Topics in Algorithmic Game Theory and Economics

Pieter Kleer
Max Planck Institute for Informatics
Saarland Informatics Campus

December 16, 2020

Lecture 6
Finite games III - Computation of CE and CCE
Hierarchy of equilibrium concepts

Finite (cost minimization) game $\Gamma = (N, (S_i)_{i \in N}, (C_i)_{i \in N})$ consists of:

- Finite set $N$ of players.
- Finite strategy set $S_i$ for every player $i \in N$.
- Cost function $C_i : \times jS_j \rightarrow \mathbb{R}$ for every $i \in N$.
Two-player games with mixed strategies (recap)

Two-player game $(A, B)$ given by matrices $A, B \in \mathbb{R}^{m \times n}$.
- Row player Alice and column player Bob independently choose strategy $x \in \Delta_A$ and $y \in \Delta_B$.

Gives product distribution $\sigma_{x,y} : S_A \times S_B \rightarrow [0, 1]$ over strategy profiles:
- $\sigma_{x,y}(a_k, b_\ell) = \sigma_{k\ell} = x_k y_\ell$ for $k = 1, \ldots, m$ and $\ell = 1, \ldots, n$.

Example

Distribution over strategy profiles is given by

$$
\begin{pmatrix}
  x_1 y_1 & x_1 y_2 & x_1 y_3 \\
  x_2 y_1 & x_2 y_2 & x_2 y_3
\end{pmatrix}
$$

<table>
<thead>
<tr>
<th></th>
<th>$b_1$</th>
<th>$b_2$</th>
<th>$b_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>(0, 2)</td>
<td>(1, 0)</td>
<td>(2, 1)</td>
</tr>
<tr>
<td>$a_2$</td>
<td>(3, 0)</td>
<td>(0, 1)</td>
<td>(1, 4)</td>
</tr>
</tbody>
</table>

Then expected cost (for Alice) $C_A(\sigma_{x,y})$ is

$$
x^T A y = \mathbb{E}_{(a_k, b_\ell) \sim \sigma_{x,y}}[C_A(a_k, b_\ell)] = \sum_{(a_k, b_\ell) \in S_A \times S_B} \sigma_{k\ell} C_A(a_k, b_\ell)
$$

- Remember that $A_{k\ell} = C_A(a_k, b_\ell)$. 

Beyond mixed strategies
Equilibrium concepts as distributions over $S_A \times S_B$

We have seen the following equilibrium concepts:

- **PNE**: Strategy profile $s = (s_A, s_B) \in S_A \times S_B$.
  - Gives *indicator distribution* $\sigma$ over $S_A \times S_B$ with
    \[
    \sigma(t) = \begin{cases} 
    1 & t = s \\ 
    0 & t \neq s
    \end{cases}.
    \]

- **MNE**: Mixed strategies $(x, y)$ with $x \in \Delta_A$, $y \in \Delta_B$.
  - Gives *product distribution* $\sigma$ over $S_A \times S_B$, where
    \[
    \sigma(a_k, b_\ell) = \sigma_{k\ell} = x_k y_\ell.
    \]

- **(C)CE**: (Coarse) correlated equilibrium will be defined as *general distribution* $\sigma$ over $S_A \times S_B$.
  - I.e., not induced by specific player actions.
Alice and Bob both approach an intersection.

\begin{tabular}{c|cc}
  & Stop & Go \\
\hline
Stop & (0, 0) & (3, −1) \\
Go & (−1, 3) & (4, 4) \\
\end{tabular}

- Two PNEs: (Stop, Go), (Go, Stop).
- One MNE: Both players randomize over Stop and Go.

Distributions over strategy profiles \((a, b)\) for these equilibria are

\[
\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}.
\]
Sensible ‘equilibrium’ would be the strategy profile distribution

\[ \sigma = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}. \]

Cannot be achieved as mixed equilibrium.

- There are no \( x \in \Delta_A, y \in \Delta_B \) such that \( \sigma_{k\ell} = x_k y_\ell \) for all \( k, \ell \in \{1, 2\} \).

Idea is to introduce traffic light (mediator or trusted third party).

- Traffic light samples/draws one of the two strategy profiles from distribution.
- Gives realization as recommendation to the players.
  - Tells Alice to go and Bob to stop (or vice versa)

*Conditioned on this recommendation, the best thing for a player to do is to follow it.*
Correlated equilibrium (CE), informal

Correlated equilibrium $\sigma : S_A \times S_B \rightarrow [0, 1]$ can be seen as follows.

