Topics in Algorithmic Game Theory and Economics

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Lecture 2
Congestion Games I - Computation of PNE
Introduction

Congestion games can be used to model, e.g.,
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- Traffic/routing games,
- Scheduling games,
- Broadcast games,
- Cost-sharing games.
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*Studied extensively in the last twenty years in the area of AGT.*
Atomic selfish routing game (example)

Given is directed graph $G = (V, E)$ with origin $o$ and destination $d$. 

![Diagram of the directed graph with nodes $o$, $v$, $w$, and $d$ connected by edges $e_1$, $e_2$, $e_3$, $e_4$, and $e_5$.]
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For strategy profile $s = (s_1, s_2, \ldots, s_n) \in \mathcal{P}^n$, with $x_e = x_e(s)$ number of players using $e \in E$,

$$C_i(s) = \sum_{e \in s_i} c_e(x_e).$$
Suppose we have $n = 4$ players and edges $E = \{ e_1, \ldots, e_4 \}$.
- Remember that player places one unit of flow on a path.
- Cost of player $i$ in profile $s$ is given by (with $s_i \in \{ T, B \} = \mathcal{P}$)

$$C_i(s) = \sum_{e \in s_i} c_e(x_e).$$

\[
\begin{align*}
C_1(s) &= 5 + 2 = 7 \\
C_2(s) &= 5 + 2 = 7 \\
C_3(s) &= 5 + 2 = 7 \\
C_4(s) &= 5 + 2 = 7
\end{align*}
\]
Atomic selfish routing game (cont’d)

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\[
\begin{align*}
C_1(s) &= 1 + 3 \cdot 1 = 4 \\
C_2(s) &= 5 + 2 = 7 \\
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\begin{align*}
C_1(s) &= 2 + 3 \cdot 2 = 8 \\
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\[
\begin{align*}
C_1(s) &= 3 + 3 \cdot 3 = 12 \\
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  C_1(s) &= 4 + 3 \cdot 4 = 16 \\
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\[ s = (T, T, T, T) \]
(Atomic) congestion game $\Gamma = (N, E, (S_i)_{i \in N}, (c_e)_{e \in E})$: 

- **Set of players** $N = \{1, \ldots, n\}$.
- **Set of resources** $E = \{e_1, \ldots, e_m\}$.
- **Strategy set** $S_i \subseteq 2^E = \{X : X \subseteq E\}$ for all $i \in N$.
- **Cost function** $c_e : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ for $e \in E$.

Although the word 'congestion' hints at these functions being non-decreasing, this is not required.

Player places one unit of unsplittable load on a strategy with the goal of minimizing her cost. For strategy profile $s = (s_1, s_2, \ldots, s_n) \in S_1 \times S_2 \times \cdots \times S_n$, $C_i(s) = \sum_{e \in s_i} c_e(x_e)$, where $x_e = x_e(s)$ is the number of players using $e \in E$, i.e., the load.
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Broadcast game (example)

Given is undirected graph $G = (V, E)$.

- Edges $e \in E$ are resources with cost function $c_e$.
- Players place one unit of unsplittable load on spanning tree of $G$.
  - Spanning trees are the strategies of the players.

![Graph G](image-url)
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Example of base (graphic) matroid congestion game.
We will focus on pure Nash equilibria in congestion games.

**Definition (Pure Nash equilibrium (PNE))**

A strategy profile $s \in \times_i S_i$ is a pure Nash equilibrium if for every $i \in N$, $C_i(s_1, \ldots, s_i, \ldots, s_n) \leq C_i(s_1, \ldots, s'_i, \ldots, s_n)$ for every $s'_i \in S_i$. 

**Why focus on PNE?**

There always exists at least one!
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**Potential function method**
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Show existence of **potential function** $\Phi : \times_i S_i \rightarrow \mathbb{R}$ tracking improvements in player costs.

That is, $\Phi$ has the property that if, in strategy profile $s = (s_1, \ldots, s_n)$,
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- Player $i$ has improving move by switching to $s'_i \in S_i$, i.e.,
  $$C_i(s'_i, s_{-i}) < C_i(s).$$

- Then also
  $$\Phi(s'_i, s_{-i}) < \Phi(s).$$

Algorithm 3: Better response dynamics

Input: Strategy profile $s_0 \in \times_i S_i$.

Output: Pure Nash equilibrium $s^*$. 

1. $k = 0$.
2. while $s_k$ is not a pure Nash equilibrium do
   1. Select player $i \in N$ and $s'_i \in S_i$ such that $C_i(s'_i, s_{-i}) < C_i(s)$.
   2. $s_{k+1} \leftarrow (s'_i, s_{k-i})$.
   3. $k \leftarrow k + 1$.
3. return $s^* \leftarrow s_k$. 


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**ALGORITHM 4:** Better response dynamics

**Input**: Strategy profile \( s^0 \in \times_i S_i \).
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\[
\begin{align*}
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  \text{while } s^k \text{ is not a pure Nash equilibrium do} \\
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  &\quad s^{k+1} \leftarrow (s'_i, s^k_{-i}). \\
  &\quad k \leftarrow k + 1. \\
\end{align*}
\]

**return** \( s^* \leftarrow s^k \)
Better response dynamics always terminate (converge) in finite number of steps, given the existence of the function $\Phi$. 

