Topics in Algorithmic Game Theory and Economics

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Max Planck Institute for Informatics (D1)
Saarland Informatics Campus

December 2, 2020

Lecture 4
Finite games - Existence and Computation of MNE
Finite game

Finite game $\Gamma = (N, (S_i)_{i \in N}, (C_i)_{i \in N})$ consists of:

- Finite set $N$ of players.
- Finite strategy set $S_i$ for every player $i \in N$.
- Cost function $C_i: S_j \rightarrow \mathbb{R}$ for every $i \in N$.

Matching pennies:

Alice and Bob both choose side of a penny. $(a, b)$ denotes cost for Alice (A) and Bob (B) in given profile.

<table>
<thead>
<tr>
<th>Alice</th>
<th>Bob</th>
</tr>
</thead>
<tbody>
<tr>
<td>Head</td>
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</tr>
<tr>
<td>Tails</td>
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No PNE:

- $A - \rightarrow (Head, Tails)$
- $B - \rightarrow (Tails, Tails)$
- $A - \rightarrow (Head, Head)$
- $B - \rightarrow (Tails, Head)$

Game does have mixed Nash equilibrium (MNE). Both randomize over their strategies $\{\text{Head}, \text{Tails}\}$.

Mixed strategies $\sigma_A = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $\sigma_B = \left(\frac{1}{2}, \frac{1}{2}\right)$. 

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We focus on two-player games (for sake of notation).
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A mixed strategy is a probability distribution over \( S_i \) for \( i \in \{\text{Alice, Bob}\} \).
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- Interpretation: Alice plays strategy \( a_1 \) with prob. \( x_1 \), etc...
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**Example**

Strategies of Alice and Bob are given by:

\[
\Delta_{\text{Alice}} = \{ (x_1, x_2) : x_1 + x_2 = 1, x_1, x_2 \geq 0 \},
\]

\[
\Delta_{\text{Bob}} = \{ (y_1, y_2, y_3) : y_1 + y_2 + y_3 = 1, y_1, y_2, y_3 \geq 0 \}.
\]

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For \( x \in \Delta_A, y \in \Delta_B \), we get **product distribution** \( \sigma_{x,y} : S_A \times S_B \to [0, 1] \) over strategy profiles,
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**Example (cont’d)**

Distribution over strategy profiles is given by

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Then expected cost \( C_i(\sigma_{x,y}) = C_i(x, y) \), of \( i \in \{\text{Alice, Bob}\} \) is
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\[ C_i(x, y) = \mathbb{E}_{(a_k, b_\ell) \sim \sigma_{x,y}}[C_i(a_k, b_\ell)] = \sum_{(a_k, b_\ell) \in S_A \times S_B} x_k y_\ell C_i(a_k, b_\ell) \]
Matrix representation

Matrix representation of cost functions $C_i : \Delta_A \times \Delta_B \rightarrow \mathbb{R}$ for $i \in \{\text{Alice, Bob}\}$ given by $A, B \in \mathbb{R}^{m \times n}$ defined as
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Example (cont’d)

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 4 \end{pmatrix}.$$ 

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Matrix representation of cost functions $C_i : \Delta_A \times \Delta_B \rightarrow \mathbb{R}$ for $i \in \{\text{Alice, Bob}\}$ given by $A, B \in \mathbb{R}^{m \times n}$ defined as

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Example (cont’d)

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 4 \end{pmatrix}.$$ 

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Mixed Nash equilibrium
Hierarchy of equilibrium concepts

- **CCE**: Computationally tractable
- **CE**: Exists in any finite game, but hard to compute
- **MNE**: Exists in any finite game, but hard to compute
- **PNE**: Exists in any congestion game
Mixed Nash equilibrium (2-player case)

For two-player game \((A, B)\), we have

\[
C_A(x, y) = x^T Ay = \sum_{k=1}^{m} \sum_{\ell=1}^{n} A_{k\ell} x_k y_\ell, \quad C_B(x, y) = x^T By = \sum_{k=1}^{m} \sum_{\ell=1}^{n} B_{k\ell} x_k y_\ell
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Definition (Mixed Nash equilibrium)

Pair \((x^*, y^*) \in \Delta_A \times \Delta_B\) is **mixed Nash equilibrium (MNE)** if neither Alice nor Bob can deviate to other mixed strategy and improve cost:

\[
C_A(x^*, y^*) \leq C_A(x', y^*) \quad \forall x' \in \Delta_A
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For \(\epsilon > 0\), pair \((x^*, y^*)\) is \(\epsilon\)-approximate MNE (or simply \(\epsilon\)-MNE) if

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C_A(x^*, y^*) \leq C_A(x', y^*) + \epsilon \quad \forall x' \in \Delta_A
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Will see later that is suffices to have these conditions only for pure strategies:

- One strategy is played with probability 1.
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Example

Alice has $S_A = \{a_1, a_2\}$ and $S_B = \{b_1, b_2, b_3\}$.

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 3 & 3 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 4 & 2 \\ 2 & 0 & 4 \end{pmatrix}.$$
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Suppose that $x = (0.5, 0.5)$ and $y = (0.3, 0.4, 0.3)$, then

$$C_B(x, y) = x^T B y = (0.5 \ 0.5) \begin{pmatrix} 2 & 4 & 2 \\ 2 & 0 & 4 \end{pmatrix} \begin{pmatrix} 0.3 \\ 0.4 \\ 0.3 \end{pmatrix} = 2.3$$
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Is $(x, y)$ MNE? For $y' = (0.3, 0.7, 0)$, $C_B(x, y') = x^T By' = 2 < 2.3.$
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(Row) vector $x^T B = (2, 2, 3)^T$ gives (expected) cost for Bob per column.
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- Should only give positive probability to \( b_1, b_2 \) (given Alice plays \( x \)).
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(Row) vector $x^T B = (2, 2, 3)^T$ gives (expected) cost for Bob per column.

- Bob assigns positive probability to $b_3$: not optimal.
- Should only give positive probability to $b_1, b_2$ (given Alice plays $x$).

In MNE, players only have positive probability on rows/columns that minimize expected cost per row/column (given other’s strategy).
Column $b_j$ is best response against $x$ for Bob if $(x^T B)_j = \min_k (x^T B)_k.$
Definition

Column $b_j$ is **best response against** $x$ for Bob if $(x^T B)_j = \min_k (x^T B)_k$. Row $a_i$ is **best response against** $y$ for Alice if $(Ay)_i = \min_k (Ay)_k$. 

Example (cont'd)

An MNE is given by $x^* = (1, 0)$, $y^* = (0.5, 0, 0.5)$. $(x^* )^T B = (2, 4, 2)^T$. We have $y^* _1, y^* _3 > 0$ and $(x^T B)_1, (x^T B)_3$ are min. $Ay^* = (2, 2)$. We have $x^* _1 > 0$ and $(Ay^*)_1$ is minimum.
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Example (cont'd) An MNE is given by $x^* = (1, 0), y^* = (0.5, 0, 0.5)$. 

$(x^T B)_j$ is expected cost for Bob in column $j$ given Alice plays $x$.

$(Ay)_i$ is expected cost for Alice in row $i$ given Bob plays $y$. 

We have $y^* 1, y^* 3 > 0$ and $(x^T B)_1, (x^T B)_3$ are min.
Definition

Column $b_j$ is **best response** against $x$ for Bob if $(x^TB)_j = \min_k (x^TB)_k$. Row $a_i$ is **best response** against $y$ for Alice if $(Ay)_i = \min_k (Ay)_k$.