- Mediator (third party) draws sample $x = (x_A, x_B) \sim \sigma$.
  - $\sigma$ is known to Alice and Bob, but not $x$.
- Gives **private recommendation** $x_A$ to Alice, and $x_B$ to Bob.
  - Alice and Bob do not know each other's recommendation!
    - Game of Chicken is the exception to the rule.
- Recommendations give players some info on which $x$ was drawn.

*Player assumes all other players play private recommendation, i.e., Alice assumes Bob follows his recommendation (and vice versa).*

*In CE, no player has incentive to deviate given its recommendation.*

**Remark**

We will later see **no-regret** algorithms whose output is a coarse correlated equilibrium (similar algorithms exist converging to CE). Therefore, for (C)CE, it's not always necessary that all players know the distribution $\sigma$ up front, nor that there is an actual third party that samples from it.
Suppose Alice gets second row $a_2$ as recommendation.
This gives Alice a (conditional) probability distribution $\rho$ for column privately recommended to Bob:

- Column $b_1$ with probability
  \[
  \frac{2}{8} \cdot \frac{2}{8} + \frac{1}{8} \cdot \frac{1}{8} + \frac{3}{8} \cdot \frac{3}{8} = \frac{2}{6}.
  \]

- Column $b_2$ with probability
  \[
  \frac{1}{8} \cdot \frac{2}{8} + \frac{1}{8} \cdot \frac{1}{8} + \frac{3}{8} \cdot \frac{3}{8} = \frac{1}{6}.
  \]

- Column $b_3$ with probability
  \[
  \frac{3}{8} \cdot \frac{2}{8} + \frac{1}{8} \cdot \frac{1}{8} + \frac{3}{8} \cdot \frac{3}{8} = \frac{3}{6}.
  \]

Assuming distribution $\rho$ over Bob’s recommendation, notion of CE says Alice should have no incentive to deviate to first row $a_1$ (in expectation).

\[
\mathbb{E}_\rho[\text{Row } a_2] = 3 \times \frac{2}{6} + 0 \times \frac{1}{6} + 1 \times \frac{3}{6} = \frac{9}{6}.
\]

\[
\mathbb{E}_\rho[\text{Row } a_1] = 0 \times \frac{2}{6} + 1 \times \frac{1}{6} + 2 \times \frac{3}{6} = \frac{7}{6}.
\]

$\sigma$ as above is not a CE!
(Coarse) correlated equilibrium

**Definition (Correlated equilibrium (CE))**

A distribution $\sigma$ on $\times_i S_i$ is a **correlated equilibrium** if for every $i \in N$ and $t_i \in S_i$, and every unilateral deviation $t'_i \in S_i$, it holds that

$$\mathbb{E}_{x \sim \sigma} [C_i(x) | x_i = t_i] \leq \mathbb{E}_{x \sim \sigma} [C_i(t'_i, x_{-i}) | x_i = t_i].$$

Set-up for coarse correlated equilibrium is similar, but you do not get private recommendation from mediator.

**Definition (Coarse correlated equilibrium (CCE))**

A distribution $\sigma$ on $\times_i S_i$ is a **coarse correlated equilibrium** if for every $i \in N$, and every unilateral deviation $t'_i \in S_i$, it holds that

$$\mathbb{E}_{x \sim \sigma} [C_i(x)] \leq \mathbb{E}_{x \sim \sigma} [C_i(t'_i, x_{-i})].$$

Exercise: Prove that every CE is also CCE.

(Hint: Use “Law of total expectation”, i.e., $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X | Y]]$.)
Final remark
Remember MNE is pair of mixed strategies \((x, y)\) that yields product distribution \(\sigma\) over strategy profiles.

MNE through the lens of CE
MNE is special case of CE, where recommendation of mediator gives no extra information.

- Conditional distribution \(\rho\) that Alice constructs for Bob’s private recommendation is the same for every row recommended to her.
  - It is just the mixed strategy \(y\) of Bob!
  - That is, the recommendation is not relevant for Alice.

Exercise: Check this yourself!
Computation of correlated equilibrium
Once again, linear programming comes to the rescue...

**Theorem**

For a given finite game \( \Gamma \), there is a linear program that computes a correlated equilibrium \( \sigma : \times_i S_i \rightarrow [0, 1] \) in time polynomial in \( |\times_i S_i| \) and the input size of the cost functions.

