Lecture 4
Finite games - Existence and Computation of MNE
Finite game

Finite game $\Gamma = (N, (S_i)_{i \in N}, (C_i)_{i \in N})$ consists of:

- Finite set $N$ of players.
- Finite strategy set $S_i$ for every player $i \in N$.
- Cost function $C_i : \times_j S_j \rightarrow \mathbb{R}$ for every $i \in N$.

Matching pennies

Alice and Bob both choose side of a penny.

- $(a, b)$ denotes cost for Alice (A) and Bob (B) in given profile.

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No PNE: (Head, Head) $\xrightarrow{B}$ (Head, Tails) $\xrightarrow{A}$ (Tails, Tails) $\xrightarrow{B}$ (Tails, Head) $\xrightarrow{A}$ (Head, Head).

Game does have mixed Nash equilibrium (MNE).

- Both randomize over their strategies $\{\text{Head, Tails}\}$.
- Mixed strategies $\sigma_A = (1/2, 1/2)$ and $\sigma_B = (1/2, 1/2)$.
Mixed strategies

We focus on two-player games (for sake of notation). Players are
- **Row player** Alice (A) with strategy set $S_A = \{a_1, \ldots, a_m\}$, and
- **Column player** Bob (B) with strategy set $S_B = \{b_1, \ldots, b_n\}$.

**Definition (Mixed strategy)**

A **mixed strategy** is a probability distribution over $S_i$ for $i \in \{Alice, Bob\}$. The collection of all mixed strategies will be denoted by $\Delta_i$, i.e.,

$$
\Delta_{Alice} = \{(x_1, \ldots, x_m) : \sum_i x_i = 1, x_i \geq 0 \text{ for } i = 1, \ldots, m\},
$$
$$
\Delta_{Bob} = \{(y_1, \ldots, y_n) : \sum_j y_j = 1, y_j \geq 0 \text{ for } j = 1, \ldots, n\}.
$$

- Interpretation: Alice plays strategy $a_1$ with prob. $x_1$, etc...

**Example**

Strategies of Alice and Bob are given by:

- $\Delta_{Alice} = \{(x_1, x_2) : x_1 + x_2 = 1, x_1, x_2 \geq 0\}$,
- $\Delta_{Bob} = \{(y_1, y_2, y_3) : y_1 + y_2 + y_3 = 1, y_1, y_2, y_3 \geq 0\}$.

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For $x \in \Delta_A, y \in \Delta_B$, we get **product distribution** $\sigma_{x,y}: S_A \times S_B \rightarrow [0, 1]$ over strategy profiles,

$\sigma_{x,y}(a_k, b_\ell) = x_k y_\ell$ for $k = 1, \ldots, m$ and $\ell = 1, \ldots, n$.

**Example (cont’d)**

Distribution over strategy profiles is given by

$\begin{pmatrix}
  x_1 y_1 & x_1 y_2 & x_1 y_3 \\
  x_2 y_1 & x_2 y_2 & x_2 y_3
\end{pmatrix}$

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Then **expected cost** $C_i(\sigma_{x,y}) = C_i(x, y)$, of $i \in \{\text{Alice, Bob}\}$ is

$$C_i(x, y) = \mathbb{E}_{(a_k, b_\ell) \sim \sigma_{x,y}} \left[ C_i(a_k, b_\ell) \right] = \sum_{(a_k, b_\ell) \in S_A \times S_B} x_k y_\ell C_i(a_k, b_\ell)$$
Matrix representation

Matrix representation of cost functions $C_i : \Delta_A \times \Delta_B \to \mathbb{R}$ for $i \in \{\text{Alice, Bob}\}$ given by $A, B \in \mathbb{R}^{m \times n}$ defined as

$$A_{k\ell} = C_A(a_k, b_\ell) \quad \text{and} \quad B_{k\ell} = C_B(a_k, b_\ell) \quad \text{for} \quad k = 1, \ldots, m \quad \text{and} \quad \ell = 1, \ldots, n.$$ 

Example (cont’d)