(E.g., if $x^TB = (7, 1, 3)^T$, then $(x^TB)_1 = 7, (x^TB)_2 = 1, (x^TB)_3 = 3$.)

- $(x^TB)_j$ is expected cost for Bob in column $j$ given Alice plays $x$.
- $(Ay)_i$ is expected cost for Alice in row $i$ given Bob plays $y$.

Definition (MNE, best response version)

Mixed strategies $(x^*, y^*)$ form MNE if Alice and Bob only assign positive probability to best responses.
Column $b_j$ is **best response** against $x$ for Bob if $(x^T B)_j = \min_k (x^T B)_k$.
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(E.g., if $x^T B = (7, 1, 3)^T$, then $(x^T B)_1 = 7, (x^T B)_2 = 1, (x^T B)_3 = 3$.)

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**Definition (MNE, best response version)**

Mixed strategies $(x^*, y^*)$ form MNE if Alice and Bob only assign positive probability to best responses. That is, pair $(x^*, y^*)$ is MNE if

- $x^*_i > 0 \implies (Ay^*)_i = \min_k (Ay^*)_k \quad \forall i = 1, \ldots, m,$
- $y^*_j > 0 \implies ((x^*)^T B)_j = \min_k ((x^*)^T B)_k \quad \forall j = 1, \ldots, n.$
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Column $b_j$ is **best response against** $x$ for Bob if $(x^T B)_j = \min_k (x^T B)_k$. Row $a_i$ is **best response against** $y$ for Alice if $(Ay)_i = \min_k (Ay)_k$.

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**Example (cont’d)**

An MNE is given by $x^* = (1, 0), y^* = (0.5, 0, 0.5)$. 

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An MNE is given by $x^* = (1, 0), y^* = (0.5, 0, 0.5)$.
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(10/31)
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Example (cont’d)

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Finally, we write $e^k \in \Delta_A$ for **pure strategy** in which Alice plays $a_k \in S_A$ with probability 1.
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e_j^k = \begin{cases} 
1 & \text{if } j = k \\
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$$e^k_j = \begin{cases} 
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- If Alice plays $e^k \in S_A$, then $C_A(e^k, y) = (e^k)^T Ay = (Ay)_k$. 

Analogous definitions for Bob. For Alice, one has $e^k \in \mathbb{R}^m$ and for Bob $e^\ell \in \mathbb{R}^n$. We abuse notation and do not always state the dimension of these vectors.

Definition (MNE, pure strategy version)

Mixed strategies $(x^*, y^*)$ form MNE if

$$x^*^T Ay^* \leq (e^k_i)^T Ay^* = 1, \ldots, m,$$

$$x^*^T By^* \leq (x^*^T A e^j_j = 1, \ldots, n).$$

That is, players both have no improving move to pure strategy. I.e., suffices to focus on pure strategies in definition on Slide 8.

Exercise: Prove that this definition is equivalent to that on Slide 8.
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That is, players both have no improving move to pure strategy.

- I.e., suffices to focus on pure strategies in definition on Slide 8.
- Exercise: Prove that this definition is equivalent to that on Slide 8.
A mixed strategy $\sigma_i : S_i \rightarrow [0, 1]$ of player $i \in N$ is a probability distribution over pure strategies in $S_i$, i.e., coming from

$$\Delta_i = \left\{ \tau : \tau(t) \geq 0 \ \forall t \in S_i \ \text{and} \ \sum_{t \in S_i} \tau(t) = 1 \right\}.$$
Definition (Mixed Nash equilibrium (MNE))

A mixed strategy \( \sigma_i : S_i \rightarrow [0, 1] \) of player \( i \in N \) is a probability distribution over pure strategies in \( S_i \), i.e., coming from

\[
\Delta_i = \left\{ \tau : \tau(t) \geq 0 \ \forall t \in S_i \ \text{and} \ \sum_{t \in S_i} \tau(t) = 1 \right\}.
\]

A collection of mixed strategies \( (\sigma_i)_{i \in N} \), with \( \sigma_i \in \Delta_i \), is a mixed Nash equilibrium if

\[
C_i(\sigma) := \mathbb{E}_{s \sim \sigma} [C_i(s)] \leq \mathbb{E}_{(s_{-i}) \sim (\sigma_{-i})} [C_i(s'_i, s_{-i})] \ \forall s'_i \in S_i.
\] (1)
Mixed Nash equilibrium (general)

**Definition (Mixed Nash equilibrium (MNE))**

A mixed strategy \( \sigma_i : S_i \to [0, 1] \) of player \( i \in N \) is a probability distribution over pure strategies in \( S_i \), i.e., coming from

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A collection of mixed strategies \((\sigma_i)_{i \in N}, \) with \( \sigma_i \in \Delta_i \), is a mixed Nash equilibrium if

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C_i(\sigma) := \mathbb{E}_{s \sim \sigma} [C_i(s)] \leq \mathbb{E}_{(s_i \sim (\sigma_i))} [C_i(s_i', s_{-i})] \ \forall s_i' \in S_i.
\]

(1)

Here

- \( \sigma : \times j S_j \to \mathbb{R}_{\geq 0} \) is given by \( \sigma(t) = \prod_j \sigma_j(t_j) \), and
- \( \sigma_{-i} : \times j \neq i S_j \rightarrow \mathbb{R}_{\geq 0} \) is given by \( \sigma_{-i}(t_{-i}) = \prod_{j \neq i} \sigma_j(t_j) \).
Existence and computational complexity
Existence ("Nobel" Prize in Economics in 1994)

Theorem (Nash’s theorem, 1950)

Any finite game $\Gamma$ has a mixed Nash equilibrium.
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Theorem (Nash’s theorem, 1950)

Any finite game $\Gamma$ has a mixed Nash equilibrium.

Theorem (Brouwer’s fixed point theorem)

Let $D \subseteq \mathbb{R}^m$ be compact and convex, and let $f : D \rightarrow D$ be a continuous function. Then there exists an $x^* \in D$ such that $f(x^*) = x^*$. 
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Let $D \subseteq \mathbb{R}^m$ be compact and convex, and let $f : D \rightarrow D$ be a continuous function. Then there exists an $x^* \in D$ such that $f(x^*) = x^*$.

Convex means that line segments between points are included in $D$. 

Convex

Not convex
Existence (“Nobel” Prize in Economics in 1994)

**Theorem (Nash’s theorem, 1950)**

*Any finite game* \( \Gamma \) *has a mixed Nash equilibrium.*

**Theorem (Brouwer’s fixed point theorem)**

*Let* \( D \subseteq \mathbb{R}^m \) *be compact and convex, and let* \( f : D \rightarrow D \) *be a continuous function. Then there exists an* \( x^* \in D \) *such that* \( f(x^*) = x^* \).*

**Compact** means **bounded** and **closed**.

- Satisfied by sets of mixed strategies \( \Delta_i \) that we will be looking at.
Existence ("Nobel" Prize in Economics in 1994)

Theorem (Nash’s theorem, 1950)

Any finite game $\Gamma$ has a mixed Nash equilibrium.

Theorem (Brouwer’s fixed point theorem)

Let $D \subseteq \mathbb{R}^m$ be compact and convex, and let $f : D \to D$ be a continuous function. Then there exists an $x^* \in D$ such that $f(x^*) = x^*$.

