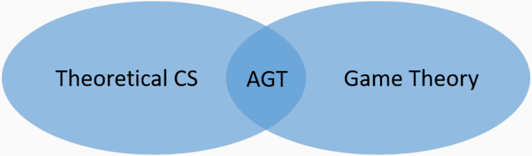
Topics in Algorithmic Game Theory and Economics

Game Theory from the Computer Scientist's point of view



Can we compute an "equilibrium" outcome of a game in polynomial time? (And more...)

Information

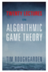
- Lectures: Wednesday, 14:15-16:00
- Homework: 4 or 5 homework sets
  - Half of points needed to qualify for exam.
- Exam: Oral examination, February 23-24, 2021
  - Covering lecture material <u>and</u> homework exercises.
- Tutorials: Doodle link given during break to check availability
- TA: Golnoosh Shahkarami

Material

- Books (for first part, until Christmas break):
  - Algorithmic Game Theory (Nisan, Roughgarden, Tardos, Vazirani)
  - Twenty Lectures on Algorithmic Game Theory (Roughgarden)

Nisan, Noam
Algorithmic game theory
Cambridge 2008
hide

- **print:** NIS n 2008:2 1.Ex
- **e-book, ip-range UdS**



Roughgarden, Tim
Twenty Lectures in Algorithmic Game Theory
Cambridge 2016
hide

- **print:** ROU t 2016:1 1.Ex
- **e-book, ip-range UdS**

- Some (elementary) background material for self-study:
  - Linear programming
  - Probability theory
  - Matroids

Tutorial "0" next week about background material.

Topics in Algorithmic Game Theory and Economics

Pieter Kleer

Max Planck Institute for Informatics (D1)
Saarland Informatics Campus

November 11, 2020

Lecture 1
Introduction and Overview

What is game theory?

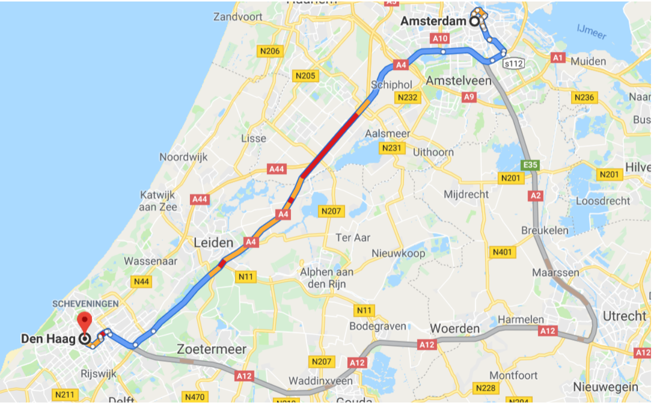
Study of mathematical models of strategic interaction among (rational) players that influence each other's outcome.

- Road users in traffic networks.
- Selfish routing of internet traffic.
- Online selling platforms.
- Auctions.

Two examples

Traffic networks

Drivers who want to get from work to home as fast as possible, not caring about the travel time of other drivers.



- Outcome is a driver's travel time from work to home.

Traffic networks (cont'd)

- Users influence each other's outcome:
  - Traffic slows down if many drivers on a road segment.
  - Drivers use traffic app to determine 'quickest' route.

Traffic networks (cont'd)

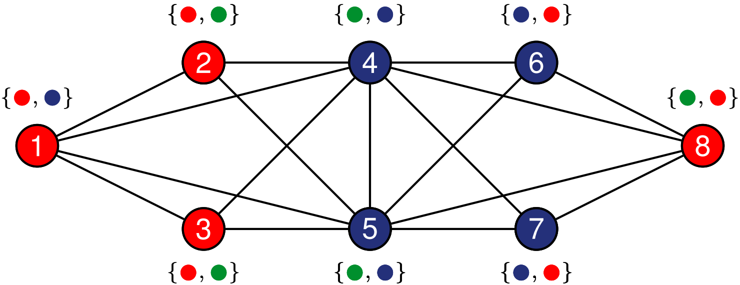
Some questions that come up:

- Assuming that drivers are selfish, how does traffic spread out over the road network?
  - So-called equilibrium flow.
  - Can we compute these equilibrium flows?
- How inefficient is such a traffic situation?
  - Compared to centralized solution in which we assign routes to drivers, with the goal of minimizing the total travel time.
    - Something, say, the government would like to achieve.

Conflicting interests:

- Road users want to get home as quickly as possible.
  - Goal: Minimize individual travel time.
- Government wants road network to be used efficiently.
  - Goal: Minimize total travel time in the network

Coordination games



- Undirected graph $G = (V, E)$; nodes in V are players,
- Strategy sets $C_i \subseteq \{1, \ldots, c\}$ for $i \in V$,
- Weights $w_e \geq 0$ for $e \in E$.
  - *Assume here $w_e = 1$ for $e \in E$.*

Choose strategy that maximizes sum of edge weights to neighbors with same color.

