) \).

It holds that

\[ \mathbb{E}[X_\tau] = \int_0^T \mathbb{P}[X_\tau > x]dx + \int_T^{\infty} \mathbb{P}[X_\tau > x]dx \]

when all distributions \( X_i \) are continuous.
\[ \mathbb{E}_{X_1,\ldots,X_m}[w(e^*)] \geq \frac{\mathbb{E}[\max_j X_j]}{2} \quad (=T) \]

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- See [background material](#) for discrete version of this claim:
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\[
\mathbb{E}[X] = \sum_{k=0}^\infty \mathbb{P}[X \geq k].
\]
$$E[X_\tau] = \int_0^T P[X_\tau > x]dx + \int_T^\infty P[X_\tau > x]dx, \quad T = \frac{\mathbb{E}[\max_j X_j]}{2}$$
\[ E[X_\tau] = \int_0^T P[X_\tau > x]dx + \int_T^\infty P[X_\tau > x]dx, \quad T = \frac{\mathbb{E}[^{\max_j}X_j]}{2} \]

Let \( \rho = P[^{\max_j}X_j \geq T] \).
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Let \( p = \mathbb{P}[\max_j X_j \geq T] \).

- \( 1 - p \) is probability that we do not select anything.
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- $1 - p$ is probability that we do not select anything.
- For any $i = 1, \ldots, m$, probability that we have not selected an element in step $i$ is then at least $1 - p$. 

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It is not hard to see that

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\int_0^T \mathbb{P}[X_\tau > x]dx \geq \int_0^T \mathbb{P}[X_\tau > T]dx \geq \int_0^T p \cdot dx = pT. \tag{5}
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Furthermore, for \( x \geq T \) it holds that
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\geq (1 - p) \mathbb{P}[\max_j X_j > x] \quad \text{(union bound)}
\]
\[ \mathbb{E}[X_\tau] \geq pT + (1 - p) \int_T^\infty \mathbb{P}[\max_j X_j > x] \, dx, \quad T = \frac{\mathbb{E}[\max_j X_j]}{2} \]
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Note that

\[ \mathbb{E}[\max_j X_j] = \int_0^T \mathbb{P}[\max_j X_j > x]dx + \int_T^\infty \mathbb{P}[\max_j X_j > x]dx = 2T \]

by definition of \( T \).
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Plugging this into the main inequality above gives

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This completes the proof.
Some remarks

Theorem (Kleinberg and Weinberg, 2012)

The KW-algorithm selects an element $e^*$ with the property that

$$\mathbb{E}_{X_1, \ldots, X_m}[w(e^*)] \geq 1/2 \cdot \mathbb{E}[\max_j X_j].$$
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- Algorithm is optimal trade-off between weight of selected elements and probability of selecting an element.
  - Higher threshold would give better weight of selected element, but prob. that we can select one gets smaller.
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Some remarks

Theorem (Kleinberg and Weinberg, 2012)

*The KW-algorithm selects an element $e^*$ with the property that*

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Matroid prophet inequality
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Selecting indep. set from matroid $\mathcal{M} = (E, \mathcal{I})$ with arrival order $\sigma$.

Set $S = \emptyset$.

- For $i = 1, \ldots, m$, a realization $w_i \sim X_i$ is generated.
  - All realizations $w_i$ are shown to the adversary.
- For $i = 1, \ldots, m$:
  - Adversary chooses $\sigma(i) \in E$, and reveals it and its weight $w_i$.
  - Online algorithm $A$ decides whether to accept or reject $\sigma(i)$, where acceptance is only allowed if $S + \sigma(i) \in \mathcal{I}$.

Theorem (Kleinberg-Weinberg, 2012)
There is an online algorithm $A$ for selecting multiple elements subject to a matroid constraint (under adversarial arrival order), with
$$\text{ALG}(A) \geq \frac{1}{2} \cdot \text{OPT},$$
where $\text{OPT} = \mathbb{E}(y_1, \ldots, y_m) \sim X_1 \times \cdots \times X_m$ is offline optimum.
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Algorithm sets threshold in step $i$ based on **marginal contribution** of $\sigma(i)$. 