Brouwer’s theorem says that $f$ has a fixed point.
Existence ("Nobel" Prize in Economics in 1994)

Theorem (Nash’s theorem, 1950)

Any finite game $\Gamma$ has a mixed Nash equilibrium.

Theorem (Brouwer’s fixed point theorem)

Let $D \subseteq \mathbb{R}^m$ be compact and convex, and let $f : D \to D$ be a continuous function. Then there exists an $x^* \in D$ such that $f(x^*) = x^*$.

Brouwer’s theorem fails if $f$ is not continuous.
Proof of Nash’s theorem

Show that MNEs correspond to fixed points of some function. Brouwer’s theorem then gives existence (proof is not constructive).

Proof given for 2-player games. (To save on notation.)

Proof: Consider set $D = \Delta A \times \Delta B$. (Convex and compact.)

Remember $\Delta A = \{ (x_1, \ldots, x_m) : \sum_k x_k = 1, x_k \geq 0 \}$,

$\Delta B = \{ (y_1, \ldots, y_n) : \sum_\ell y_\ell = 1, y_\ell \geq 0 \}$.

For $(x, y) \in \Delta A \times \Delta B$, define

$$R_A, a_k(x, y) = \max \{ 0, C_A(x, y) - C_A(e_k, y) \} \quad k = 1, \ldots, m$$

$$R_B, b_\ell(x, y) = \max \{ 0, C_B(x, y) - C_B(x, e_\ell) \} \quad \ell = 1, \ldots, n$$

Note that the $R\cdot, \cdot(x, y)$ encode MNE as follows:

$$R_z, s_z(x, y) = 0 \quad \forall z \in \{A, B\} \forall s_z \in S_z \iff (x, y) \text{ is MNE}.$$ 

Exercise: Show that $R_z, s_z(x, y)$ is a continuous function.
Proof of Nash’s theorem

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Show that MNEs correspond to fixed points of some function. Brouwer’s theorem then gives existence (proof is not constructive).

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Consider set $D = \Delta_A \times \Delta_B$. (Convex and compact.)

Remember

$\Delta_A = \{ (x_1, \ldots, x_m) : \sum x_k = 1, x_k \geq 0 \}$,

$\Delta_B = \{ (y_1, \ldots, y_n) : \sum y_\ell = 1, y_\ell \geq 0 \}$.

For $(x, y) \in \Delta_A \times \Delta_B$, define

$R_A, a_k(x, y) = \max \{ 0, C_A(x, y) - C_A(e_k, y) \}$ for $k = 1, \ldots, m$,

$R_B, b_\ell(x, y) = \max \{ 0, C_B(x, y) - C_B(x, e_\ell) \}$ for $\ell = 1, \ldots, n$.

Note that the $R \cdot, \cdot(x, y)$ encode MNE as follows:

$R_z, s_z(x, y) = 0$ for all $z \in \{ A, B \}$ for all $s_z \in S_z$ if and only if $(x, y)$ is MNE.

Exercise: Show that $R_z, s_z(x, y)$ is a continuous function.
Proof of Nash’s theorem

Show that MNEs correspond to fixed points of some function. Brouwer’s theorem then gives existence (proof is not constructive).

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Proof of Nash’s theorem

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Proof: Consider set \( D = \Delta_A \times \Delta_B \).
Proof of Nash’s theorem

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**Proof:** Consider set $D = \Delta_A \times \Delta_B$. *(Convex and compact.)*
Proof of Nash’s theorem

Show that MNEs correspond to fixed points of some function. Brouwer’s theorem then gives existence (proof is not constructive).

- *Proof given for 2-player games. (To save on notation.)*

**Proof:** Consider set $D = \Delta_A \times \Delta_B$. (*Convex and compact.*)

- Remember $\Delta_A = \{(x_1, \ldots, x_m) : \sum_k x_k = 1, x_k \geq 0\}$,
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  \[
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\]

For $(x, y) \in \Delta_A \times \Delta_B$, 

$$
R_A, a_k(x, y) = \max\{0, C_A(x, y) - C_A(e_k, y)\} \quad k = 1, \ldots, m
$$

$$
R_B, b_\ell(x, y) = \max\{0, C_B(x, y) - C_B(x, e_\ell)\} \quad \ell = 1, \ldots, n
$$

Note that the $R \cdot s(z, s) (x, y)$ encode MNE as follows:

$$
R z, s z (x, y) = 0 \quad \forall z \in \{A, B\} \forall s z \in S z \iff (x, y) \text{ is MNE}.
$$

Exercise: Show that $R z, s z (x, y)$ is a continuous function.
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$$

For $(x, y) \in \Delta_A \times \Delta_B$, define

$$
R_{A,a_k}(x, y) = \max\{0, C_A(x, y) - C_A(e^k, y)\} \quad k = 1, \ldots, m
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Note that the \( R_{\cdot, \cdot}(x, y) \) encode MNE as follows:
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\end{align*}
\]

Note that the $R_{\cdot, \cdot}(x, y)$ encode MNE as follows:

\[
R_{z,s_z}(x, y) = 0 \quad \forall z \in \{A, B\} \quad \forall s_z \in S_z \quad \Leftrightarrow \quad (x, y) \text{ is MNE}.
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Proof of Nash’s theorem

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$$

Exercise: Show that $R_{z,s_z}(x, y)$ is a continuous function.
\[ \begin{align*}
R_{A,a_k}(x, y) &= \max\{0, C_A(x, y) - C_A(e^k, y)\} \quad k = 1, \ldots, m \\
R_{B,b_\ell}(x, y) &= \max\{0, C_B(x, y) - C_B(x, e^\ell)\} \quad \ell = 1, \ldots, n
\end{align*} \]
We use these functions to define mapping $f : \Delta_A \times \Delta_B \rightarrow \Delta_A \times \Delta_B$.
We use these functions to define mapping $f : \Delta_A \times \Delta_B \rightarrow \Delta_A \times \Delta_B$ by $f(x, y) = (x', y') = (x'_1, \ldots, x'_m, y'_1, \ldots, y'_n)$, where
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$$f(x, y) = (x', y') = (x'_1, \ldots, x'_m, y'_1, \ldots, y'_n),$$

where

$$x'_i := \frac{x_i + R_{A,a_i}(x, y)}{\sum_{k=1}^{m} x_k + R_{A,a_k}(x, y)}$$
\[ R_{A,a_k}(x, y) = \max\{0, C_A(x, y) - C_A(e^k, y)\} \quad k = 1, \ldots, m \]
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We use these functions to define mapping \( f : \Delta_A \times \Delta_B \to \Delta_A \times \Delta_B \) by
\[
 f(x, y) = (x', y') = (x'_1, \ldots, x'_m, y'_1, \ldots, y'_n), \text{ where }
\]
\[
 x'_i := \frac{x_i + R_{A,a_i}(x, y)}{\sum_{k=1}^m x_k + R_{A,a_k}(x, y)} = \frac{x_i + R_{A,a_i}(x, y)}{1 + \sum_{k=1}^m R_{A,a_k}(x, y)}
\]

Exercise: Show that \( f \) is a continuous function.