- LP has one variable \( \sigma_s \) for every strategy profile \( s \in \times_i S_i \).
  - Polynomial number of variables if number of players \( |N| \) is assumed to be constant.
  
- For two-player games, note that \( |S_A \times S_B| = mn \).

Conditions in definition CE can be modeled as linear program.

- We will do the 2-player case, and focus on Alice.
Definition (Correlated equilibrium (for Alice))

Distribution $\sigma$ on $S_A \times S_B$ is correlated equilibrium if for every “recommendation” $a_k \in S_A$ and deviation $a_k'$ it holds, with $x = (x_A, x_B)$, that

$$\mathbb{E}_{x \sim \sigma} [C_A(x_A, x_B) \mid x_A = a_k] \leq \mathbb{E}_{x \sim \sigma} [C_A(a_k', x_B) \mid x_A = a_k].$$

- LP will have variables $\sigma_{k \ell}$ for $k = 1, \ldots, m$, $\ell = 1, \ldots, n$.

Linear constraints for Alice

Fix “recommended row” $a_k \in S_A$ and “deviation” $a_k' \in S_A$. Now,

$$\mathbb{E}_{x \sim \sigma} [C_A(x_A, x_B) \mid x_A = a_k] = \sum_{\ell=1,\ldots,n} C_A(a_k, b_\ell) \mathbb{P}[x = (x_A, x_B) \mid x_A = a_k]$$

$$= \sum_{\ell=1,\ldots,n} C_A(a_k, b_\ell) \frac{\sigma_{k \ell}}{\sum_r \sigma_{kr}}$$

$$= \frac{1}{\sum_r \sigma_{kr}} \sum_{\ell=1,\ldots,n} C_A(a_k, b_\ell) \sigma_{k \ell}$$

$$\mathbb{E}_{x \sim \sigma} [C_A(a_k', x_B) \mid x_A = a_k] = \sum_{\ell=1,\ldots,n} C_A(a_k', b_\ell) \mathbb{P}[x = (x_A, x_B) \mid x_A = a_k]$$

$$= \frac{1}{\sum_r \sigma_{kr}} \sum_{\ell=1,\ldots,n} C_A(a_k', b_\ell) \sigma_{k \ell}$$
Conditions in (1) for Alice and Bob are equivalent to

\[
\sum_{\ell=1,\ldots,n} C_A(a_k, b_\ell)\sigma_{k\ell} \leq \sum_{\ell=1,\ldots,n} C_A(a_{k'}, b_\ell)\sigma_{k\ell} \quad \forall a_k, a_{k'} \in S_A
\]

\[
\sum_{k=1,\ldots,m} C_B(a_k, b_\ell)\sigma_{k\ell} \leq \sum_{k=1,\ldots,m} C_B(a_k, b_{\ell'})\sigma_{k\ell} \quad \forall b_\ell, b_{\ell'} \in S_B
\]

- Note that these are linear constraints in variables \(\sigma_{k\ell}\).

Linear program is now as follows

\[
\begin{align*}
\text{max} & \quad 0 \\
\text{s.t.} & \quad \sum_{\ell=1,\ldots,n} C_A(a_k, b_\ell)\sigma_{k\ell} \leq \sum_{\ell=1,\ldots,n} C_A(a_{k'}, b_\ell)\sigma_{k\ell} & \forall a_k, a_{k'} \in S_A \\
& \sum_{k=1,\ldots,m} C_B(a_k, b_\ell)\sigma_{k\ell} \leq \sum_{k=1,\ldots,m} C_B(a_k, b_{\ell'})\sigma_{k\ell} & \forall b_\ell, b_{\ell'} \in S_B \\
& \sum_{k,\ell} \sigma_{k\ell} = 1 \\
& \sigma_{k\ell} \geq 0 & \forall a_k \in S_A, b_\ell \in S_B
\end{align*}
\]
This is a feasibility LP, i.e., the goal is to find a feasible solution of
the linear system above.

We know at least one solution exists by Nash’s theorem
  • Remember that every MNE is also CE.