Why? 

If player $i$ makes an improving move in step $k$, then $\Phi(s_{k+1}) < \Phi(s_k)$. 

This means 

$\cdots < \Phi(s_{k+1}) < \Phi(s_k) < \Phi(s_{k-1}) < \cdots < \Phi(s_1) < \Phi(s_0)$. 

There are only finitely many strategy profiles. 

Remember that we assume that $S_i$ is finite for every $i \in \mathbb{N}$.

Theorem (Rosenthal, 1973) 

Every (finite) congestion game possesses a pure Nash equilibrium. It can be computed by better response dynamics. 

What is the potential function $\Phi$?
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Remember that $C_i(s) = \sum_{e \in s_i} c_e(x_e)$.  

- $x_e = x_e(s)$ total number of players using resource $e$ in $s$. 

Rosenthal’s potential

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**Lemma (Rosenthal’s potential)**

Rosenthal’s potential satisfies, for every $i \in N$ and $s'_i \in S_i$,

$$C_i(s) - C_i(s'_i, s_{-i}) = \Phi(s) - \Phi(s'_i, s_{-i}).$$
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The Rosenthal (potential) function $\Phi : \times_i S_i \rightarrow \mathbb{R}$ is given by

$$\Phi(s) = \sum_{e \in E} \sum_{k=1}^{x_e(s)} c_e(k).$$

Remember that $C_i(s) = \sum_{e \in S_i} c_e(x_e)$.

- $x_e = x_e(s)$ total number of players using resource $e$ in $s$.

Lemma (Rosenthal’s potential)

Rosenthal’s potential satisfies, for every $i \in N$ and $s'_i \in S_i$,

$$C_i(s) - C_i(s'_i, s_{-i}) = \Phi(s) - \Phi(s'_i, s_{-i}).$$

- Proof (sketch) on Slide 12 for symmetric singleton games.
- Exercise: Generalize the proof to general congestion games.
Rosenthal’s potential (example)

Remember, for strategy profile \( s \),

\[
\Phi(s) = \sum_{e \in E} \sum_{k=1}^{x_e(s)} c_e(k).
\]

\( c_1(x) = x \)
\( c_2(x) = 3x \)
\( c_3(x) = 5 \)
\( c_4(x) = 2 \)

\[
\Phi(s) = 0 + 0 + [c_3(1) + c_3(2) + c_3(3) + c_3(4)]
+ [c_4(1) + c_4(2) + c_4(3) + c_4(4)]
= 28
\]

\[
C_1(s) = 5 + 2 = 7 \\
C_2(s) = 5 + 2 = 7 \\
C_3(s) = 5 + 2 = 7 \\
C_4(s) = 5 + 2 = 7
\]
Rosenthal’s potential (example)

Remember, for strategy profile $s$,

$$\Phi(s) = \sum_{e \in E} \sum_{k=1}^{x_e(s)} c_e(k).$$

\[
\begin{align*}
\Phi(s) &= [c_1(1)] \\
&\quad + [c_2(1)] \\
&\quad + [c_3(1) + c_3(2) + c_3(3)] \\
&\quad + [c_4(1) + c_4(2) + c_4(3)] \\
&= 25
\end{align*}
\]

\[
\begin{align*}
C_1(s) &= 1 + 3 \cdot 1 = 4 \\
C_2(s) &= 5 + 2 = 7 \\
C_3(s) &= 5 + 2 = 7 \\
C_4(s) &= 5 + 2 = 7
\end{align*}
\]
Rosenthal’s potential (example)

Remember, for strategy profile $s$,

$$\Phi(s) = \sum_{e \in E} \sum_{k=1}^{x_e(s)} c_e(k).$$

$$C_1(s) = 2 + 3 \cdot 2 = 8$$
$$C_2(s) = 2 + 3 \cdot 2 = 8$$
$$C_3(s) = 5 + 2 = 7$$
$$C_4(s) = 5 + 2 = 7$$

$$\Phi(s) = [c_1(1) + c_1(2)] + [c_2(1) + c_2(2)] + [c_3(1) + c_3(2)] + [c_4(1) + c_4(2)] = 26$$
Rosenthal’s potential (example)

Remember, for strategy profile \( s \),

\[
\Phi(s) = \sum_{e \in E} \sum_{k=1}^{x_e(s)} c_e(k).
\]

\( c_1(x) = x \)
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\( c_3(x) = 5 \)
\( c_4(x) = 2 \)

\( C_1(s) = 3 + 3 \cdot 3 = 12 \)
\( C_2(s) = 3 + 3 \cdot 3 = 12 \)
\( C_3(s) = 3 + 3 \cdot 3 = 12 \)
\( C_4(s) = 5 + 2 = 7 \)