If \((x^*, y^*)\) is MNE, then
\[
 R_{z,s}(x, y) = 0 \quad \forall z \in \{A, B\} \quad \forall s \in S_z,
\]
and so \( x'_i = x^* \) and \( y'_i = y^* \).
In other words, \((x^*, y^*)\) is a fixed point of \( f \).
\begin{align*}
R_{A,a_k}(x, y) &= \max\{0, C_A(x, y) - C_A(e^k, y)\} \quad k = 1, \ldots, m \\
R_{B,b_\ell}(x, y) &= \max\{0, C_B(x, y) - C_B(x, e^\ell)\} \quad \ell = 1, \ldots, n
\end{align*}

We use these functions to define mapping \( f : \Delta_A \times \Delta_B \to \Delta_A \times \Delta_B \) by 
\( f(x, y) = (x', y') = (x'_1, \ldots, x'_m, y'_1, \ldots, y'_n) \), where

\[
x'_i := \frac{x_i + R_{A,a_i}(x, y)}{\sum_{k=1}^m x_k + R_{A,a_k}(x, y)} = \frac{x_i + R_{A,a_i}(x, y)}{1 + \sum_{k=1}^m R_{A,a_k}(x, y)} \quad i = 1, \ldots, m
\]
\[ R_{A,a_k}(x, y) = \max\{0, C_A(x, y) - C_A(e^k, y)\} \quad k = 1, \ldots, m \]
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We use these functions to define mapping \( f : \Delta_A \times \Delta_B \to \Delta_A \times \Delta_B \) by
\[
  f(x, y) = (x', y') = (x_1', \ldots, x_m', y_1', \ldots, y_n')
\]
where
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x_i' := \frac{x_i + R_{A,a_i}(x, y)}{\sum_{k=1}^{m} x_k + R_{A,a_k}(x, y)} = \frac{x_i + R_{A,a_i}(x, y)}{1 + \sum_{k=1}^{m} R_{A,a_k}(x, y)} \quad i = 1, \ldots, m
\]
and \( y' \in \Delta_2 \) by
\[
y_j' := \frac{y_j + R_{B,b_j}(x, y)}{\sum_{\ell=1}^{n} y_\ell + R_{B,b_\ell}(x, y)}
\]

Exercise: Show that \( f \) is a continuous function.

If \((x^*, y^*)\) is MNE, then \( R_{z,s_z}(x, y) = 0 \) \( \forall z \in \{A, B\} \forall s_z \in S_z \), and so \( x' = x^* \) and \( y' = y^* \). In other words, \((x^*, y^*)\) is fixed point of \( f \).
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\[
    f(x, y) = (x', y') = (x'_1, \ldots, x'_m, y'_1, \ldots, y'_n),
\]

where

\[
    x'_i := \frac{x_i + R_{A,a_i}(x, y)}{\sum_{k=1}^{m} x_k + R_{A,a_k}(x, y)} = \frac{x_i + R_{A,a_i}(x, y)}{1 + \sum_{k=1}^{m} R_{A,a_k}(x, y)} \quad i = 1, \ldots, m
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\]
Exercise: Show that \( f \) is a continuous function.
\[ R_{A,a_k}(x, y) = \max \{0, C_A(x, y) - C_A(e^k, y)\} \quad k = 1, \ldots, m \]
\[ R_{B,b_\ell}(x, y) = \max \{0, C_B(x, y) - C_B(x, e^\ell)\} \quad \ell = 1, \ldots, n \]

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\[
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y'_j & := \frac{y_j + R_{B,b_j}(x, y)}{\sum_{\ell=1}^n y_{\ell} + R_{B,b_\ell}(x, y)} = \frac{y_j + R_{B,b_j}(x, y)}{1 + \sum_{\ell=1}^n R_{B,b_\ell}(x, y)} & j = 1, \ldots, n
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We use these functions to define mapping \( f : \Delta_A \times \Delta_B \to \Delta_A \times \Delta_B \) by
\[
(f(x, y))(i, j) = (x'_i, y'_j) = \left(x'_1, \ldots, x'_m, y'_1, \ldots, y'_n\right),
\]
where
\[
x'_i := \frac{x_i + R_{A,a_i}(x, y)}{\sum_{k=1}^m x_k + R_{A,a_k}(x, y)} = \frac{x_i + R_{A,a_i}(x, y)}{1 + \sum_{k=1}^m R_{A,a_k}(x, y)} \quad i = 1, \ldots, m
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and
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y'_j := \frac{y_j + R_{B,b_j}(x, y)}{\sum_{\ell=1}^n y_\ell + R_{B,b_\ell}(x, y)} = \frac{y_j + R_{B,b_j}(x, y)}{1 + \sum_{\ell=1}^n R_{B,b_\ell}(x, y)} \quad j = 1, \ldots, n
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Exercise: Show that \( f \) is a continuous function.

If \( (x^*, y^*) \) is MNE, then \( R_{z,s_z}(x, y) = 0 \quad \forall z \in \{A, B\} \forall s_z \in S_z \),
\[
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If \((x^*, y^*)\) is MNE, then \( R_{z,s_z}(x, y) = 0 \ \forall z \in \{A, B\} \ \forall s_z \in S_z \), and so \( x' = x^* \) and \( y' = y^* \).
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and
\[ y_j' := \frac{y_j + R_{B,b_j}(x, y)}{\sum_{\ell=1}^{n} y_\ell + R_{B,b_\ell}(x, y)} = \frac{y_j + R_{B,b_j}(x, y)}{1 + \sum_{\ell=1}^{n} R_{B,b_\ell}(x, y)} \quad j = 1, \ldots, n \]

Exercise: Show that \( f \) is a continuous function.

If \((x^*, y^*)\) is MNE, then \( R_{z,s_z}(x, y) = 0 \forall z \in \{A, B\} \forall s_z \in S_z \), and so \( x' = x^* \) and \( y' = y^* \). In other words, \((x^*, y^*)\) is fixed point of \( f \).
Other direction remains: If \((x^*, y^*)\) is fixed point of \(f\), then it is MNE.
Other direction remains: If \((x^*, y^*)\) is fixed point of \(f\), then it is MNE. Suffices to show that \(R_{z,s_z}(x, y) = 0\) \(\forall z \in \{A, B\} \forall s_z \in S_z\).
Other direction remains: If \((x^*, y^*)\) is fixed point of \(f\), then it is MNE. 

Suffices to show that \(R_{z,s_z}(x, y) = 0\) \(\forall z \in \{A, B\} \ \forall s_z \in S_z\).

\[
R_{A,a_i}(x, y) = \max\{0, C_A(x, y) - C_A(e^i, y)\} \quad i = 1, \ldots, m
\]

\[
x'_i := \frac{x_i + R_{A,a_i}(x, y)}{1 + \sum_{k=1}^{m} R_{A,a_k}(x, y)} \quad i = 1, \ldots, m
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\]

Note that

\[
C_A(x, y) = \sum_{k} x_k C_A(e^k, y)
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\[
\begin{align*}
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x_i' &= \frac{x_i + R_{A,a_i}(x, y)}{1 + \sum_{k=1}^{m} R_{A,a_k}(x, y)} \quad i = 1, \ldots, m
\end{align*}
\]

Note that

\[
C_A(x, y) = \sum_k x_k C_A(e^k, y) \leq \max_{k: x_k > 0} C_A(e^k, y) \sum_k x_k
\]
Other direction remains: If \((x^*, y^*)\) is fixed point of \(f\), then it is MNE. Suffices to show that \(R_{z,sz}(x, y) = 0 \ \forall z \in \{A, B\} \ \forall sz \in S_z\). 

\[
R_{A,a_i}(x, y) = \max\{0, C_A(x, y) - C_A(e^i, y)\} \quad i = 1, \ldots, m
\]

\[
x'_i := \frac{x_i + R_{A,a_i}(x, y)}{1 + \sum_{k=1}^m R_{A,a_k}(x, y)} \quad i = 1, \ldots, m
\]