*Why not use the LP for computing MNE? We would need additional
non-linear constraint enforcing that \( \sigma \) is product distribution.*
For general finite game $\Gamma = (N, (S_i), (C_i))$, linear program is as follows.

\[
\begin{align*}
\text{max} & \quad 0 \\
\text{s.t.} & \quad \sum_{s_i \in S_i} C_i(s_i, s_{-i})\sigma(s_i, s_{-i}) \\
& \quad \leq \sum_{s_i \in S_i} C_i(s_i', s_{-i})\sigma(s_i, s_{-i}) \quad \forall i \in N \text{ and } s_i, s_i' \in S_i \\
& \quad \sum_{s \in \times_i S_i} \sigma(s) = 1 \\
& \quad \sigma(s) \geq 0 \quad \forall s \in \times_i S_i.
\end{align*}
\]

- We use notation $S_{-i} = S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_{|N|}$.
- And $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_{|N|}) \in S_{-i}$. 
No-regret dynamics
The model

Alice, with strategy set $S_A = \{a_1, \ldots, a_m\}$, plays “game” against adversary.
- Adversary will be used to represent other players later on.
- (Looking ahead: Players will converge to CCE.)

The game dynamics

Game is repeated for $T$ rounds. In every round $t = 1, \ldots, T$:
- Alice picks prob. distr. $p(t) = (p_1(t), \ldots, p_m(t))$ over $\{a_1, \ldots, a_m\}$.
- Adversary picks cost vector $c(t) : \{a_1, \ldots, a_m\} \rightarrow [0, 1]$.
- Strategy $a(t)$ is drawn according to $p(t)$.
  - Alice incurs cost $c(t)(a(t))$ and gets to know cost vector $c(t)$.

**Goal of Alice is to minimize average cost** $\frac{1}{T} \sum_{t=1}^{T} c(t)(a(t))$

against a benchmark. **What should the benchmark be?**
Best choices in hindsight

Would be natural to compare against best choices in hindsight, i.e.,

$$\frac{1}{T} \sum_{t=1}^{T} \min_{a \in S_A} c^{(t)}(a^{(t)}) .$$

- Alice’s cost if she would have put all prob. mass on strategy minimizing cost vector $c^{(t)}$, in every step $t$.

- Said differently, Alice’s best choice if adversary would have to announce cost vector first.

“Regret” of Alice, for given realization $a^{(1)}, \ldots, a^{(T)}$, would then be defined as

$$\alpha(T) = \frac{1}{T} \left( \sum_{t=1}^{T} c^{(t)}(a^{(t)}) - \sum_{t=1}^{T} \min_{a \in S_A} c^{(t)}(a) \right)$$

- Alice has no (or vanishing) regret if, in expectation, $\alpha(T) \to 0$ when $T \to \infty$.

We next illustrate that, under the definition $\alpha(T)$, vanishing regret cannot be achieved. (We will give an alternative definition afterwards.)
\[ \alpha(T) = \frac{1}{T} \left( \sum_{t=1}^{T} c(t)(a(t)) - \sum_{t=1}^{T} \min_{a \in S_A} c(t)(a) \right) \]

**Example**

Suppose Alice has two actions \( a \) and \( b \). In every round, when Alice chooses \( p(t) = (p_a(t), p_b(t)) \), adversary sets

\[ c(t) = (c(t)(a), c(t)(b)) = \begin{cases} (1, 0) & p_a(t) \geq 1/2 \\ (0, 1) & p_b(t) > 1/2 \end{cases} \]

- Expected cost in round \( t \) is at least \( 1/2 \).
- Best choice in hindsight gives cost of zero in round \( t \).

*Expected regret \( \alpha(T) \) is at least \( 1/2 \) for every \( T \).*

*Is there another “sensible” regret definition yielding non-trivial results?*
Best fixed strategy in hindsight

Another possibility is to compare with best fixed strategy in hindsight, i.e.,

$$\min_{a \in S_A} \frac{1}{T} \sum_{t=1}^{T} c^{(t)}(a).$$

- We interchange “minimum” and “summation”.
- Cost if Alice would have been allowed to switch to (fixed) strategy $a$ in every step.
  - This is still w.r.t. to adversarial cost vectors chosen by adversary based on prob. distributions $p^{(t)}$.

**Definition (Regret)**

For given prob. distr. $p^{(1)}, \ldots, p^{(T)}$ and adversarial cost vectors $c^{(1)}, \ldots, c^{(T)}$, the (time-averaged) regret of Alice is defined as

$$\rho(T) = \frac{1}{T} \left( \sum_{t=1}^{T} c^{(t)}(a^{(t)}) - \min_{a \in S_A} \sum_{t=1}^{T} c^{(t)}(a) \right),$$

where $a^{(t)}$ is sample according to distribution $p^{(t)}$. Alice has no regret (w.r.t. chosen distributions) if $\rho(T) \to 0$ when $T \to \infty$, in expectation.
More general, “Alice” is called an online decision making algorithm.