\( \Phi(s) = [c_1(1) + c_1(2) + c_1(3)] + [c_2(1) + c_2(2) + c_2(3)] + [c_3(1)] + [c_4(1)] = 31 \)
Rosenthal’s potential (example)

Remember, for strategy profile $s$,

$$\Phi(s) = \sum_{e \in E} \sum_{k=1}^{x_e(s)} c_e(k).$$

$C_1(s) = 4 + 3 \cdot 4 = 16$
$C_2(s) = 4 + 3 \cdot 4 = 16$
$C_3(s) = 4 + 3 \cdot 4 = 16$
$C_3(s) = 4 + 3 \cdot 4 = 16$

$\Phi(s) = [c_1(1) + c_1(2) + c_1(3) + c_1(4)]$
$+ [c_2(1) + c_2(2) + c_2(3) + c_2(4)]$
$+ 0$
$+ 0$
$= 40$
Symmetric singleton game $\Gamma = (N, E, (S_i), (c_e))$ given by

$$S_i = \{\{e_1\}, \{e_2\}, \ldots, \{e_m\}\}$$
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That is, every player has to choose one resource from the set $E$. 
Symmetric singleton game \( \Gamma = (N, E, (S_i), (c_e)) \) given by

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**Remember** \( \Phi(s) = \sum_{e \in E} \sum_{k=1}^{x_e(s)} c_e(k) \) and \( C_i(s) = \sum_{e \in S_i} c_e(x_e) \).
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\[
\begin{align*}
C_i(s) - C_i(s'_i, s_{-i}) &= c_2(4) - c_1(3) \\
&= c_2(x_2(s)) - c_1(x_1(s) + 1)
\end{align*}
\]
Symmetric singleton game \( \Gamma = (N, E, (S_i), (c_e)) \) given by
\[
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That is, every player has to choose one resource from the set \( E \).

\textbf{Remember} \( \Phi(s) = \sum_{e \in E} \sum_{k=1}^{x_e(s)} c_e(k) \) \textbf{and} \( C_i(s) = \sum_{e \in s_i} c_e(x_e) \).
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\[
C_i(s) - C_i(s', s_{-i}) = c_2(4) - c_1(3) = c_2(x_2(s)) - c_1(x_1(s) + 1)
\]

\[
\Phi(s) - \Phi(s', s_{-i}) = [c_1(1) + c_1(2)] + [c_2(1) + c_2(2) + c_2(3) + c_2(4)] + c_3(1) + c_4(1) - [c_1(1) + c_1(2) + c_1(3)] - [c_2(1) + c_2(2) + c_2(3)] - c_3(1) - c_4(1) = c_2(4) - c_1(3) = c_2(x_2(s)) - c_1(x_1(s) + 1)
\]
PNE always exists and can be computed by better response dynamics.
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*In fact, we showed that congestion games are exact potential games.*
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In fact, we showed that congestion games are exact potential games.

**Definition (Exact potential game)**

Finite game $\Gamma = (N, (S_i), (C_i))$ is exact potential game if there exists function $\Phi : \times_i S_i \rightarrow \mathbb{R}$ such that

$$C_i(s) - C_i(s_i', s_{-i}) = \Phi(s) - \Phi(s_i', s_{-i})$$

for every $i \in N$ and $s_i' \in S_i$. 
Brief overview

PNE always exists and can be computed by better response dynamics.

In fact, we showed that congestion games are exact potential games.

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Finite game $\Gamma = (N, (S_i), (C_i))$ is exact potential game if there exists function $\Phi : \times_i S_i \rightarrow \mathbb{R}$ such that

$$C_i(s) - C_i(s_i', s_{-i}) = \Phi(s) - \Phi(s_i', s_{-i})$$

for every $i \in N$ and $s_i' \in S_i$.

Theorem

The class of congestion games is ‘isomorphic’ to the class of exact potential games.
Algorithmic questions

Of interest to the computer scientist:

Do better response dynamics converge in poly-time to PNE? If not, can we compute PNE in polynomial time by other means? For both questions: In general NO, but in certain special cases YES. Polynomial in parameters needed to specify player costs in game. In general $n^k$ numbers are needed for this where $k = \max_i |S_i|$. Many special cases can be represented more compactly. For positive answers to the above questions, we usually get $\text{poly}(n, m, |c|)$-running time.

How to study computational complexity of PNE in congestion games? Interpret it as local search problem w.r.t. Rosenthal's potential.
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\[ nk \]

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*How to study computational complexity of PNE in congestion games?*
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How to study computational complexity of PNE in congestion games?

Interpret it as local search problem w.r.t. Rosenthal’s potential.
In all statements on previous slides, we can replace ‘better’ by ‘best’.
Remark (for Homework 1)

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Best response dynamics

In better response dynamics algorithm, always choose strategy yielding best improvement in cost.
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Best response dynamics

In better response dynamics algorithm, always choose strategy yielding best improvement in cost.

ALGORITHM 7: Best response dynamics

Input : Strategy profile $s^0 \in \times_i S_i$.
Output: Pure Nash equilibrium $s^*$.