Coordination games (cont'd)

Game-theoretical problem:

- Find coloring in which no player has an incentive to deviate to another color.
  - 'Stable' equilibrium outcome.
  - Known as (pure) Nash equilibrium.

Centralized (classical) optimization problem:

- Find coloring maximizing total weight of unicolored edges.
  - Socially optimal outcome.
  - Maximizing overall "happiness" of players.

A recurring theme (informal)

Discrete (or continuous) optimization problem over set S of strategy vectors (or profiles) with objective function $C : S \to \mathbb{R}$.

- **Classical (centralized) optimization:** Find

$$s^* = \text{argmin}\{C(t) : t \in S\}.$$

- **Game theory variant:** Find “equilibrium” solution $s \in S$.
  - Will see some equilibrium concepts later on.

Fundamental questions in Algorithmic Game Theory (AGT)

- Equilibrium computation
  - Can we compute equilibrium in polynomial time?
- Inefficiency of equilibria
  - How much worse can $C(s)$ be compared to $C(s^*)$?
  - Price of Anarchy (PoA)/Price of Stability (PoS).

Games and equilibrium concepts

Mathematical formulation

Finite game $\Gamma = (N, (\mathcal{S}_i)_{i\in N}, (C_i)_{i\in N})$ consists of:

- Finite set N of players of size n.
- Finite strategy set $\mathcal{S}_i$ for every player $i \in N$.
- Cost function $C_i : \times_i \mathcal{S}_i \to \mathbb{R}$ for every $i \in N$.
    - Player goal is to choose strategy minimizing cost.
    - Or to maximize utility $U_i = -C_i$.

Assuming the players are rational, which strategy profiles can one expect to see as an outcome of the game?
(All players have full information.)

Equilibrium concepts

Some solution/equilibrium concepts:

- Dominant strategies,
- Pure Nash equilibrium,
- Mixed Nash equilibrium,
- (Coarse) correlated equilibrium, and more...

Natural questions that come up:

- *Does a solution concept always exist?*
- *Can we compute it in polynomial time, i.e., efficiently?*
- *Are there natural player dynamics converging to it?*
  - And how long do these dynamics take to converge?

Prisoner's dilemma

Famous thought experiment.

Prisoner's dilemma

Alice and Bob committed a crime. Police wants a confession.

		Bob	
		Silent	*Betray*
Alice	*Silent*	$(1, 1)$	$(3, 0)$
	Betray	$(0, 3)$	$(2, 2)$

- (a, b) refers to years of prison time they get.

- Problem is that Alice and Bob are not allowed to communicate.
- See also, e.g., "Golden Balls/Split or Steal" on YouTube.
  - Similar game where communication is possible.

Dominant strategies

Definition (Dominant strategy)

A strategy $t_i \in \mathcal{S}_i$ is dominant for player $i \in N$ if

$$C_i(s_1, \ldots, t_i, \ldots, s_n) \leq C_i(s_1, \ldots, t_i', \ldots, s_n)$$

for every $t_i' \in \mathcal{S}_i$ and any strategy vector

$$s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n) \in \times_{j \neq i} \mathcal{S}_j$$

of the other players. Strategy profile $t \in \times_i \mathcal{S}_i$ is called dominant if every player plays a dominant strategy.

- *No matter what the other players do, it is best to play t_i.*
- Does not always exist.

Pure Nash equilibrium

Definition (Pure Nash equilibrium (PNE))

A strategy profile $s \in \times_i \mathcal{S}_i$ is a pure Nash equilibrium if for every $i \in N$,

$$C_i(s_1, \ldots, s_i, \ldots, s_n) \leq C_i(s_1, \ldots, s_i', \ldots, s_n)$$

for every $s_i' \in \mathcal{S}_i$. *In short, $C_i(s) \leq C_i(s_i', s_{-i})$.*

- *Given strategies s_{-i} of other players, it's best to play s_i.*
  - s_i is best response to s_{-i}.
  - Switch from profile s to (s_i', s_{-i}) is called unilateral player deviation.
- PNE is natural outcome of better/best response dynamics (BRD)
  - Players take turns and switch to strategy that improves their cost.
  - *Remember coordination game example.*

PNE not guaranteed to exist in general games.

- Existence is known for special class of congestion games.
  - Next lectures.

Matching pennies

PNE is not guaranteed to exist, already in very simple games.

Matching pennies

Alice and Bob both choose side of a penny.

		Bob	
		Head	*Tails*
Alice	*Head*	$(0,1)$	$(1,0)$
	Tails	$(1,0)$	$(0,1)$

- Alice wants both coins to be on the same side.
- Bob wants both coins to be on different sides.