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 4 \end{pmatrix}.$$ 

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Expected cost under mixed strategies $x \in \Delta_A, y \in \Delta_B$ is then

$$C_{\text{Alice}}(x, y) = x^T Ay = \sum_{k=1}^{m} \sum_{\ell=1}^{n} A_{k\ell} x_k y_\ell, \quad C_{\text{Bob}}(x, y) = x^T By = \sum_{k=1}^{m} \sum_{\ell=1}^{n} B_{k\ell} x_k y_\ell.$$ 

Short overview

Two-player game $(A, B)$ is given by matrices $A, B \in \mathbb{R}^{m \times n}$, with player Alice choosing mixed strategy $x$ over rows, and player Bob mixed strategy $y$ over columns. Expected costs are given by $x^T Ay$ and $x^T By$, respectively.
Mixed Nash equilibrium
Hierarchy of equilibrium concepts

- PNE: Exists in any congestion game
- MNE: Exists in any finite game, but hard to compute
- CE: Computationally tractable
- CCE: Not shown in the diagram

- Exists in any congestion game
- Computationally tractable
Mixed Nash equilibrium (2-player case)

For two-player game \((A, B)\), we have

\[
C_A(x, y) = x^T Ay = \sum_{k=1}^{m} \sum_{\ell=1}^{n} A_{k\ell} x_k y_\ell, \quad C_B(x, y) = x^T By = \sum_{k=1}^{m} \sum_{\ell=1}^{n} B_{k\ell} x_k y_\ell
\]

Definition (Mixed Nash equilibrium)

Pair \((x^*, y^*) \in \Delta_A \times \Delta_B\) is mixed Nash equilibrium (MNE) if neither Alice nor Bob can deviate to other mixed strategy and improve cost:

\[
C_A(x^*, y^*) \leq C_A(x', y^*) \quad \forall x' \in \Delta_A
\]
\[
C_B(x^*, y^*) \leq C_B(x^*, y') \quad \forall y' \in \Delta_B
\]

For \(\epsilon > 0\), pair \((x^*, y^*)\) is \(\epsilon\)-approximate MNE (or simply \(\epsilon\)-MNE) if

\[
C_A(x^*, y^*) \leq C_A(x', y^*) + \epsilon \quad \forall x' \in \Delta_A
\]
\[
C_B(x^*, y^*) \leq C_B(x^*, y') + \epsilon \quad \forall y' \in \Delta_B
\]

Will see later that is suffices to have these conditions only for pure strategies: One strategy is played with probability 1.
Example

Alice has $S_A = \{a_1, a_2\}$ and $S_B = \{b_1, b_2, b_3\}$.

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 3 & 3 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 4 & 2 \\ 2 & 0 & 4 \end{pmatrix}.$$ 

Suppose that $x = (0.5, 0.5)$ and $y = (0.3, 0.4, 0.3)$, then

$$C_B(x, y) = x^T By = (0.5 \quad 0.5) \begin{pmatrix} 2 & 4 & 2 \\ 2 & 0 & 4 \end{pmatrix} \begin{pmatrix} 0.3 \\ 0.4 \\ 0.3 \end{pmatrix} = 2.3$$

Is $(x, y)$ MNE? For $y' = (0.3, 0.7, 0)$, $C_B(x, y') = x^T By' = 2 < 2.3$.

(Row) vector $x^T B = (2, 2, 3)^T$ gives (expected) cost for Bob per column.

- Bob assigns positive probability to $b_3$: not optimal.
- Should only give positive probability to $b_1, b_2$ (given Alice plays $x$).

_In MNE, players only have positive probability on rows/columns that minimize expected cost per row/column (given other’s strategy)._
Definition

Column $b_j$ is best response against $x$ for Bob if $(x^T B)_j = \min_k (x^T B)_k$.
Row $a_i$ is best response against $y$ for Alice if $(Ay)_i = \min_k (Ay)_k$.