$k = 0$.
while $s^k$ is not a pure Nash equilibrium do
    Select player $i \in N$ and $s'_i \in S_i$ such that
    $C_i(s'_i, s_{-i}) = \min_{t_i \in S_i} C_i(t_i, s_{-i})$.
    $s^{k+1} \leftarrow (s'_i, s^k_{-i})$.
    $k \leftarrow k + 1$.
end
return $s^* \leftarrow s^k$. 
Some positive results to algorithmic questions

1 Special cases where response dynamics converge quickly:
   - Better response dynamics in singleton congestion games.
   - Best response dynamics in base matroid congestion games.
   - Homework 1.
Some positive results to algorithmic questions

1. Special cases where response dynamics converge quickly:
   - Better response dynamics in singleton congestion games.
   - Best response dynamics in base matroid congestion games.
   - Homework 1.

2. Special case where PNE can be computed by other means:
   - Symmetric network congestion games.
A singleton congestion game $\Gamma = (N, E, (S_i), (c_e))$ has the property that $S_i \subseteq \{\{e_1\}, \{e_2\}, \ldots, \{e_m\}\}$, i.e., every possible strategy consists of a single resource.
### Definition
A singleton congestion game $\Gamma = (N, E, (S_i), (c_e))$ has the property that $S_i \subseteq \{\{e_1\}, \{e_2\}, \ldots, \{e_m\}\}$, i.e., every possible strategy consists of a single resource.

### Theorem (Ieong et al., 2005)
For singleton congestion games, better response dynamics (BRD) terminate in at most $n^2 m$ steps (with $n$ #players and $m$ #resources).

*Proof on next slide.*
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Singleton congestion games

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**Lemma**

If cost functions \((c_e)\) are integer-valued,
Singleton congestion games

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If cost functions \( (c_e) \) are integer-valued, then Rosenthal’s potential \( \Phi \) is integer-valued,
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Lemma
If cost functions $(c_e)$ are integer-valued, then Rosenthal's potential $\Phi$ is integer-valued, and BRD converge in at most $\Phi_{\text{max}} - \Phi_{\text{min}}$ steps.
Singleton congestion games

**Definition**

A singleton congestion game \( \Gamma = (N, E, (S_i), (c_e)) \) has the property that \( S_i \subseteq \{\{e_1\}, \{e_2\}, \ldots, \{e_m\}\} \), i.e., every possible strategy consists of a single resource.

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**Lemma**

If cost functions \( (c_e) \) are integer-valued, then Rosenthal’s potential \( \Phi \) is integer-valued, and BRD converge in at most \( \Phi_{\text{max}} - \Phi_{\text{min}} \) steps.

- \( \Phi_{\text{max}}, \Phi_{\text{min}} \) are max. and min. attained by \( \Phi \), respectively.
- For any strategy profile \( s \), it holds that \( \Phi_{\text{min}} \leq \Phi(s) \leq \Phi_{\text{max}} \).
**Proof idea:** Show that cost functions can be replaced by ‘nice’ (polynomially bounded, integer) cost functions while preserving improving moves.
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Step 1: Defining the ‘nice’ cost functions.
Consider $C = \bigcup_{e \in E} \{c_e(1), \ldots, c_e(n)\}$. 

Example ($n = 3$ and $m = 2$):

- $c_1(1) = 3, c_1(2) = 10, c_1(3) = 1000$
- $c_2(1) = 5, c_2(2) = 1000, c_2(3) = 1004$

We have $C = \{3, 5, 10, 1000, 1004\}$.

Then $\tilde{c}_1(1) = 1, \tilde{c}_1(2) = 3, \tilde{c}_1(3) = 4, \tilde{c}_2(1) = 2, \tilde{c}_2(2) = 4, \tilde{c}_2(3) = 5$.

We have $\tilde{C} = \{1, 2, 3, 4, 5\}$. 
Proof idea: Show that cost functions can be replaced by ‘nice’ (polynomially bounded, integer) cost functions while preserving improving moves. Then apply lemma from previous slide.

Step 1: Defining the ‘nice’ cost functions.
Consider \( C = \bigcup_{e \in E} \{c_e(1), \ldots, c_e(n)\} \). Note that \( |C| \leq nm \).
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Step 1: Defining the ‘nice’ cost functions.
Consider $C = \bigcup_{e \in E} \{c_e(1), \ldots, c_e(n)\}$. Note that $|C| \leq nm$.

- Costs of resources for given loads $x_e \in \{1, \ldots, n\}$. 

Example ($n = 3$ and $m = 2$):

- $c_1(1) = 3$, $c_1(2) = 10$, $c_1(3) = 1000$,
- $c_2(1) = 5$, $c_2(2) = 1000$, $c_2(3) = 1004$.

We have $C = \{3, 5, 10, 1000, 1004\}$. Then $	ilde{c}_1(1) = 1$, $	ilde{c}_1(2) = 3$, $	ilde{c}_1(3) = 4$, $	ilde{c}_2(1) = 2$, $	ilde{c}_2(2) = 4$, $	ilde{c}_2(3) = 5$.