Note that

\[
C_A(x, y) = \sum_k x_k C_A(e^k, y) \leq \max_{k: x_k > 0} C_A(e^k, y) \sum_k x_k = \max_{k: x_k > 0} C_A(e^k, y)
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Other direction remains: If \((x^*, y^*)\) is fixed point of \(f\), then it is MNE. Suffices to show that \(R_{z,s_z}(x, y) = 0\ \forall z \in \{A, B\} \ \forall s_z \in S_z\).

\[
R_{A,a_i}(x, y) = \max\{0, C_A(x, y) - C_A(e^i, y)\} \quad i = 1, \ldots, m
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\[
x'_i := \frac{x_i + R_{A,a_i}(x, y)}{1 + \sum_{k=1}^{m} R_{A,a_k}(x, y)} \quad i = 1, \ldots, m
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C_A(x, y) = \sum_{k} x_k C_A(e^k, y) \leq \max_{k: x_k > 0} C_A(e^k, y) \sum_{k} x_k = \max_{k: x_k > 0} C_A(e^k, y)
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Let us look at \(x_i'\) for fixed point \((x^*, y^*)\):

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Let us look at \(x^\prime_i\) for fixed point \((x^*, y^*)\):

- \(x^*_i = \frac{x^*_i}{1 + \sum_{k=1}^m R_{A,a_k}(x^*, y^*)} \quad x^*_i > 0 \iff \frac{1}{1 + \sum_{k=1}^m R_{A,a_k}(x^*, y^*)} = 1\)
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- This gives \(\sum_{k=1}^{m} R_{A, a_k}(x^*, y^*) = 0\).
- \(R_{A, a_k}\) is always non-negative \(\Rightarrow R_{A, a_k}(x^*, y^*) = 0\) for \(k = 1, \ldots, m\).
Theorem (Nash’s theorem, 1950)

Any finite game \( \Gamma \) has a mixed Nash equilibrium.

Can we compute an MNE efficiently?

Assuming cost functions are rational (think of \( A, B \in \mathbb{Q}^{m \times n} \)), MNE is always rational when \( n = 2 \), but MNE can be irrational when \( n \geq 3 \).

Irrational numbers are, e.g., \( \pi \), \( e \) (Euler’s number), etc.

(Context: Suppose \( f(z) = z^2 + z - 2 \), then \( f(z) = z \) is solved by \( z^* = \pm \sqrt{2} \).)

For \( n \geq 3 \), Rational \( \epsilon \)-approximate MNE still exists for any \( \epsilon > 0 \).

Algorithms are known to compute approx. equilibrium. E.g., Scarf’s algorithm (1967) for approximating fixed points.

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Computing MNE will be referred to as problem \( \text{NASH} \).

Some (informal) intuition

Consider function/search problem version of NP:
For problem X, decide whether solution exists. If YES, output one.

Is \( \text{NASH} \) NP-complete?
Not likely.

“Deciding” whether Nash equilibrium exists is trivial.
\( \text{NASH} \) is complete for complexity class PPAD (already for $n = 2$).

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*How to study computational complexity of MNE in 2-player games?*
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Theorem (Chen and Deng, 2006)

Computing MNE in 2-player games is PPAD-complete

- Same is true for approximate equilibria when $n \geq 3$. 

Theorem (Lipton, Markakis and Mehta, 2003)

There is an $O^*(n^{24} \log(n)/\epsilon^2)$ algorithm known for computing $\epsilon$-approximate MNE in 2-player game. Quasi-polynomial in $n$.

Theorem (Rubinstein, 2016)

There exists a constant $\epsilon > 0$ such that, assuming the "Exponential Time Hypothesis for PPAD", computing $\epsilon$-approximate MNE in 2-player game requires time at least $n \log(1 - o(1))^{\Omega(n)}$. 

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Theorem (Lipton, Markakis and Mehta, 2003)

There is an \( O^*(n^{24 \log(n)/\epsilon^2}) \) algorithm known for computing \( \epsilon \)-approximate MNE in 2-player game.

- Quasi-polynomial in \( n \).

Theorem (Rubinstein, 2016)

There exists a constant \( \epsilon > 0 \) such that, assuming the “Exponential Time Hypothesis for PPAD”, computing \( \epsilon \)-approximate MNE in 2-player game requires time at least \( n^{\log^{1-o(1)}(n)} \).
Two-player zero-sum games
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Two-player game is called zero-sum if $A + B = 0$, i.e., $A = -B$. 
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- Row player (Alice) tries to maximize utility $x^T Cy$;
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Think of it as that Bob has to pay $x^T Cy$ to Alice.
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Algorithmic aspects of MNE:
- Can be modeled as optimal solution of linear program (LP).
  - Solvable in polynomial time.
  - *(Any LP can be written as zero-sum game as well.)*
- Certain player dynamics can “learn” it: Fictitious Play
  - Holds for more classes of games, but not in general.
Value of zero-sum game

What can Alice guarantee to get from Bob?
Value of zero-sum game

What can Alice guarantee to get from Bob?
- Suppose Alice plays mixed strategy $x$.

Theorem (Von Neumann, 1928)
Consider a two-player zero-sum game given by matrix $C$. Then

$$v_A = \max_x \min_y x^T Cy = \min_y \max_x x^T Cy = v_B.$$ 

The number $v = v_A = v_B$ is called the value of the game.

Often referred to as the "Minimax theorem".
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- Suppose Alice plays mixed strategy $x$. What should Bob do?

Choose $y$ such that $x^T Cy$ is minimal, i.e., strategy attaining $\min_{y \in \Delta_B} x^T Cy$.

So what should Alice do?

Choose $x$ maximizing $\min_{y \in \Delta_B} x^T Cy$.

Alice can guarantee to get $v_A = \max_x \min_y x^T Cy$.

Similarly, Bob can guarantee to pay no more than $v_B = \min_y \max_x x^T Cy$.

Exercise: Show that $v_A \leq v_B$.

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$(x^*, y^*)$ is MNE if and only $x^*$ optimal for Alice and $y^*$ optimal for Bob.
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Any MNE yield the same utility/loss for Alice/Bob, namely $v = v_A = v_B$. 

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- Exercise: Prove these corollaries
Two-player zero-sum games
Computing MNE using linear programming
Optimal strategy $x^*$ for Alice is solution to optimization problem.
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$$\max \min_y x^T Cy$$

subject to $x \in \Delta_A$

The dual of this program precisely computes optimal strategy for Bob!

In fact, strong duality can be used to prove the minimax theorem.

Theorem MNE can be computed in polynomial time in $2$-player zero-sum game.
LP formulation for optimal strategy

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- We assume that the $C$ is $m \times n$ matrix, i.e., $m$ rows, $n$ columns.

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$$\sum_{i=1}^m x_i = 1$$

$$x_i \geq 0 \quad i = 1, \ldots, m$$

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\end{align*}$$

- For any fixed $x$, the number $\min_y x^T Cy$ is attained for some pure strategy $e^k$ for $k = 1, \ldots, n$, where

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(\textit{Note that } C e^k \text{ is precisely the } k\text{-th column of the matrix } C. )
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**Theorem**

MNE can be computed in polynomial time in 2-player zero-sum game.
Two-player zero-sum games

Fictitious play
Simultaneous fictitious play (Brown, 1951)

*Introduced as algorithm for approximating value of zero-sum game.*

Game is played *repeatedly.*
Simultaneous fictitious play (Brown, 1951)

*Introduced as algorithm for approximating value of zero-sum game.* Game is played *repeatedly*. In every round:

\[ S_A = \{ a_1, \ldots, a_m \} \text{ (rows)} \quad \text{and} \quad S_B = \{ b_1, \ldots, b_n \} \text{ (columns)}. \]

**Definition (Empirical distribution)**

Let \( r_t \) be row chosen by Alice in step \( t = 1, \ldots, T - 1 \).