---

**Example (Graphic matroid)**

![Graphic matroid diagram]
KW-algorithm for matroid constraint

Algorithm sets threshold in step $i$ based on marginal contribution of $\sigma(i)$.

- Let $y' = (y'_1, \ldots, y'_m) \geq 0$ be given weights, and let $B'$ be a max. weight base under $y'$. 

Example (Graphic matroid)

\begin{figure}[h]
    \centering
    \begin{tikzpicture}
        \node (a) at (0,0) {$a$};
        \node (b) at (1,0) {$b$};
        \node (c) at (2,0) {$c$};
        \node (d) at (3,0) {$d$};
        \node (e) at (4,0) {$e$};
        \node (f) at (5,0) {$f$};
        \draw (a) -- (b);
        \draw (b) -- (c);
        \draw (c) -- (d);
        \draw (d) -- (e);
        \draw (e) -- (f);
    \end{tikzpicture}
\end{figure}
KW-algorithm for matroid constraint

Algorithm sets threshold in step \( i \) based on \textit{marginal contribution} of \( \sigma(i) \).

- Let \( y' = (y'_1, \ldots, y'_m) \geq 0 \) be given weights, and let \( B' \) be a max. weight base under \( y' \).
- For given independent set \( S \in \mathcal{I} \), we can augment \( S \) with elements \( R(S) \subseteq B' \) so that \( S \cup R(S) \) is base of \( \mathcal{M} \).
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```
 a  b
 f  c
 e  d
```
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Example (Graphic matroid)

![Diagram of a graphic matroid with labeled weights and a base $B'$]
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Example (Graphic matroid)

![Graph with weights](image)

$S$ selected so far
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Example (Graphic matroid)

![Diagram of a graphic matroid with edges and weights labeled, showing augmentation $R(S)$]
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Example (Graphic matroid)

![Graphic matroid example](image)
Assume that \( \sigma = (e_1, \ldots, e_m) \).
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**KW-algorithm with initial $S = \emptyset$**

For $i = 1, \ldots, m$: If $S \cup \{e_i\} \in \mathcal{I}$ do the following.

- Set threshold
  $$T_i = \mathbb{E}_{y' \sim X_1 \times \cdots \times X_m}[y'(R(S)) - y'(R(S \cup \{e_i\}))].$$

- Set $S \leftarrow S \cup \{e_i\}$ if $w_i \geq T_i$. 

Roughly speaking, $T_i$ is expected gain of adding $e_i$ to $S$. If revealed realization $w_i$ exceeds expected gain, add it to $S$. In order to determine $T_i$, we take expectation over all elements (and not just those that have not yet arrived). $T_i$ does not use realized weights $w_1, \ldots, w_{i-1}$ revealed so far.

Computational remark: If the $X_i$ are discrete (with finite support), $T_i$ can be computed exactly (in possibly exponential time). For continuous distributions, usually approximation is needed (by means of repeatedly sampling vectors $y'$ from $X_1 \times \cdots \times X_m$ and computing average).
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Remarks

Theorem (Kleinberg and Weinberg, 2012)

KW-algorithm for matroids gives prophet inequality with \( \alpha = \frac{1}{2} \).

Result also extends to intersection of \( p \) matroid constraints, where one then gets \( \alpha = \frac{1}{4p - 2} \).

Can be used to model, e.g., setting where edges of bipartite graph arrive online (with known distributions).

Strategyproof mechanism?

For single element setting, conversion of respective KW-algorithm into strategyproof mechanism is easy. This is not the case for the matroid setting.

Adaptive vs. non-adaptive threshold-based algorithms. KW-algorithm is adaptive in the sense that threshold \( T_i \) in step \( i \) depends on arrival order \( \sigma \) and elements \( S \) selected so far. Does not necessarily yield strategyproof (online) mechanism.
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- They show that one can hope at best for a prophet inequality with

$$\alpha = \Omega \left( \frac{\log \log(m)}{\log(m)} \right).$$
Beyond matroids
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For general downward-closed set systems, lower bound from last week also applies to Bayesian setting (with adversarial arrivals).

Theorem (Babaioff et al. (2007), Rubinstein (2016))