(E.g., if $x^T B = (7, 1, 3)^T$, then $(x^T B)_1 = 7, (x^T B)_2 = 1, (x^T B)_3 = 3$.)
- $(x^T B)_j$ is expected cost for Bob in column $j$ given Alice plays $x$.
- $(Ay)_i$ is expected cost for Alice in row $i$ given Bob plays $y$.

Definition (MNE, best response version)

Mixed strategies $(x^*, y^*)$ form MNE if Alice and Bob only assign positive probability to best responses. That is, pair $(x^*, y^*)$ is MNE if

$$x^*_i > 0 \implies (Ay^*)_i = \min_k (Ay^*)_k \quad \forall i = 1, \ldots, m,$$

$$y^*_j > 0 \implies ((x^*)^T B)_j = \min_k ((x^*)^T B)_k \quad \forall j = 1, \ldots, n.$$ 

Example (cont’d)

An MNE is given by $x^* = (1, 0), y^* = (0.5, 0, 0.5)$.
- $(x^*)^T B = (2, 4, 2)^T$. We have $y^*_1, y^*_3 > 0$ and $(x^T B)_1, (x^T B)_3$ are min.
- $Ay^* = (2, 2)$. We have $x^*_1 > 0$ and $(Ay^*)_1$ is minimum.
Finally, we write $e^k \in \Delta_A$ for pure strategy in which Alice plays $a_k \in S_A$ with probability 1. That is,

$$e^k_j = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

- If Alice plays $e^k \in S_A$, then $C_A(e^k, y) = (e^k)^T Ay = (Ay)_k$.
- Analogous definitions for Bob.

For Alice, one has $e^k \in \mathbb{R}^m$ and for Bob $e^\ell \in \mathbb{R}^n$. We abuse notation and do not always state the dimension of these vectors.

**Definition (MNE, pure strategy version)**

Mixed strategies $(x^*, y^*)$ form MNE if

$$(x^*)^T Ay^* \leq (e^i)^T Ay^* \quad i = 1, \ldots, m,$$

$$(x^*)^T By^* \leq (x^*)^T Ae^j \quad j = 1, \ldots, n.$$ 

That is, players both have no improving move to pure strategy.

- I.e., suffices to focus on pure strategies in definition on Slide 8.
- Exercise: Prove that this definition is equivalent to that on Slide 8.
Mixed Nash equilibrium (general)

Definition (Mixed Nash equilibrium (MNE))

A mixed strategy $\sigma_i : S_i \rightarrow [0, 1]$ of player $i \in N$ is a probability distribution over pure strategies in $S_i$, i.e., coming from

$$\Delta_i = \left\{ \tau : \tau(t) \geq 0 \ \forall t \in S_i \text{ and } \sum_{t \in S_i} \tau(t) = 1 \right\}.$$

A collection of mixed strategies $(\sigma_i)_{i \in N}$, with $\sigma_i \in \Delta_i$, is a mixed Nash equilibrium if

$$C_i(\sigma) := \mathbb{E}_{s \sim \sigma} [C_i(s)] \leq \mathbb{E}_{(s_{-i}) \sim (\sigma_{-i})} [C_i(s'_i, s_{-i})] \ \forall s'_i \in S_i. \quad (1)$$

Here

- $\sigma : \times_j S_j \rightarrow \mathbb{R}_{\geq 0}$ is given by $\sigma(t) = \prod_j \sigma_j(t_j)$, and
- $\sigma_{-i} : \times_{j \neq i} S_j \rightarrow \mathbb{R}_{\geq 0}$ is given by $\sigma_{-i}(t_{-i}) = \prod_{j \neq i} \sigma_j(t_j)$. 

Existence and computational complexity
Existence ("Nobel" Prize in Economics in 1994)

**Theorem (Nash’s theorem, 1950)**

Any finite game $\Gamma$ has a mixed Nash equilibrium.

**Theorem (Brouwer’s fixed point theorem)**

Let $D \subseteq \mathbb{R}^m$ be compact and convex, and let $f : D \rightarrow D$ be a continuous function. Then there exists an $x^* \in D$ such that $f(x^*) = x^*$.