Empirical distribution over \( S_A \) in round \( t \) is given by

\[ \bar{x}_i(t) = \frac{|\{ j : r_j = a_i, 1 \leq j \leq t-1 \}|}{t-1} \]

for \( i = 1, \ldots, m \).

(Fraction of rounds in which Alice chose row \( i \).)

Analogous definition for Bob (with chosen column \( c_t \) in round \( t \)).
Simultaneous fictitious play (Brown, 1951)

*Introduced as algorithm for approximating value of zero-sum game.*

Game is played *repeatedly*. In every round:

- Alice (A) and Bob (B) play a pure strategy.
Simultaneous fictitious play (Brown, 1951)

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Game is played *repeatedly*. In every round:

- Alice (A) and Bob (B) play a pure strategy.
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Game is played **repeatedly**. In every round:

- Alice (A) and Bob (B) play a pure strategy.
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*Informally speaking, empirical distributions “converge” to MNE.*
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Let $S_A = \{a_1, \ldots, a_m\}$ (rows) and $S_B = \{b_1, \ldots, b_n\}$ (columns).
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Let $S_A = \{a_1, \ldots, a_m\}$ (rows) and $S_B = \{b_1, \ldots, b_n\}$ (columns).

**Definition (Empirical distribution)**

Let $r_t$ be row chosen by Alice in step $t = 1, \ldots, T - 1$. 
Simultaneous fictitious play (Brown, 1951)

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Game is played *repeatedly*. In every round:

- Alice (A) and Bob (B) play a pure strategy.
- They base their decision on *history* of the other player.
  - Choose best response w.r.t. empirical distribution (so far) of strategies chosen by the other.

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Game is played repeatedly. In every round:
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- Analogous definition for Bob (with chosen column $c_t$ in round $t$).
Example

Suppose the matrix $C$ has $n = 6$ rows, and that Alice plays $(a_1, a_1, a_4, a_6, a_4, a_5, a_2, a_3, a_4)$ in first $t - 1 = 9$ rounds. Then

$$
\bar{x}(t) = \bar{x}(10) = \frac{1}{9} (2, 1, 1, 3, 1, 1) = \left(\frac{2}{9}, \frac{1}{9}, \frac{1}{9}, \frac{3}{9}, \frac{1}{9}, \frac{1}{9}\right).
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Suppose the matrix $C$ has $n = 6$ rows, and that Alice plays $(a_1, a_1, a_4, a_6, a_4, a_5, a_2, a_3, a_4)$ in first $t - 1 = 9$ rounds. Then

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The idea of fictitious play is that Alice believes Bob plays every round according to some (unknown to her) probability distribution $y$. 
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- She uses empirical distribution $\bar{y}(t)$ as guess for $y$ in step $t$.
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The idea of fictitious play is that Alice believes Bob plays every round according to some (unknown to her) probability distribution $y$.

- She uses empirical distribution $\bar{y}(t)$ as guess for $y$ in step $t$.
- Alice chooses best response row $r_t \in S_A$ with respect to $\bar{y}(t)$:

$$r_t \in \arg\max_j \{(e^i)^T C \bar{y}(t) : i = 1, \ldots, m\}.$$
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Suppose the matrix $C$ has $n = 6$ rows, and that Alice plays $(a_1, a_1, a_4, a_6, a_4, a_5, a_2, a_3, a_4)$ in first $t - 1 = 9$ rounds. Then 
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The idea of fictitious play is that Alice believes Bob plays every round according to some (unknown to her) probability distribution $y$.

- She uses empirical distribution $\bar{y}(t)$ as guess for $y$ in step $t$.
- Alice chooses best response row $r_t \in S_A$ with respect to $\bar{y}(t)$:
  \[
  r_t \in \text{argmax}_j \{ (e^i)^T C\bar{y}(t) : i = 1, \ldots, m \}.
  \]

Bob is doing the same w.r.t Alice (for unknown distribution $x$).
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- He uses empirical distribution $\bar{x}(t)$ as guess for $x$ in step $t$.
- Bob chooses best response column $c_t \in S_B$ with respect to $\bar{x}(t)$:
Example

Suppose the matrix $C$ has $n = 6$ rows, and that Alice plays $(a_1, a_1, a_4, a_6, a_4, a_5, a_2, a_3, a_4)$ in first $t - 1 = 9$ rounds. Then

$$\bar{x}(t) = \bar{x}(10) = \frac{1}{9} (2, 1, 1, 3, 1, 1) = \left(\frac{2}{9}, \frac{1}{9}, \frac{1}{9}, \frac{3}{9}, \frac{1}{9}, \frac{1}{9}\right).$$

The idea of fictitious play is that Alice believes Bob plays every round according to some (unknown to her) probability distribution $y$.

- She uses empirical distribution $\bar{y}(t)$ as guess for $y$ in step $t$.
- Alice chooses best response row $r_t \in S_A$ with respect to $\bar{y}(t)$:
  $$r_t \in \arg\max_i \{(e^i)^T C \bar{y}(t) : i = 1, \ldots, m\}.$$  

Bob is doing the same w.r.t Alice (for unknown distribution $x$).

- He uses empirical distribution $\bar{x}(t)$ as guess for $x$ in step $t$.
- Bob chooses best response column $c_t \in S_B$ with respect to $\bar{x}(t)$:
  $$c_t \in \arg\min_j \{\bar{x}(t)^T Ce^j : j = 1, \ldots, n\}.$$
Example

Suppose the matrix $C$ has $n = 6$ rows, and that Alice plays $(a_1, a_1, a_4, a_6, a_4, a_5, a_2, a_3, a_4)$ in first $t - 1 = 9$ rounds. Then

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The idea of fictitious play is that Alice believes Bob plays every round according to some (unknown to her) probability distribution $y$.

- She uses empirical distribution $\bar{y}(t)$ as guess for $y$ in step $t$.
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- He uses empirical distribution $\bar{x}(t)$ as guess for $x$ in step $t$.
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$$c_t \in \arg\min_j \{ \bar{x}(t)^T C e^j : j = 1, \ldots, n\}.$$
ALGORITHM 1: Fictitious play (with index tie-breaking rule)

**Input**: $m \times n$ matrix $C$; initial row $r$, column $c$; round total $T \in \mathbb{N}$.

**Output**: Empirical distributions $\bar{x}(T), \bar{y}(T)$.

\[
\bar{x}(1) = e_r \text{ and } \bar{y}(1) = e_c.
\]

for $t = 2, \ldots, T$ do

Choose $r_t \in \text{argmax}\{(e^i)^T C \bar{y}(t) : i = 1, \ldots, m\}$

Choose $c_t \in \text{argmin}\{\bar{x}(t)^T C e^j : j = 1, \ldots, n\}$

(Choose lowest indexed row/column in case of multiple best responses.)