- Such an algorithm can use cost vectors $c^{(1)}, \ldots, c^{(t)}$, distributions $p^{(1)}, \ldots, p^{(t)}$, and realizations $a^{(t)}$, to define distribution $p^{(t+1)}$.
- Adversary can use the same information, including the chosen $p^{(t+1)}$, to define adversarial cost vector $c^{(t+1)}$.

Theorem

There exist no-regret (online decision making) algorithms with

$$\rho(T) = O \left( \frac{1}{T} \left( \sum_{t=1}^{T} c^{(t)}(a^{(t)}) - \min_{a \in S_A} \sum_{t=1}^{T} c^{(t)}(a) \right) \right)$$

where $m$ is the number of strategies.

- $T = O(\log(m)/\epsilon^2)$ steps enough to get regret below $\epsilon$.
- Will later see Multiplicative Weights (MW) algorithm achieving this.
No-regret dynamics

Convergence to (approximate) CCE
No-regret player dynamics

Let $\Gamma = (N, (S_i), (C_i))$, with $C_i : \times_j S_j \rightarrow [0, 1]$, and assume every $i \in N$ is equipped with no-regret algorithm $A_i$.

- At this point, consider the $A_i$ as “blackbox” algorithms.
- We write $m_i = |S_i|$ for number of strategies of player $i \in N$.

No-regret (player) dynamics

In every round $t = 1, \ldots, T$, every player $i \in N$ does the following:

- Use $A_i$ to compute prob. distr. $p_i^{(t)} = (p_{i,1}^{(t)}, \ldots, p_{i,m_i}^{(t)})$ over $S_i$.
- The adversarial cost vector $c_i^{(t)} : S_i \rightarrow [0, 1]$ is defined as

$$c_i^{(t)}(a) = \mathbb{E}_{s_i \sim \sigma_i^{(t)}} C_i \left( a, s_i^{(t)} \right) \quad \forall a \in S_i$$

where $\sigma_{-i}^{(t)}$ is product distribution formed by the $p_j^{(t)}$ with $j \in N \setminus \{i\}$.

- Strategy $a_i^{(t)} \sim p_i^{(t)}$ is drawn, and player $i$ incurs corresponding cost.

That is, $\sigma_i^{(t)} : S_{-i} \rightarrow [0, 1]$ is prob. distribution given by $\sigma_{-i}^{(t)}(s_{-i}) = \prod_{j \neq i} p_j^{(t)}$, where $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_{|N|}) \in S_{-i}$.

- Remember $S_{-i} = S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_{|N|}$.
No-regret (player) dynamics

In every round $t = 1, \ldots, T$, every player $i \in N$ does the following:

- Use $\mathcal{A}_i$ to compute prob. distr. $p_i^{(t)} = (p_{i,1}^{(t)}, \ldots, p_{i,m_i}^{(t)})$ over $S_i$.
- The adversarial cost vector $c_i^{(t)} : S_i \rightarrow [0, 1]$ is defined as
  \[
  c_i^{(t)}(a) = \mathbb{E}_{s \sim \sigma_i^{(t)}} C_i \left( a, s_{-i}^{(t)} \right) \quad \forall a \in S_i
  \]
  where $\sigma_i^{(t)}$ is product distribution formed by the $p_j^{(t)}$ with $j \in N \setminus \{i\}$.
- Strategy $a^{(t)} \sim p_i^{(t)}$ is drawn, and player $i$ incurs corresponding cost.

Set $\sigma_i^{(t)} = \prod_j p_j^{(t)}$ and let $\sigma_T = \frac{1}{T} \sum_{t=1}^{T} \sigma_i^{(t)}$ be the time average of all production distributions obtained in steps $t = 1, \ldots, T$.

**Theorem**

The time average $\sigma_T$ is a $\rho_i(T)$-approximate CCE, i.e., it satisfies

\[
\mathbb{E}_{s \sim \sigma_T} [C_i(s)] \leq \mathbb{E}_{s \sim \sigma_T} [C_i(s', s_{-i})] + \rho_i(T)
\]

for $i \in N$ and fixed $s_i' \in S_i$. 
Theorem

The time average $\sigma_T$ is a $\rho_i(T)$-approximate CCE, i.e., it satisfies

$$E_{s \sim \sigma_T}[C_i(s)] \leq E_{s \sim \sigma_T}[C_i(s'_i, s_{-i})] + \rho_i(T)$$

for $i \in N$ and fixed $s'_i \in S_i$.