Brouwer’s theorem says that $f$ has a fixed point.
Proof of Nash’s theorem

Show that MNEs correspond to fixed points of some function. Brouwer’s theorem then gives existence (proof is not constructive).

- **Proof given for 2-player games. (To save on notation.)**

**Proof:** Consider set $D = \Delta_A \times \Delta_B$. (*Convex and compact.*)

- Remember $\Delta_A = \{(x_1, \ldots, x_m) : \sum_k x_k = 1, x_k \geq 0\}$, $\Delta_B = \{(y_1, \ldots, y_n) : \sum_\ell y_\ell = 1, y_\ell \geq 0\}$.

For $(x, y) \in \Delta_A \times \Delta_B$, define

$$R_{A,a_k}(x, y) = \max\{0, C_A(x, y) - C_A(e^k, y)\} \quad k = 1, \ldots, m$$

$$R_{B,b_\ell}(x, y) = \max\{0, C_B(x, y) - C_B(x, e^\ell)\} \quad \ell = 1, \ldots, n$$

Note that the $R_{.,.}(x, y)$ encode MNE as follows:

$$R_{Z,s_z}(x, y) = 0 \quad \forall z \in \{A, B\} \quad \forall s_z \in S_z \quad \Leftrightarrow \quad (x, y) \text{ is MNE.}$$

Exercise: Show that $R_{Z,s_z}(x, y)$ is a continuous function.
\[ R_{A,a_k}(x, y) = \max\{0, C_A(x, y) - C_A(e^k, y)\} \quad k = 1, \ldots, m \]
\[ R_{B,b_\ell}(x, y) = \max\{0, C_B(x, y) - C_B(x, e^\ell)\} \quad \ell = 1, \ldots, n \]

We use these functions to define mapping \( f : \Delta_A \times \Delta_B \to \Delta_A \times \Delta_B \) by
\[
f(x, y) = (x', y') = (x'_1, \ldots, x'_m, y'_1, \ldots, y'_n),\]
where
\[
x'_i := \frac{x_i + R_{A,a_i}(x, y)}{\sum_{k=1}^m x_k + R_{A,a_k}(x, y)} = \frac{x_i + R_{A,a_i}(x, y)}{1 + \sum_{k=1}^m R_{A,a_k}(x, y)} \quad i = 1, \ldots, m
\]
and \( y' \in \Delta_2 \) by
\[
y'_j := \frac{y_j + R_{B,b_j}(x, y)}{\sum_{\ell=1}^n y_\ell + R_{B,b_\ell}(x, y)} = \frac{y_j + R_{B,b_j}(x, y)}{1 + \sum_{\ell=1}^n R_{B,b_\ell}(x, y)} \quad j = 1, \ldots, n
\]

Exercise: Show that \( f \) is a continuous function.

If \((x^*, y^*)\) is MNE, then \( R_{z,s_z}(x, y) = 0 \ \forall z \in \{A, B\} \ \forall s_z \in S_z\), and so \( x' = x^* \) and \( y' = y^* \). In other words, \((x^*, y^*)\) is fixed point of \( f \).
Other direction remains: If \((x^*, y^*)\) is fixed point of \(f\), then it is MNE.

Suffices to show that \(R_{z,s_z}(x, y) = 0\) \(\forall z \in \{A, B\} \ \forall s_z \in S_z\).

\[
R_{A,a_i}(x, y) = \max\{0, C_A(x, y) - C_A(e^i, y)\} \quad i = 1, \ldots, m
\]

\[
x_i' := \frac{x_i + R_{A,a_i}(x, y)}{1 + \sum_{k=1}^m R_{A,a_k}(x, y)} \quad i = 1, \ldots, m
\]

Note that

\[
C_A(x, y) = \sum_k x_k C_A(e^k, y) \leq \max_{k: x_k > 0} C_A(e^k, y) \sum_k x_k = \max_{k: x_k > 0} C_A(e^k, y)
\]

- There exists \(\bar{i}\) with \(x_{\bar{i}} > 0\) such that \(R_{A,a_{\bar{i}}}(x, y) = 0\).