Update empirical distributions $(\bar{x}(t), \bar{y}(t))$ to $(\bar{x}(t + 1), \bar{y}(t + 1))$

end

return $\bar{x}(T), \bar{y}(T)$
Fictitious play algorithm

**ALGORITHM 2:** Fictitious play (with index tie-breaking rule)

**Input:** $m \times n$ matrix $C$; initial row $r$, column $c$; round total $T \in \mathbb{N}$.

**Output:** Empirical distributions $\bar{x}(T), \bar{y}(T)$.

$\bar{x}(1) = e_r$ and $\bar{y}(1) = e_c$.

for $t = 2, \ldots, T$ do

Choose $r_t \in \arg\max\{(e^i)^T C \bar{y}(t) : i = 1, \ldots, m\}$

Choose $c_t \in \arg\min\{\bar{x}(t)^T C e^j : j = 1, \ldots, n\}$

*(Choose lowest indexed row/column in case of multiple best responses.)*

Update empirical distributions $(\bar{x}(t), \bar{y}(t))$ to $(\bar{x}(t + 1), \bar{y}(t + 1))$

end

return $\bar{x}(T), \bar{y}(T)$

Observe that we specify a **tie-breaking rule** that decides which column/row to choose, in case there are multiple best responses.
Theorem (Robinson, 1951)

Utility/cost of Alice/Bob converges to value $v$ of the game.

That is, as $t \to \infty$, it holds that

$$\max_i (e_i) C \bar{y}(t) \to v,$$
$$\min_j \bar{x}(t) T C e_j \to v,$$
$$\bar{x}(t) T C \bar{y}(t) \to v.$$ 

Empirical distributions $(\bar{x}(t), \bar{y}(t))$ "converge" to MNE as $t \to \infty$.

Convergence in the sense that $(\bar{x}(t), \bar{y}(t))$ is $\epsilon(t)$-approximate equilibrium, where $\epsilon(t) \to 0$ as $t \to \infty$.

Convergence time of Fictitious Play still not fully understood!

Some notes on fictitious play

Simple way to compute value and $\epsilon$-MNE.

Avoiding the need to solve LPs.

Players do not need to know each other's empirical distribution.

Alice only needs to know vector $(C \bar{y}(t))$ in round $t$.

Bob only needs to know (row) vector $(\bar{x}(t) T C)$ in round $t$.

Fictitious play can be defined for any two-player game $(A, B)$.

Convergence fails beyond zero-sum games.
Theorem (Robinson, 1951)

Utility/cost of Alice/Bob converges to value \( v \) of the game. That is, as \( t \to \infty \), it holds that

\[
\max_i (e_i) C \bar{y}(t) \to v,
\]
\[
\min_j \bar{x}(t) T C e_j \to v,
\]
and
\[
\bar{x}(t) T C \bar{y}(t) \to v.
\]

Empirical distributions \((\bar{x}(t), \bar{y}(t))\) "converge" to MNE as \( t \to \infty \).

Convergence in the sense that \((\bar{x}(t), \bar{y}(t))\) is \( \epsilon(t) \)-approximate equilibrium, where \( \epsilon(t) \to 0 \) as \( t \to \infty \).

Convergence time of Fictitious Play still not fully understood!

Some notes on fictitious play

Simple way to compute value and \( \epsilon \)-MNE.

Avoiding the need to solve LPs.

Players do not need to know each other's empirical distribution.

Alice only needs to know vector \((C \bar{y}(t))\) in round \( t \).

Bob only needs to know (row) vector \((\bar{x}(t) T C)\) in round \( t \).

Fictitious play can be defined for any two-player game \((A, B)\).

Convergence fails beyond zero-sum games.
Theorem (Robinson, 1951)

Utility/cost of Alice/Bob converges to value $v$ of the game. That is, as $t \to \infty$, it holds that

$$\max_i(e^i)C\tilde{y}(t) \to v,$$

Empirical distributions $(\tilde{x}(t), \tilde{y}(t))$ "converge" to MNE as $t \to \infty$.

Convergence in the sense that $(\tilde{x}(t), \tilde{y}(t))$ is $\epsilon(t)$-approximate equilibrium, where $\epsilon(t) \to 0$ as $t \to \infty$.

Convergence time of Fictitious Play still not fully understood!

Some notes on fictitious play

Simple way to compute value and $\epsilon$-MNE. Avoiding the need to solve LPs. Players do not need to know each other's empirical distribution. Alice only needs to know vector $(C\tilde{y}(t))$ in round $t$. Bob only needs to know (row) vector $(\tilde{x}(t)^TC)$ in round $t$.

Fictitious play can be defined for any two-player game $(A, B)$.

Convergence fails beyond zero-sum games.
Theorem (Robinson, 1951)

Utility/cost of Alice/Bob converges to value $v$ of the game. That is, as $t \to \infty$, it holds that

$$\max_i (e^i) C\bar{y}(t) \to v, \quad \min_j \bar{x}(t)^T Ce_j \to v,$$
Theorem (Robinson, 1951)

Utility/cost of Alice/Bob converges to value $v$ of the game. That is, as $t \to \infty$, it holds that

$$\max_i (e^i) C \bar{y}(t) \to v, \quad \min_j \bar{x}(t)^T C e_j \to v, \quad \text{and} \quad \bar{x}(t)^T C \bar{y}(t) \to v.$$
Theorem (Robinson, 1951)

Utility/cost of Alice/Bob converges to value $v$ of the game. That is, as $t \to \infty$, it holds that

$$\max_i (\mathbf{e}_i) C \bar{y}(t) \to v, \quad \min_j \bar{x}(t)^T C \mathbf{e}_j \to v, \quad \text{and} \quad \bar{x}(t)^T C \bar{y}(t) \to v.$$  

Empirical distributions $(\bar{x}(t), \bar{y}(t))$ “converge” to MNE as $t \to \infty$. 

Theorem (Robinson, 1951)

Utility/cost of Alice/Bob converges to value $v$ of the game. That is, as $t \to \infty$, it holds that

- $\max_i (e^i) C \tilde{y}(t) \to v$, $\min_j \bar{x}(t)^T C e_j \to v$, and $\bar{x}(t)^T C \tilde{y}(t) \to v$.

Empirical distributions $(\bar{x}(t), \tilde{y}(t))$ “converge” to MNE as $t \to \infty$.
- Convergence in the sense that $(\bar{x}(t), \tilde{y}(t))$ is $\epsilon(t)$-approximate equilibrium, where $\epsilon(t) \to 0$ as $t \to \infty$. 
Theorem (Robinson, 1951)

Utility/cost of Alice/Bob converges to value $v$ of the game. That is, as $t \to \infty$, it holds that

$$\max_i (e^i) C\bar{y}(t) \to v, \quad \min_j \bar{x}(t)^T C e_j \to v, \quad \text{and} \quad \bar{x}(t)^T C\bar{y}(t) \to v.$$ 

Empirical distributions $(\bar{x}(t), \bar{y}(t))$ “converge” to MNE as $t \to \infty$.

- Convergence in the sense that $(\bar{x}(t), \bar{y}(t))$ is $\epsilon(t)$-approximate equilibrium, where $\epsilon(t) \to 0$ as $t \to \infty$.

Convergence time of Fictitious Play still not fully understood!
Theorem (Robinson, 1951)

Utility/cost of Alice/Bob converges to value \( v \) of the game. That is, as \( t \to \infty \), it holds that

\[
\max_i(e^i) C \bar{y}(t) \to v, \quad \min_j \bar{x}(t)^T C e_j \to v, \quad \text{and} \quad \bar{x}(t)^T C \bar{y}(t) \to v.
\]

Empirical distributions \((\bar{x}(t), \bar{y}(t))\) “converge” to MNE as \( t \to \infty \).