Is there another sensible "equilibrium"?

Yes, randomize over both strategies.

Mixed Nash equilibrium

Definition (Mixed Nash equilibrium (MNE))

A mixed strategy $\sigma_i : \mathcal{S}_i \to [0, 1]$ of player $i \in N$ is a probability distribution over pure strategies in $\mathcal{S}_i$, i.e.,

$$\Delta_i = \left\{ \tau : \tau(t) \geq 0 \ \ \forall t \in \mathcal{S}_i \ \ \text{and} \ \ \sum_{t \in \mathcal{S}_i} \tau(t) = 1 \right\}.$$

A collection of mixed strategies $\sigma = (\sigma_i)_{i \in N}$, with $\sigma_i \in \Delta_i$, is a mixed Nash equilibrium if

$$\mathbb{E}_{x \sim \sigma}\left[C_i(x)\right] \leq \mathbb{E}_{(x_i', x_{-i}) \sim (\sigma_i', \sigma_{-i})}\left[C_i(x_i', x_{-i})\right] \ \ \forall \sigma_i' \in \Delta_i. \tag{1}$$

Theorem (Nash's theorem, 1950)

Any finite game Γ has a mixed Nash equilibrium.

Remark on definition MNE

In definition of MNE, it is sufficient to look at pure strategies σ_i' in (1).

- Pure strategy (distribution): One strategy played with probability 1.
- Exercise: Prove the remark above.

Good news:

- There is a sensible equilibrium concept that always exists.

Bad news:

- Might not be unique.
  - Many equilibrium concepts suffer from this
- Turns out to be ‘difficult’ to compute (in general).

Is there an equilibrium concept that always exists and is computable?

Game of Chicken

Game of Chicken

Alice and Bob both approach an intersection.

		Bob	
		Stop	*Go*
Alice	*Stop*	$(0, 0)$	$(3, -1)$
	Go	$(-1, 3)$	$(4, 4)$

- Two PNEs: (Stop, Go), (Go, Stop)
- One MNE: Both players randomize over Stop and Go.

Distributions over strategy profiles (a, b) for these equilibria are

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}.$$

- Sensible ‘equilibrium’ would be the strategy profile distribution

$$\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}.$$

- Cannot be achieved as mixed equilibrium.
    - Cannot be achieved as a product distribution of mixed strategies.

Idea is to introduce traffic light (mediator or trusted third party).

- Traffic light samples/draws one of the two strategy profiles from distribution.
- Gives realization as recommendation to the players.
    - Tells Alice to go and Bob to stop (or vice versa)

Conditioned on this recommendation, the best thing for a player to do is to follow it.

Correlated equilibrium

Definition (Correlated equilibrium (CE))

A distribution σ on $\times_i \mathcal{S}_i$ is a correlated equilibrium if for every $i \in N$ and $x_i \in \mathcal{S}_i$, and every unilateral deviation $x_i' \in \mathcal{S}_i$, it holds that

$$\mathbb{E}_{x\sim\sigma}\left[C_i(x) \mid x_i\right] \leq \mathbb{E}_{x\sim\sigma}\left[C_i(x_i', x_{-i}) \mid x_i\right].$$

Theorem (Computation of CE, informal)

A correlated equilibrium can be computed 'efficiently' (i.e., this concept is computationally tractable).

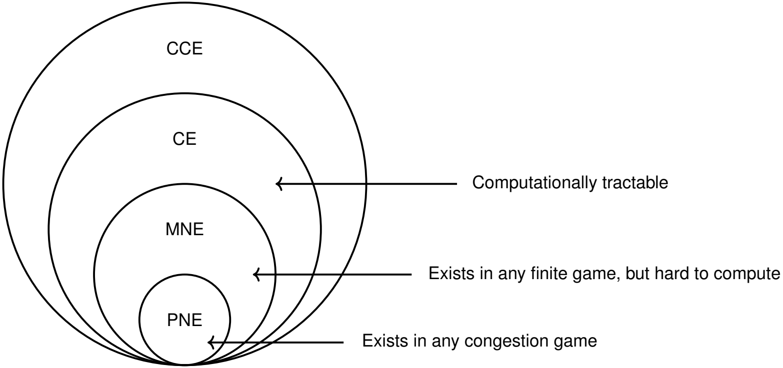
Definition (Coarse correlated equilibrium (CCE))

A distribution σ on $\times_i \mathcal{S}_i$ is a coarse correlated equilibrium if for every $i \in N$, and every unilateral deviation $x_i' \in \mathcal{S}_i$, it holds that

$$\mathbb{E}_{x\sim\sigma}\left[C_i(x)\right] \leq \mathbb{E}_{x\sim\sigma}\left[C_i(x_i', x_{-i})\right].$$

Hierarchy of equilibrium concepts

The concepts we have seen so far all are subsets of each other.