Let us look at \(x_i'\) for fixed point \((x^*, y^*)\):

- \(x_i^* = \frac{x_i^*}{1 + \sum_{k=1}^m R_{A,a_k}(x^*, y^*)}\) \(\iff 1 = \frac{1}{1 + \sum_{k=1}^m R_{A,a_k}(x^*, y^*)}\)

- This gives \(\sum_{k=1}^m R_{A,a_k}(x^*, y^*) = 0\).

- \(R_{A,a_k}\) is always non-negative \(\Rightarrow R_{A,a_k}(x^*, y^*) = 0\) for \(k = 1, \ldots, m\).
Computation of MNE

Theorem (Nash’s theorem, 1950)

Any finite game $\Gamma$ has a mixed Nash equilibrium.

Can we compute an MNE efficiently?

Assuming cost functions are rational (think of $A, B \in \mathbb{Q}^{m \times n}$),

- MNE is always rational when $n = 2$, but
- MNE can be irrational when $n \geq 3$.
  - Irrational numbers are, e.g., $\pi$, e (Euler’s number), etc.

(Context: Suppose $f(z) = z^2 + z - 2$, then $f(z) = z$ is solved by $z^* = \pm \sqrt{2}$.)

For $n \geq 3$,

- Rational $\epsilon$-approximate MNE still exists for any $\epsilon > 0$.
- Algorithms are known to compute approx. equilibrium.
  - E.g., Scarf’s algorithm (1967) for approximating fixed points.
  - Probably hard to compute in general (similar to upcoming discussion for $n = 2$).
Complexity of computing MNE \((n = 2)\)

For \(n = 2\), proof(s) of Brouwer’s theorem give no algorithm.

- (Combinatorial) algorithms are known, e.g., Lemke-Howson algorithm.
  - Worst-case running time is exponential (in \#strategies).

**How to study computational complexity of MNE in 2-player games?**

*Computing MNE will be referred to as problem NASH.*

**Some (informal) intuition**

Consider function/search problem version of NP:

- For problem X, decide whether solution exists. If YES, output one.

Is NASH NP-complete? **Not likely.**

- “Deciding” whether Nash equilibrium exists is trivial.

NASH is complete for complexity class PPAD (already for \(n = 2\)).

- “Polynomial Parity Arguments on Directed graphs”
- See Chapter 20 [R2016] for this class, and more..
Theorem (Chen and Deng, 2006)

*Computing MNE in 2-player games is PPAD-complete*

- Same is true for approximate equilibria when $n \geq 3$.

What about approximate equilibria in 2-player games?

Assuming game is normalized ($0 \leq A_{ij}, B_{ij} \leq 1$) and $m = n$, we have:

Theorem (Lipton, Markakis and Mehta, 2003)

*There is an $O^*(n^{24 \log(n)}/\epsilon^2)$ algorithm known for computing $\epsilon$-approximate MNE in 2-player game.*

- Quasi-polynomial in $n$.

Theorem (Rubinstein, 2016)

*There exists a constant $\epsilon > 0$ such that, assuming the “Exponential Time Hypothesis for PPAD”, computing $\epsilon$-approximate MNE in 2-player game requires time at least $n^{\log^{1-o(1)}(n)}$.***
Two-player zero-sum games
Two-player game is called **zero-sum** if \( A + B = 0 \), i.e., \( A = -B \).
- Minimizing cost under \( A \) is same as maximizing cost under \( B \).

**Viewpoint that we take:** Given is \( m \times n \) matrix \( C \).
- Row player (Alice) tries to maximize **utility** \( x^T Cy \);
- Column player (Bob) tries to minimize **cost** \( x^T Cy \).