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Consider $C = \bigcup_{e \in E} \{c_e(1), \ldots, c_e(n)\}$. Note that $|C| \leq nm$.
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For $e \in E$, define $\tilde{c}_e : \{1, \ldots, n\} \to \{1, \ldots, nm\}$ by

$$\tilde{c}_e(i) = r \Leftrightarrow r - 1 \text{ distinct values } c_f(j) \in C \text{ for which } c_f(j) < c_e(i).$$
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$\tilde{c}_e(i) = r \iff r - 1$ distinct values $c_f(j) \in C$ for which $c_f(j) < c_e(i)$.

- That is, $c_e(i)$ is the $r$-th smallest number in $C$. 

Example ($n = 3$ and $m = 2$):

- $c_1(1) = 3$,
- $c_1(2) = 10$,
- $c_1(3) = 1000$,
- $c_2(1) = 5$,
- $c_2(2) = 1000$,
- $c_2(3) = 1004$.

We have $C = \{3, 5, 10, 1000, 1004\}$.

Then $\tilde{c}_1(1) = 1$, $\tilde{c}_1(2) = 3$, $\tilde{c}_1(3) = 4$,

$\tilde{c}_2(1) = 2$, $\tilde{c}_2(2) = 4$, $\tilde{c}_2(3) = 5$.

We have $\tilde{C} = \{1, 2, 3, 4, 5\}$. 
Proof idea: Show that cost functions can be replaced by ‘nice’ (polynomially bounded, integer) cost functions while preserving improving moves. Then apply lemma from previous slide.

**Step 1: Defining the ‘nice’ cost functions.**
Consider $C = \bigcup_{e \in E} \{c_e(1), \ldots, c_e(n)\}$. Note that $|C| \leq nm$.

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Example ($n = 3$ and $m = 2$):
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Consider \( C = \bigcup_{e \in E} \{ c_e(1), \ldots, c_e(n) \} \). Note that \( |C| \leq nm \).
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\[
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\]

That is, \( c_e(i) \) is the \( r \)-th smallest number in \( C \).

Example (\( n = 3 \) and \( m = 2 \)):
- \( c_1(1) = 3, c_1(2) = 10, c_1(3) = 1000, c_2(1) = 5, c_2(2) = 1000, c_2(3) = 1004 \).
Proof idea: Show that cost functions can be replaced by ‘nice’ (polynomially bounded, integer) cost functions while preserving improving moves. Then apply lemma from previous slide.

Step 1: Defining the ‘nice’ cost functions.
Consider \( C = \bigcup_{e \in E} \{ c_e(1), \ldots, c_e(n) \} \). Note that \( |C| \leq nm \).

- Costs of resources for given loads \( x_e \in \{1, \ldots, n\} \).

For \( e \in E \), define \( \tilde{c}_e : \{1, \ldots, n\} \to \{1, \ldots, nm\} \) by

\[ \tilde{c}_e(i) = r \Leftrightarrow r - 1 \text{ distinct values } c_f(j) \in C \text{ for which } c_f(j) < c_e(i). \]

- That is, \( c_e(i) \) is the \( r \)-th smallest number in \( C \).

Example (\( n = 3 \) and \( m = 2 \)):

- \( c_1(1) = 3, c_1(2) = 10, c_1(3) = 1000, c_2(1) = 5, c_2(2) = 1000, c_2(3) = 1004 \). We have \( C = \{3, 5, 10, 1000, 1004\} \).
Proof idea: Show that cost functions can be replaced by ‘nice’ (polynomially bounded, integer) cost functions while preserving improving moves. Then apply lemma from previous slide.

**Step 1: Defining the ‘nice’ cost functions.**
Consider \( C = \bigcup_{e \in E} \{ c_e(1), \ldots, c_e(n) \} \). Note that \( |C| \leq nm \).

- Costs of resources for given loads \( x_e \in \{1, \ldots, n\} \).

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- \( c_1(1) = 3, c_1(2) = 10, c_1(3) = 1000, c_2(1) = 5, c_2(2) = 1000, c_2(3) = 1004. \) We have \( C = \{3, 5, 10, 1000, 1004\} \).
- Then \( \tilde{c}_1(1) = 1, \tilde{c}_1(2) = 3, \tilde{c}_1(3) = 4, \tilde{c}_2(1) = 2, \tilde{c}_2(2) = 4, \tilde{c}_2(3) = 5. \)
Proof idea: Show that cost functions can be replaced by ‘nice’ (polynomially bounded, integer) cost functions while preserving improving moves. Then apply lemma from previous slide.

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Improving moves are preserved under this transformation from $c_e$ to $\tilde{c}_e$. 

Exercise: Show that this transformation fails for non-singleton congestion games (i.e., in general (1) is not true).

Player $i$ has improving move from resource $e_2$ to $e_1$ under cost functions $(c_e)$ if and only if it is an improving move under the $(\tilde{c}_e)$. 