- Exercise: Prove that this is indeed a hierarchy.
  - Every PNE is an MNE, every MNE is a CE, etc.

Rough outline until Christmas

Congestion and potential games

- Existence of PNE.
- Computational complexity.
  - Complexity of computing PNE.
  - Complexity of best response dynamics.
- Inefficiency of equilibria.
  - Price of Anarchy/Stability.

General 2-player and n-player games

- Existence of MNE (Nash's theorem).
  - Discussion on computational complexity.
- Computation of approximate mixed Nash equilibria.
- Computation of (coarse) correlated equilibria.
  - Linear programming approach.
  - Decentralized dynamics.
- Inefficiency of MNE/CE/CCE.
  - Roughgarden's smoothness framework.

Background (prerequisite) material

Some tools from combinatorics, probability theory and optimization

Linear programming

Optimize linear function over set of linear constraints, e.g.,

$$\begin{array}{ll} \max & x_1 + 3x_2 \\ \text{subject to} & x_1 + x_2 \leq 5 \\ & 3x_1 + x_2 \leq 2 \\ & x_1, x_2 \geq 0 \\ & x_1, x_2 \in \mathbb{R}. \end{array}$$

In general,

$$\begin{array}{ll} \max & c^T x \\ \text{subject to} & Ax \leq b \\ & x \geq 0 \end{array}$$

Theorem (Linear programming, informal)

There is a polynomial time algorithm for finding an optimal solution to a linear program.

- Might have seen this in, e.g., course “Optimization”.

Probability theory

Basic knowledge about probability theory is assumed, in particular, we sometimes use concentration inequalities.

- Markov's inequality
- Chebyshev's inequality
- Chernoff bounds

Matroids

Generalization of linear independence of vectors in, e.g., $\mathbb{R}^n$.

Let $E = \{v_1, \ldots, v_k\}$ be collection of vectors $v_i \in \mathbb{R}^n$ for all i.

- Assume that $k > n$ and $\text{span}(E) = \mathbb{R}^n$.

Subset of vectors $X \subseteq E$ is called linearly independent if, for $\gamma_i \in \mathbb{R}$,

$$\sum_{v_i \in X} \gamma_i \cdot v_i = 0 \quad \Rightarrow \quad \gamma_i = 0 \ \forall i.$$

- No $v_i \in X$ can be written as linear combination of other vectors.

Example

$$E = \{v_1, v_2, v_3, v_2\} = \left\{ \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \end{pmatrix}, \begin{pmatrix} 17 \\ 34 \end{pmatrix}, \begin{pmatrix} -4 \\ -2 \end{pmatrix} \right\}$$

Is $X = \{v_1, v_2, v_3\}$ independent? NO, because $v_3 = 3v_1 + 4v_2$.

- Maximal independent sets are bases (of $\mathbb{R}^n$).

Let E be finite set of elements (think of, e.g., vectors).

Matroid

Set system $\mathcal{M} = (E, \mathcal{I})$ with non-empty $\mathcal{I} \subseteq 2^E$ is matroid if:

- *Downward-closed*: $A \in \mathcal{I}$ and $B \subseteq A \Rightarrow B \in \mathcal{I}$,
- *Augmentation property*:

$$A, C \in \mathcal{I} \text{ and } |C| > |A| \Rightarrow \exists e \in C \setminus A \text{ such that } A \cup \{e\} \in \mathcal{I}.$$

Sets in $\mathcal{I}$ are called independent sets.

Linear matroid: Let $E = \{v_i : i = 1, \ldots, k\} \subseteq \mathbb{R}^n$ and take

$$W \in \mathcal{I} \quad \Leftrightarrow \quad \text{vectors in } W \text{ are linearly independent.}$$

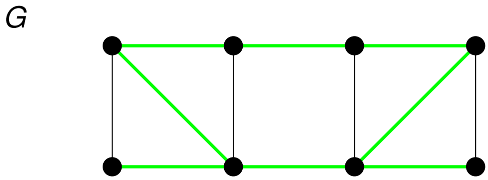
- Downward-closed property easy to check.
- For augmentation property, note that if $|C| \geq |A| + 1$ and every $v_i \in C$ is a linear combination of vectors in A, then $\text{span}(C) \subseteq \text{span}(A)$, and hence

$$|C| = \dim(\text{span}(C)) \leq \dim(\text{span}(A)) = |A|,$$

which gives a contradiction.

Graphic matroid: Let $G = (V, E)$ be undirected graph and consider matroid $\mathcal{M} = (E, \mathcal{I})$, with ground the edges E of G, given by

$$W \in \mathcal{I} \quad \Leftrightarrow \quad \text{subgraph with edges of } W \text{ has no cycle.}$$



- Bases (i.e., maximal independent sets) of the graphic matroid are spanning trees of G.