*Think of it as that Bob has to pay \( x^T Cy \) to Alice.*

**Algorithmic aspects of MNE:**
- Can be modeled as optimal solution of **linear program (LP).**
  - Solvable in polynomial time.
  - *(Any LP can be written as zero-sum game as well.)*
- Certain player dynamics can “learn” it: **Fictitious Play**
  - Holds for more classes of games, but not in general.
Value of zero-sum game

What can Alice guarantee to get from Bob?
- Suppose Alice plays mixed strategy $x$. What should Bob do?
- Choose $y$ such that $x^T Cy$ is minimal, i.e., strategy attaining
  $\min_{y \in \Delta_B} x^T Cy$.
- So what should Alice do? Choose $x$ maximizing $\min_{y \in \Delta_B} x^T Cy$.
- Alice can guarantee to get $v_A = \max_x \min_y x^T Cy$.

Similarly, Bob can guarantee to pay no more than $v_B = \min_y \max_x x^T Cy$.
- Exercise: Show that $v_A \leq v_B$

Theorem (Von Neumann, 1928)

Consider a two-player zero-sum game given by matrix $C$. Then

$$ v_A = \max_x \min_y x^T Cy = \min_y \max_x x^T Cy = v_B. $$

The number $v = v_A = v_B$ is called the value of the game.

- Often referred to as the “Minimax theorem”
Theorem (Minimax)

Consider a two-player zero-sum game given by matrix $C$. Then

$$v_A = \max_x \min_y x^T Cy = \min_y \max_x x^T Cy = v_B.$$  

We say that $x^*$ is optimal for Alice if $v_A$ is attained for $x^*$, i.e.,

$$\max_x \min_y x^T Cy = \min_y (x^*)^T Cy,$$

and, similarly, $y^*$ is optimal for Bob if $v_B$ is attained for $y^*$, i.e.,

$$\min_y \max_x x^T Cy = \max_x x^T Cy^*.$$

Corollary

$(x^*, y^*)$ is MNE if and only $x^*$ optimal for Alice and $y^*$ optimal for Bob.

- Computing MNE comes down to computing optimal strategies.

Corollary

Any MNE yield the same utility/loss for Alice/Bob, namely $v = v_A = v_B$.

- Exercise: Prove these corollaries.
Two-player zero-sum games

Computing MNE using linear programming
Optimal strategy $x^*$ for Alice is solution to optimization problem.

- We assume that the $C$ is $m \times n$ matrix, i.e., $m$ rows, $n$ columns.

$$\begin{align*}
\text{max} & \quad w \\
\text{subject to} & \quad w \leq \sum_{i=1}^{m} C_{ik} x_i \quad k = 1, \ldots, n \\
& \quad \sum_{i=1}^{m} x_i = 1 \\
& \quad x_i \geq 0 \quad i = 1, \ldots, m \\
& \quad w \in \mathbb{R}
\end{align*}$$

Problem above is indeed LP, with variables $(x_1, \ldots, x_m, w)$.

- First $m$ variables of optimum give optimal strategy $x^*$.
- Variable $w$ of optimum gives value $v = v_A$ of the game.

*The dual of this program precisely computes optimal strategy for Bob!*  
*In fact, strong duality can be used to prove the minimax theorem.*

**Theorem**

*MNE can be computed in polynomial time in 2-player zero-sum game.*
Two-player zero-sum games

Fictitious play
Simultaneous fictitious play (Brown, 1951)

**Introduced as algorithm for approximating value of zero-sum game.**

Game is played **repeatedly**. In every round:
- Alice (A) and Bob (B) play a pure strategy.
- They base their decision on **history** of the other player.
  - Choose best response w.r.t. empirical distribution (so far) of strategies chosen by the other.

*Informally speaking, empirical distributions “converge” to MNE.*

Let $S_A = \{a_1, \ldots, a_m\}$ (rows) and $S_B = \{b_1, \ldots, b_n\}$ (columns).