**Improving moves are preserved under this transformation from \( c_e \) to \( \tilde{c}_e \).**

In example above, for strategy profile \( s \) (on the left),

\[
C_i(s'_i, s_{-i}) < C_i(s) \iff \tilde{C}_i(s'_i, s_{-i}) < \tilde{C}_i(s) \tag{1}
\]
Improving moves are preserved under this transformation from $c_e$ to $\tilde{c}_e$.

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$$C_i(s'_i, s_{-i}) < C_i(s) \iff \tilde{C}_i(s'_i, s_{-i}) < \tilde{C}_i(s) \quad (1)$$

which here means,

$$c_1(x_1(s) + 1) < c_2(x_2(s)) \iff \tilde{c}_1(x_1(s) + 1) < \tilde{c}_2(x_2(s)).$$
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Exercise: Show that this transformation fails for non-singleton congestion games (i.e., in general (1) is not true).
Step 2: BRD analysis in ‘nice’ game.
Rosenthal’s potential

\[ \tilde{\Phi}(s) = \sum_{e \in E} \sum_{k=1}^{x_e(s)} \tilde{c}_e(x_e) \]

is integer-valued
**Step 2: BRD analysis in ‘nice’ game.**

Rosenthal’s potential

\[
\hat{\Phi}(s) = \sum_{e \in E} \sum_{k=1}^{x_e(s)} \tilde{c}_e(x_e)
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is integer-valued and satisfies

\[
0 \leq \hat{\Phi} \leq n^2 m.
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- First note that \( \tilde{c}_e(x_e) \leq nm \) for any load \( x_e \in \{1, \ldots, n\} \).
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- First note that \(\tilde{c}_e(x_e) \leq nm\) for any load \(x_e \in \{1, \ldots, n\}\).
  - Because \(|C| \leq nm\).
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- First note that \( \tilde{c}_e(x_e) \leq nm \) for any load \( x_e \in \{1, \ldots, n\} \).
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- Also, \( \tilde{\Phi} \) is sum of at most \( n \) values in \( \tilde{C} = \bigcup_{e \in E} \{ \tilde{c}_e(1), \ldots, \tilde{c}_e(n) \} \).
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  - E.g., \( \tilde{\Phi}(s) = [\tilde{c}_1(1) + \tilde{c}_1(2)] + [\tilde{c}_2(1) + \tilde{c}_2(2) + \tilde{c}_2(3) + \tilde{c}_2(4)] \).
    - Sum of \( n = 6 \) terms.
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\[ \tilde{\Phi}(s) = \sum_{e \in E} \sum_{k=1}^{x_e(s)} \tilde{c}_e(x_e) \]

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    - Sum of \( n = 6 \) terms.
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\tilde{\Phi}(s) = \sum_{e \in E} \sum_{k=1}^{x_e(s)} \tilde{c}_e(x_e)
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    - Sum of \(n = 6\) terms.
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Then apply lemma from Slide 17.
Symmetric network congestion games

I.e., the “atomic selfish routing game” example from earlier.

Resources are edges of given directed graph $G = (V, E)$.

Common strategy set of players is set of all $o, d$-paths in $G$.

Theorem (Best response dynamics)

Best response dynamics might take an exponential (in $n$) number of steps to terminate (i.e, to converge to a PNE).

Is there another way to compute a PNE?

Theorem (Fabrikant et al., 2004)

There exists a poly$(n, m)$-time algorithm for computing a PNE in a symmetric network congestion game when the cost functions are non-negative and non-decreasing.

Idea: Compute strategy profile $s$ minimizing Rosenthal's potential.

Convince yourself this is indeed a pure Nash equilibrium.
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- Convince yourself this is indeed a pure Nash equilibrium.
Reduction to flow problem

If every player chooses $o, d$-path, resulting in strategy profile $s$, we obtain a so-called $o, d$-flow of size $n$. Every player routes one unit of flow over some path. Resulting loads $x_e(s) = f_e$ satisfy the linear (in)equalities $F = \{f \in \mathbb{R}^{|E|} : \sum_{(w,v) \in E} f_{wv} = \sum_{(v,w) \in E} f_{vw} \forall v \in V \setminus \{o,d\} \}$. $\sum_{(o,w) \in E} f_{ow} = n \sum_{(w,d) \in E} f_{wd} \geq 0 \forall (v,w) \in E$. 
Reduction to flow problem

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Reduction to flow problem

If every player chooses \( o, d \)-path, resulting in strategy profile \( s \), we obtain a so-called \( o, d \)-flow of size \( n \).

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Resulting loads \(x_e(s) = f_e\) satisfy the linear (in)equalities

\[
\mathcal{F} = \left\{ f \in \mathbb{R}_{\geq 0}^{\left|E\right|} : \begin{align*}
\sum_{w: (w,v) \in E} f_{wv} &= \sum_{w: (v,w) \in E} f_{vw} \quad \forall v \in V \setminus \{o, d\} \\
\sum_{w: (o,w) \in E} f_{ow} &= n \\
\sum_{w: (w,d) \in E} f_{wd} &= n \\
f_{vw} &\geq 0 \quad \forall (v, w) \in E \right\}.
\]

High-level idea: Instead of computing a strategy profile \( s^* \in \times_i S_i \) minimizing

\[
\Phi(s) = \sum_{e \in E} \sum_{k=1}^{x_e(s)} c_e(k),
\]
compute an integral o-, d-flow (or load profile) \( f^* \in F \) that minimizes

\[
\bar{\Phi}(f) = \sum_{e \in E} f_e \sum_{k=1}^{x_e} c_e(k),
\]

Map o-, d-flow \( f^* \) to strategy profile \( s^* \) minimizing \( \Phi \).