- Convergence in the sense that \((\bar{x}(t), \bar{y}(t))\) is \( \epsilon(t) \)-approximate equilibrium, where \( \epsilon(t) \to 0 \) as \( t \to \infty \).

Convergence time of Fictitious Play still not fully understood!

Some notes on fictitious play
Theorem (Robinson, 1951)

Utility/cost of Alice/Bob converges to value $v$ of the game. That is, as $t \to \infty$, it holds that

- $\max_i(e^i)C\bar{y}(t) \to v$, $\min_j \bar{x}(t)^TCe_j \to v$, and $\bar{x}(t)^TC\bar{y}(t) \to v$.

Empirical distributions $(\bar{x}(t), \bar{y}(t))$ “converge” to MNE as $t \to \infty$.

- Convergence in the sense that $(\bar{x}(t), \bar{y}(t))$ is $\epsilon(t)$-approximate equilibrium, where $\epsilon(t) \to 0$ as $t \to \infty$.

*Convergence time of Fictitious Play still not fully understood!*

Some notes on fictitious play

- Simple way to compute value and $\epsilon$-MNE.
Theorem (Robinson, 1951)

Utility/cost of Alice/Bob converges to value \( v \) of the game. That is, as \( t \to \infty \), it holds that

- \( \max_i (e_i^T) C \bar{y}(t) \to v \),
- \( \min_j \bar{x}(t)^T C e_j \to v \),
- and \( \bar{x}(t)^T C \bar{y}(t) \to v \).

Empirical distributions \((\bar{x}(t), \bar{y}(t))\) “converge” to MNE as \( t \to \infty \).

- Convergence in the sense that \((\bar{x}(t), \bar{y}(t))\) is \( \epsilon(t) \)-approximate equilibrium, where \( \epsilon(t) \to 0 \) as \( t \to \infty \).

Convergence time of Fictitious Play still not fully understood!

Some notes on fictitious play

- Simple way to compute value and \( \epsilon \)-MNE.
  - Avoiding the need to solve LPs.
Theorem (Robinson, 1951)

Utility/cost of Alice/Bob converges to value $v$ of the game. That is, as $t \to \infty$, it holds that

- $\max_i (e^i) C\bar{y}(t) \to v$, $\min_j \bar{x}(t)^T C e_j \to v$, and $\bar{x}(t)^T C\bar{y}(t) \to v$.

Empirical distributions $(\bar{x}(t), \bar{y}(t))$ “converge” to MNE as $t \to \infty$.

- Convergence in the sense that $(\bar{x}(t), \bar{y}(t))$ is $\epsilon(t)$-approximate equilibrium, where $\epsilon(t) \to 0$ as $t \to \infty$.

Convergence time of Fictitious Play still not fully understood!

Some notes on fictitious play

- Simple way to compute value and $\epsilon$-MNE.
  - Avoiding the need to solve LPs.
- Players do not need to know each other’s empirical distribution.
Theorem (Robinson, 1951)

Utility/cost of Alice/Bob converges to value $v$ of the game. That is, as $t \to \infty$, it holds that

\[ \max_i (e^i) C \bar{y}(t) \to v, \quad \min_j \bar{x}(t)^T C e_j \to v, \quad \text{and} \quad \bar{x}(t)^T C \bar{y}(t) \to v. \]

Empirical distributions $(\bar{x}(t), \bar{y}(t))$ “converge” to MNE as $t \to \infty$.

- Convergence in the sense that $(\bar{x}(t), \bar{y}(t))$ is $\epsilon(t)$-approximate equilibrium, where $\epsilon(t) \to 0$ as $t \to \infty$.

Convergence time of Fictitious Play still not fully understood!

Some notes on fictitious play

- Simple way to compute value and $\epsilon$-MNE.
  - Avoiding the need to solve LPs.
- Players do not need to know each other’s empirical distribution.
  - Alice only needs to know vector $(C \bar{y}(t))$ in round $t$. 
Theorem (Robinson, 1951)

Utility/cost of Alice/Bob converges to value \( v \) of the game. That is, as \( t \to \infty \), it holds that
\[
\max_i (e^i) C\bar{y}(t) \to v, \ \min_j \bar{x}(t)^T Ce_j \to v, \ \text{and} \ \bar{x}(t)^T C\bar{y}(t) \to v.
\]

Empirical distributions \((\bar{x}(t), \bar{y}(t))\) “converge” to MNE as \( t \to \infty \).
- Convergence in the sense that \((\bar{x}(t), \bar{y}(t))\) is \( \epsilon(t) \)-approximate equilibrium, where \( \epsilon(t) \to 0 \) as \( t \to \infty \).

Convergence time of Fictitious Play still not fully understood!

Some notes on fictitious play
- Simple way to compute value and \( \epsilon \)-MNE.
  - Avoiding the need to solve LPs.
- Players do not need to know each other’s empirical distribution.
  - Alice only needs to know vector \((C\bar{y}(t))\) in round \( t \).
  - Bob only needs to know (row) vector \((\bar{x}(t)^T C)\) in round \( t \).
Theorem (Robinson, 1951)

Utility/cost of Alice/Bob converges to value $v$ of the game. That is, as $t \to \infty$, it holds that

$$\max_i (e^i)^T C \tilde{y}(t) \to v, \min_j \tilde{x}(t)^T C e_j \to v, \text{ and } \tilde{x}(t)^T C \tilde{y}(t) \to v.$$

Empirical distributions $(\tilde{x}(t), \tilde{y}(t))$ “converge” to MNE as $t \to \infty$.

- Convergence in the sense that $(\tilde{x}(t), \tilde{y}(t))$ is $\epsilon(t)$-approximate equilibrium, where $\epsilon(t) \to 0$ as $t \to \infty$.

Convergence time of Fictitious Play still not fully understood!

Some notes on fictitious play
- Simple way to compute value and $\epsilon$-MNE.
  - Avoiding the need to solve LPs.
- Players do not need to know each other’s empirical distribution.
  - Alice only needs to know vector $(C \tilde{y}(t))$ in round $t$.
  - Bob only needs to know (row) vector $(\tilde{x}(t)^T C)$ in round $t$.
- Fictitious play can be defined for any two-player game $(A, B)$. 
**Theorem (Robinson, 1951)**

*Utility/cost of Alice/Bob converges to value $v$ of the game. That is, as $t \to \infty$, it holds that*

- $\max_i (e^i) C\bar{y}(t) \to v$, $\min_j \bar{x}(t)^T C e_j \to v$, and $\bar{x}(t)^T C\bar{y}(t) \to v$.

*Empirical distributions $(\bar{x}(t), \bar{y}(t))$ “converge” to MNE as $t \to \infty$.*

- Convergence in the sense that $(\bar{x}(t), \bar{y}(t))$ is $\epsilon(t)$-approximate equilibrium, where $\epsilon(t) \to 0$ as $t \to \infty$.

**Convergence time of Fictitious Play still not fully understood!**

**Some notes on fictitious play**

- Simple way to compute value and $\epsilon$-MNE.
  - Avoiding the need to solve LPs.
- Players do not need to know each other’s empirical distribution.
  - Alice only needs to know vector $(C\bar{y}(t))$ in round $t$.
  - Bob only needs to know (row) vector $(\bar{x}(t)^T C)$ in round $t$.
- Fictitious play can be defined for any two-player game $(A, B)$.
  - Convergence fails beyond zero-sum games.