Mechanism Design

Mechanism design is a form of reversed game theory:

*Given a (desired) outcome, how should we design the game to obtain that outcome as a result of strategic behaviour?*

Examples:

- **Auctions**
  - Sponsored search auctions (e.g., Google)
  - Online selling platforms (e.g., eBay)
- **(Stable) matching problems**
  - Matching children to schools
  - Matching medical students to hospitals
- **Kidney exchange markets**

*We focus mostly on (online) auctions.*
Selling one item
Selling one item

**Bidders:**
- Set of bidders \(\{1, \ldots, n\}\) and one item.
- Bidder \(i\) has valuation \(v_i\) for the item.
  - Maximum amount she is willing to pay for it.
  - Private information: \(v_i\) not known to other players or seller.
- Bidder submits bid \(b_i\).
  - Vector of all bids denoted by \(b = (b_1, \ldots, b_n)\).

**Seller:** *Collects (sealed) bids.*
- Gives item to some bidder (if any).
  - Allocation rule \(x = x(b) = (x_1, \ldots, x_n)\), with
    \[
    x_i = \begin{cases} 
    1 & \text{if } i \text{ gets the item,} \\
    0 & \text{otherwise.}
    \end{cases}
    \]
  - Charges price of \(p\) to bidder \(i^*\) receiving item.
    - Pricing rule \(p = p(b)\).

**Utility of bidder \(i\):**
\[
u_i(b) = x_i(b)(v_i - p(b)) = \begin{cases} 
    v_i - p(b) & \text{if } i \text{ gets the item,} \\
    0 & \text{otherwise.}
    \end{cases}
\]
We have

- Bidders with valuations $v = (v_1, \ldots, v_n)$ and bids $b = (b_1, \ldots, b_n)$.
- Seller with allocation rule $x(b)$ and pricing rule $p(b)$.
- Utility of player given by $u_i(b) = x_i(b)(v_i - p(b))$.
- *Revenue of seller is $p$ if item is sold.*

**Definition**

A (deterministic) mechanism $(x, p)$ for selling an item to one of $n$ bidders is given by an allocation rule $x : \mathbb{R}^n \rightarrow \{0, 1\}^n$ with $\sum_i x_i \leq 1$, and pricing rule $p : \mathbb{R}^n \rightarrow \mathbb{R}$.

Goal of bidder $i$ is to maximize utility given mechanism $(x, p)$.

- Bidders will try to bid strategically.
- How should we design auction to prevent undesirable outcomes?
First price auction

Bidders report bids $b = (b_1, \ldots, b_n)$. Item is given to $i^* = \arg\max_i b_i$ and price $p = \max_i b_i$ is charged.

Example

Suppose there are three bidders

- Valuations $(v_1, v_2, v_3) = (10, 30, 25)$.
- Bids $(b_1, b_2, b_3) = (5, 22, 23)$.

Winner is bidder $i^* = 3$, with price $p = 23$. Utilities are $u = (0, 0, 2)$.

Is this a good auction format?

- Does not incentivize truthful bidding.
  - Bidders have incentive to lie (i.e., not report true valuation $v_i$).
- Bidder 2 values item the most, but does not get it.
  - Allocation rule does not maximize social welfare objective
    
    $\text{“Revenue for seller” + “Player utilities”} = \sum_i v_i x_i(b) = v_{i^*}$
Selling one item

Second price auction
Second price auction

Given bids \( b = (b_1, \ldots, b_n) \):
- Item is allocated to highest bidder \( i^* = \text{argmax}_i b_i \).
- Price charged is second-highest bid \( p = \max_{j \neq i^*} b_j \).
- Ties are broken according to some fixed tie-breaking rule.

Example
Suppose we have three bidders.
- Valuations \( (v_1, v_2, v_3) = (10, 30, 25) \).
- Bids \( (b_1, b_2, b_3) = (10, 30, 22) \).
Winner is bidder \( i^* = 2 \) and pays \( p = 22 \). Utilities are \( u = (0, 8, 0) \).

Second price auction has many desirable properties.
Desired properties

Bidders have incentive to be truthful: Reporting $v_i$ is dominant strategy.

**Definition (Strategyproof)**

Mechanism $(x, p)$ incentivizes truthful bidding if for every bidder $i$, alternative bid $b'_i$, and bids $b_{-i} = (b_1, \ldots, b_{i-1}, b_{i+1}, b_n)$ of other bidders, it holds that

$$u_i(b_{-i}, v_i) \geq u_i(b_{-i}, b'_i),$$

where $u_i(b) = x_i(b)(v_i - p(b))$.

Bidders have non-negative utility (when reporting truthfully).

**Definition (Individually rational)**

Mechanism $(x, p)$ is individually rational if for every bidder $i$ it holds

$$u_i(b) \geq 0$$

for every bid vector $b = (b_1, \ldots, b_{i-1}, v_i, b_{i+1}, \ldots, b_n)$.
Mechanism has good performance guarantee.