Proof (sketch): First note that expected cost $E_{t_i \sim p_i(t)}[c_i(t)(t_i)]$ incurred by player $i$ in round $t$ boils down to $E_{s \sim \sigma(t)}[C_i(s)]$. Then

$$E_{s \sim \sigma_T}[C_i(s)] = \frac{1}{T} \sum_{t=1}^{T} E_{a \sim p_i(t)}[c_i(t)(a)] \quad \text{(time average)}$$

$$= \min_{s_i \in S_i} \frac{1}{T} \sum_{t=1}^{T} c_i(t)(s_i) + \rho_i(T) \quad \text{(definition of $\rho_i(T)$)}$$

$$= \min_{s_i \in S_i} \frac{1}{T} \sum_{t=1}^{T} E_{s_i \sim \sigma(t)} C_i(s_i, s^{(t)}_{-i}) + \rho_i(T) \quad \text{(definition of $c_i(t)$)}$$

$$\leq \frac{1}{T} \sum_{t=1}^{T} E_{s^{(t)} \sim \sigma(t)} C_i(s'_i, s^{(t)}_{-i}) + \rho_i(T) \quad \text{(plugging in fixed $s'_i$ )}$$

$$= E_{s \sim \sigma_T} C_i(s'_i, s_{-i}) + \rho_i(T) \quad \text{(time average)}$$
No-regret dynamics

Multiplicative Weights algorithm
No-regret dynamics

No-regret (player) dynamics

In every round $t = 1, \ldots, T$, every player $i \in N$ does the following:

- Use $A_i$ to compute prob. distr. $p_i^{(t)} = (p_{i,1}^{(t)}, \ldots, p_{i,m_i}^{(t)})$ over $S_i$.
- The adversarial cost vector $c_i^{(t)} : S_i \rightarrow [0, 1]$ is defined as
  \[
  c_i^{(t)}(a) = \mathbb{E}_{s_i \sim \sigma_i} C_i \left( a, s_i^{(t)} \right) \quad \forall a \in S_i
  \]
  where $\sigma_i$ is product distribution formed by the $p_j^{(t)}$ with $j \in N \setminus \{i\}$.

- Strategy $a_i^{(t)} \sim p_i^{(t)}$ is drawn, and player $i$ incurs corresponding cost.

We next give promised MW algorithm that can be used for the $A_i$, and that has the no-regret property. That is, in expectation,

\[
\rho_i(T) = \frac{1}{T} \left( \sum_{t=1}^{T} c_i^{(t)}(a_i^{(t)}) - \min_{a \in S_i} \sum_{t=1}^{T} c_i^{(t)}(a) \right) \rightarrow 0
\]

where $a_i^{(t)} \sim p_i^{(t)}$. 
Multiplicative Weights (MW) algorithm

Fix player \( i \in N \). The MW algorithm maintains weight \( w_a^{(t)} \) for every \( a \in S_i \) and chooses distribution for round \( t \) as

\[
p_{i,a}^{(t)} = \frac{w_a^{(t)}}{\sum_{r \in S_i} w_r^{(t)}}.
\]

Weight update procedure

Given is input parameter \( \eta \in (0, 1/2] \).
- Initial weights are set at \( w_a^{(1)} = 1 \) for \( a \in S_i \) (uniform distribution over \( S_i \)).
- For round \( t = 1, \ldots, T \):
  - After seeing cost vector \( c_i^{(t)} \), weights for round \( t + 1 \) are defined as
    \[
    w_a^{(t+1)} = (1 - \eta) c_i^{(t)}(a) \cdot w_a^{(t)}
    \]
- High cost strategies get smaller (relative) weight in next round.

Theorem (Littlestone and Warmuth, 1994)

\( MW \) algorithm, with \( \eta = \sqrt{\log(m_i)/T} \), has regret \( \rho_i(T) \leq 2 \sqrt{\log(m_i)/T} \)
Overview
Final remarks

- CE can also be obtained through certain player dynamics.
  - See, e.g., Chapter 18 [R2016].
- Recall that PoA bounds, that we derived for PNE, extend to CCE by means of the smoothness framework.