Can we always do this?

Lemma: Every integral \( f \in F \) can be decomposed into \( n \) (one for each player) o-, d-paths that each contain one unit of flow. (For simplicity, we assume here that \( G = (V, E) \) is acyclic.)

Assign resulting paths to players.

This gives the desired profile \( s^* \).

Does not matter which path is assigned to which player.

Symmetry assumption is crucial here! (Think about it.)
**High-level idea:** Instead of computing a strategy profile \( s^* \in \times_i S_i \) minimizing

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Map $o,d$-flow $f^*$ to strategy profile $s^*$ minimizing $\Phi$. 
**High-level idea:** Instead of computing a strategy profile $s^* \in \times_i S_i$ minimizing

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**Lemma**

*Every integral $f \in \mathcal{F}$ can be decomposed into $n$ (one for each player) $o, d$-paths that each contain one unit of flow.*

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Assign resulting paths to players.
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Map \( o, d \)-flow \( f^* \) to strategy profile \( s^* \) minimizing \( \Phi \). Can we always do this?

**Lemma**

Every integral \( f \in F \) can be decomposed into \( n \) (one for each player) \( o, d \)-paths that each contain one unit of flow.

- (For simplicity, we assume here that \( G = (V, E) \) is acyclic.)

Assign resulting paths to players. This gives the desired profile \( s^* \).
- Does not matter which path is assigned to which player.
- Symmetry assumption is crucial here! (Think about it.)
Computing profile $s^*$ minimizing Rosenthal’s potential $\Phi$:

- Compute integral $f^* \in \mathcal{F}$ that minimizes

$$\Phi(f) = \sum_{e \in E} \sum_{k=1}^{f_e} c_e(k).$$

- Decompose $f^*$ into $n$ paths, and assign those to players.
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Remark

This high-level approach also works for other congestion games with some ‘combinatorial’ structure, e.g., (Del Pia-Michini-Ferris, 2015).
Minimum cost flow problem

Directed graph $G = (V, A)$ with origin $o$ and destination $d$; flow size $n \in \mathbb{Q}$.

- Edge $e = (v, w) \in E$ has capacity $u_{vw}$ and cost $k_{vw}$.

\[
\begin{align*}
\min & \quad \sum_{e = (v, w) \in E} k_{vw} f_{vw} \\
\text{subject to} & \quad f \in \mathcal{F} \\
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Integral flow can be found in poly-time, when capacities are integral.
Reduction to min-cost flow problem (try yourself!)

Problem is that

\[ \Phi(f) = \sum_{e \in E} \sum_{k=1}^{f_e} c_e(k) \]

is not linear in the variables \( f_e \).

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- Introduce copies with capacity 1 and cost \( c_e(1), c_e(2), \ldots, c_e(n) \).
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![Diagram](image-url)
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Every integral min-cost flow of size \( n \) in graph with copied edges corresponds to flow minimizing \( \bar{\Phi} \).
Local search
High-level idea

Given function $f : S \rightarrow \mathbb{R}$, where $S$ is a finite set.
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- Can we find ‘local’ improvement in objective value $f(x)$?

- Recall better response dynamics.
  - Essentially tries to find local improvement for Rosenthal’s potential.
Local search problems

Definition

A **local search problem** $\Pi$ is given by:

- Set of instances $\mathcal{I}$;

We are interested in "unilateral deviations" as neighborhood, and Rosenthal's potential as objective function. PNEs are then precisely the local minima.
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### Complexity Class PLS

#### Definition

A local search problem $\Pi$ belongs to the complexity class **PLS** *(polynomial local search)* if for every instance $I \in \mathcal{I}$ the following can be done in polynomial time:

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The procedure in which one repeatedly tries to find a better solution in the neighborhood is known as local search.
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Given undirected graph $G = (V, E)$ and weight function $w : E \to \mathbb{R}_{\geq 0}$, find partition $V = S \cup \bar{S}$ that maximizes

$$\alpha(S, \bar{S}) = \sum_{\{i,j\} : i \in S, j \in \bar{S}} w_{ij}.$$
Maximum cut

Max-cut

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Local Search: FLIP neighborhood

For cut \((S, \bar{S})\) its neighbourhood is given by all \((T, \bar{T})\) that can be obtained by flipping precisely one node to its other side in \((S, \bar{S})\).
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PLS-reduction

**Problem Π^1 can be reduced to Π^2** means that Π^1 can be modeled as a special case of Π^2. Hence, Π^2 is the “more difficult” problem of the two (i.e., not easier than the other).