**Definition (Empirical distribution)**

Let $r_t$ be row chosen by Alice in step $t = 1, \ldots, T - 1$. **Empirical distribution** over $S_A$ in round $t$ is given by

$$\bar{x}_i(t) = \frac{|\{j : r_j = a_i, 1 \leq j \leq t - 1\}|}{t - 1}$$

for $i = 1, \ldots, m$. (*Fraction of rounds in which Alice chose row $i$.*)

- Analogous definition for Bob (with chosen column $c_t$ in round $t$).
Example

Suppose the matrix $C$ has $n = 6$ rows, and that Alice plays $(a_1, a_1, a_4, a_6, a_4, a_5, a_2, a_3, a_4)$ in first $t - 1 = 9$ rounds. Then

$$\bar{x}(t) = \bar{x}(10) = \frac{1}{9}(2, 1, 1, 3, 1, 1) = \left(\frac{2}{9}, \frac{1}{9}, \frac{1}{9}, \frac{3}{9}, \frac{1}{9}, \frac{1}{9}\right).$$

The idea of fictitious play is that Alice believes Bob plays every round according to some (unknown to her) probability distribution $y$.

- She uses empirical distribution $\bar{y}(t)$ as guess for $y$ in step $t$.
- Alice chooses best response row $r_t \in S_A$ with respect to $\bar{y}(t)$:

$$r_t \in \arg\max_j \left\{ (e^i)^T C \bar{y}(t) : i = 1, \ldots, m \right\}.$$ 

Bob is doing the same w.r.t Alice (for unknown distribution $x$).

- He uses empirical distribution $\bar{x}(t)$ as guess for $x$ in step $t$.
- Bob chooses best response column $c_t \in S_B$ with respect to $\bar{x}(t)$:

$$c_t \in \arg\min_j \left\{ \bar{x}(t)^T C e^j : j = 1, \ldots, n \right\}.$$
**ALGORITHM 1:** Fictitious play (with index tie-breaking rule)

<table>
<thead>
<tr>
<th>Input</th>
<th>$m \times n$ matrix $C$; initial row $r$, column $c$; round total $T \in \mathbb{N}$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output</td>
<td>Empirical distributions $\bar{x}(T), \bar{y}(T)$.</td>
</tr>
</tbody>
</table>

$\bar{x}(1) = e_r$ and $\bar{y}(1) = e_c$.

for $t = 2, \ldots, T$ do

Choose $r_t \in \arg\max \{(e^i)^T C \bar{y}(t) : i = 1, \ldots, m\}$

Choose $c_t \in \arg\min \{\bar{x}(t)^T C e^j : j = 1, \ldots, n\}$

*(Choose lowest indexed row/column in case of multiple best responses.)*

Update empirical distributions $(\bar{x}(t), \bar{y}(t))$ to $(\bar{x}(t + 1), \bar{y}(t + 1))$

end

return $\bar{x}(T), \bar{y}(T)$

- Observe that we specify a **tie-breaking rule** that decides which column/row to choose, in case there are multiple best responses.
Theorem (Robinson, 1951)

Utility/cost of Alice/Bob converges to value $v$ of the game. That is, as $t \to \infty$, it holds that

$$\max_i (e^i) C \bar{y}(t) \to v, \quad \min_j \bar{x}(t)^T C e_j \to v, \quad \text{and} \quad \bar{x}(t)^T C \bar{y}(t) \to v.$$ 

Empirical distributions $(\bar{x}(t), \bar{y}(t))$ “converge” to MNE as $t \to \infty$.

- Convergence in the sense that $(\bar{x}(t), \bar{y}(t))$ is $\epsilon(t)$-approximate equilibrium, where $\epsilon(t) \to 0$ as $t \to \infty$.

Convergence time of Fictitious Play still not fully understood!

Some notes on fictitious play

- Simple way to compute value and $\epsilon$-MNE.
  - Avoiding the need to solve LPs.
- Players do not need to know each other’s empirical distribution.
  - Alice only needs to know vector $(C \bar{y}(t))$ in round $t$.
  - Bob only needs to know (row) vector $(\bar{x}(t)^T C)$ in round $t$.
- Fictitious play can be defined for any two-player game $(A, B)$.
  - Convergence fails beyond zero-sum games.