There is no randomized algorithm that, for every downward-closed set system $F = (E, I)$ with $m$ elements having known weight distribution, obtains a prophet inequality with

$$\alpha > \Omega \left( \log \log(m) \log(m) \right)^{\frac{25}{35}}$$
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Selecting single element

Sample-based threshold
What prior information is needed?

Remember that the KW-algorithm for selecting a single item uses the threshold $T = \max_j X_j^2$. Computing threshold requires full knowledge of the distributions $X_i$. Can be non-trivial depending on what the distributions look like. Does there exist an algorithm using less information?

Turns out that it suffices to have one sample $x_i$ from every $X_i$. Theorem (Rubinstein, Wang and Weinberg, 2020)

Suppose we have one sample $x_i$ from every $X_i$, and let $T = \max_j x_j$. Selecting first element with $w_i \geq T$ gives prophet inequality with $\alpha = \frac{1}{2}$. Same guarantee as KW-algorithm. Algorithms only using single sample from every $X_i$ will be called single-sample algorithms.
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- Algorithms only using single sample from every \( X_i \) will be called single-sample algorithms.
Azar, Kleinberg and Weinberg (2014) give single sample algorithms leading to constant-factor prophet inequalities for various matroid constraints.

The high-level idea is to give a reduction to the secretary problem. Samples are used to mimic "observation phase" (Phase I). Slightly stronger, order-oblivious secretary algorithm is needed. An example is the $\frac{1}{4}$-approximation we saw in Homework 3.

**Theorem (Azar, Kleinberg and Weinberg, 2014 (informal))**
Every order-oblivious $\alpha$-approximation for the secretary problem (with uniform random arrivals) gives rise to a single-sample prophet inequality with factor $\alpha$ (for worst-case arrival order).

Reduction also works for graphic matroid algorithm from last week.

**Corollary (AKW, 2014)**
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An algorithm (for adversarial arrival order $\sigma$) with samples $x_i$ from $X_i$: 

- **Preprocessing:**
  - Set $k = \frac{m}{2}$, and select uniformly at random $k$ samples from $\{x_1, \ldots, x_m\}$.
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The above algorithm gives a single-sample prophet inequality with $\alpha = \frac{1}{4}$ for selecting one element.

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From single-sample prophets to secretaries

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Prophet inequalities for I.I.D. distributions
When all distributions $X_i$ are the same

Better prophet inequalities (than $\alpha = \frac{1}{2}$) are possible when all distributions $X_i$ are the same.
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**Theorem (Correa et al., 2018)**

*In case the online algorithm only has access to weights revealed so far (but not to common distribution $X$), there is a prophet inequality with $\alpha = \frac{1}{e}$ and this is best possible.*
Secretary prophet inequalities
In the prophet secretary model, the elements in \( \{e_1, \ldots, e_m\} \) arrive in uniform random order with weight \( w_i \) drawn from known distribution \( X_i \) for \( i = 1, \ldots, m \).

Theorem (Ehsani et al., 2018 (informal))

There is a secretary prophet inequality with \( \alpha = 1 - \frac{1}{e} \approx 0.63 \) for selecting multiple elements under a matroid constraint.

Theorem (Correa, Saona and Ziliotto, 2019)

There is a secretary prophet inequality with \( \alpha = 1 - \frac{1}{e} + \frac{1}{27} \approx 0.669 \) for selecting a single element.
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Overview
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Elements with unknown weights, but assumption on arrival order.
Overview second part of course

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- Prophet inequality with $\alpha = \frac{1}{2}$ for selecting single element.
- Prophet inequality with $\alpha = \frac{1}{2}$ for matroid constraint.
- Also saw some other models (e.g., single-sample settings).