**Definition**

Let Π^1 = (I^1, F^1, Φ^1, N^1) and Π^2 = (I^2, F^2, Φ^2, N^2) be two local search problems in PLS. Π^1 is PLS-reducible to Π^2 if there are two polynomial time computable functions f and g such that:

- f maps every instance I ∈ I^1 of Π^1 to an instance f(I) ∈ I^2 of Π^2;
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- if S^2 is a local minimum of f(I), then g(S^2, I) is a local minimum of I. (Local minima map to local minima.)
“Problem $\Pi_1$ can be reduced to $\Pi_2$” means that $\Pi_1$ can be modeled as a special case of $\Pi_2$, 

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If $S_2$ is a local minimum of $f(I)$, then $g(S_2, I)$ is a local minimum of $I$.
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- \( f \) maps every instance \( l \in \mathcal{I}_1 \) of \( \Pi_1 \) to an instance \( f(l) \in \mathcal{I}_2 \) of \( \Pi_2 \);
- \( g \) maps every tuple \((S_2, l)\) with \( S_2 \in F_2(f(l)) \) to a solution \( S_1 \in F_1(l) \); (**Feasible solutions map to feasible solutions.**)
- for all \( l \in \mathcal{I}_1 \): if \( S_2 \) is a local minimum of \( f(l) \), then \( g(S_2, l) \) is a local minimum of \( l \). (**Local minima map to local minima.**)
A local search problem Π is **PLS-complete** if

- Π belongs to the complexity class PLS;
- every problem in PLS is PLS-reducible to Π.
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Implication: If there is a polynomial time algorithm that computes a local optimum for a PLS-complete problem $\Pi$, then there exists a polynomial time algorithm for finding a local optimum for every problem in PLS.

Remark

The definition of PLS does not require you to solve a PLS(-complete) problem with local search.
Theorem
Maximum cut with FLIP neighborhood is PLS-complete.

In particular, local search might take an exponential long time to converge to a local optimum.

Theorem
Computing PNE with "unilateral deviation" neighborhood and, Rosenthal's potential as objective function, is PLS-complete.

U
unilateral deviation neighborhood of \( s \in \times \)

is given by

\[ N(s) = \bigcup_i \{ (s'_i, s_{-i}) : s'_i \in S_i \} \]

e.g., all profiles that can be obtained by letting at most one player deviate to another strategy.

Reduction from Max-cut with FLIP neighborhoods.
Theorem

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- **Reduction from Max-cut with FLIP neighborhoods.**
Let $\mathcal{I} = (G, w)$ be an instance of max-cut with FLIP neighborhood on graph $G = (V, E)$, with edge-weight function $w$. Maximizing weight of cut edges is equivalent to minimizing weight of non-cut edges (also locally).
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Minimum uncut

Given undirected graph $G = (V, E)$ and weight function $w : E \rightarrow \mathbb{R}_{\geq 0}$, find partition $V = S \cup \bar{S}$ that minimizes $\sum_{\{i,j\} \in E : i,j \in S \text{ or } i,j \in \bar{S}} w_e$. 
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Let $\mathcal{I} = (G, w)$ be an instance of max-cut with FLIP neighborhood on graph $G = (V, E)$, with edge-weight function $w$.

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Why? For every cut $(T, \bar{T})$ it holds that

$$\sum_{\{i,j\} \in E: i \in T, j \in \bar{T}} w_{e} + \sum_{\{i,j\} \in E: i,j \in T \text{ or } i,j \in \bar{T}} w_{e} = \sum_{e \in E} w_{e}$$
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Nodes $a$, $b$, and $c$ with two resources $r_{ab}$ and $\bar{r}_{ab}$, $r_{bc}$ and $\bar{r}_{bc}$.

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Player $i \in V$ has two strategies ($S_i = \{t_i, \bar{t}_i\}$): 

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t_i = \{r_e\}_{e \in \delta(i)} \quad \text{and} \quad \bar{t}_i = \{\bar{r}_e\}_{e \in \delta(i)}
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![Diagram](image)
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**PNEs of game are precisely locally min-uncuts/max-cuts!**
Smoothed analysis (extra)
Smoothed analysis studies algorithmic problems under (small) perturbations of the input.

Max-cut with FLIP local search (informal)

For every $e \in E$, we introduce an (independent) random perturbation $\sigma_e \sim U[0, \phi]$, where $\phi$ is a parameter, and focus on instance with perturbed weights $w'_e = w_e + \sigma_e$.

Goal: Show that every sequence of local improvements converges to a local optimum in time polynomial in $n$ and $\phi$ (in perturbed instance).

If $\phi \to \infty$, we get completely random instance.

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Average-case analysis ($\phi \rightarrow \infty$);
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What is known for max-cut in the literature?

Theorem
Local search converges to a local optimum in at most $\phi n O(\log(n))$ steps for general graphs $G$; $\text{poly}(\phi, n)$ steps for complete graphs $G$; $\text{poly}(\phi, n)$ steps for graphs with $\Delta(G) = O(\log(n))$.

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Does (smoothed) local search for max-cut always converge in polynomial number of steps, for any graph $G$?
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