**Definition (Welfare maximization)**

Mechanism \((x, p)\) is welfare maximizer if it maximizes

\[
\sum_i v_i x_i(b) = \text{“Revenue for seller”} + \text{“Player utilities”}
\]

assuming that bidders are truthful.

- For now this just means we want to allocate item to a bidder with highest (true) valuation \(v^* = \max_i v_i\).
- *(In online setting, we are content with approximation.)*

**Definition (Computational efficiency)**

Mechanism \((x, p)\) should be implementable in polynomial time, i.e., compute allocation \(x\) and price \(p\) in polynomial time.
Proof strategyproofness (second price auction):

**Mechanism** $(x, p)$ incentivizes truthful bidding if for every $i$, alternative bid $b'_i$, and $b_{-i} = (b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)$, it holds that

$$u_i(b_1, \ldots, v_i, \ldots, b_n) \geq u_i(b_1, \ldots, b'_i, \ldots, b_n).$$

Fix $i$ and $b_{-i}$. Let $p(b) = s$ be second-highest bid.

- We compare $v_i, b'_i$ and $s$ (using case distinction).
  - Assume $v_i \neq s$ for simplicity.

**Case $s > v_i$:**
- Bidder $i$ would only win if $b'_i \geq b_{\text{max}}$, but then $u_i = v_i - p < 0$.
- For any bid $b'_i < b_{\text{max}}$ (then $i$ does not get item), we have $u_i = 0$.

**Case $s < v_i$:**
- Bidder $i$ wins. Charged price $s$ same for all $b'_i > s$. For $b'_i < s$, we have $u_i = 0$. Hence, bidding $v_i$ is an optimal choice.
Myerson’s lemma

Myerson’s lemma holds in more general settings.

There exists a nice characterization, due to Myerson (1981), specifying what type of allocation rules yield strategyproof mechanisms.

- Pricing rule follows from allocation rule.

Myerson’s lemma (very informal)

If there exists a monotone allocation rule \( x \), then there is a unique pricing rule \( p \) so that the mechanism \((x, p)\) is strategyproof (and vice versa).

Monotone allocation rule has the property that, if bidder \( i \) gets item when bidding \( b_i \), she also gets item when bidding \( b'_i \geq b_i \).

- That is, \( \{0, 1\} \)-variable \( x_i = (b_i, b_{-i}) \) is non-decreasing in bid \( b_i \).

Exercise: Show second price auction has monotone allocation rule.
Selling multiple items

Unit-demand setting
Selling multiple items

Unit-demand setting:

- Set of items $M = \{1, \ldots, m\}$
- Set of bidders $N = \{1, \ldots, n\}$
- For every $i \in N$ a private valuation function $v_i : M \to \mathbb{R}_{\geq 0}$.
  - Value $v_{ij} = v_i(j)$ is value of bidder $i$ for item $j$.
- For every $i \in N$ a bid function $b_i : M \to \mathbb{R}_{\geq 0}$.
  - Bid $b_{ij} = b_i(j)$ is maximum amount $i$ is willing to pay for item $j$.

*The goal is to assign (at most) one item to every bidder.*

Example

Non-existing edges have $b_{ij} = 0$. 

![Diagram showing items and bidders with edges and values]
Non-existing edges have $b_{ij} = v_{ij} = 0$ ($i$ is not interested in item $j$)

**Definition (Mechanism)**

A (deterministic) mechanism $(x, p)$ is given by an allocation rule

$$x : \mathbb{R}_{\geq 0}^{n \times m} \rightarrow \{0, 1\}^{n \times m},$$

with $\sum_i x_{ij} \leq 1$ and $\sum_j x_{ij} \leq 1$, and pricing rule $p : \mathbb{R}_{\geq 0}^{n \times m} \rightarrow \mathbb{R}_{\geq 0}^m$.

- For bidder $i$, we have bid vector $b_i = (b_{i1}, \ldots, b_{im})$.
- With $b = (b_1, \ldots, b_n)$, we have $x = x(b)$ and $p = p(b)$.
- Utility of bidder $i$ is

$$u_i(b) = \begin{cases} 
  v_{ij} - p_j(b) & \text{if } j \text{ is the item } i \text{ receives}, \\
  0 & \text{if } i \text{ does not get an item}.
\end{cases}$$
Desired properties

- **Strategyproof:** For every $i \in N$, bidding true valuations $v_i = (v_{i1}, \ldots, v_{im})$ is dominant strategy.
  - It should hold that
    \[
    u_i(b_{-i}, v_i) \geq u_i(b_{-i}, b'_i).
    \]
    for all $b_{-i} = (b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)$ and other bid vector $b'_i$.

- **Individual rationality:** Non-negative utility when bidding truthfully.

- **Welfare maximization:** The allocation $x$ maximizes
  \[
  \sum_{i,j} x_{ij}v_{ij}
  \]
  with $x_{ij} = 1$ if bidder $i$ gets item $j$, and zero otherwise.
  - Bipartite maximum weight matching in unit-demand setting.

- **Computationally tractable:** Allocation and pricing rules should be computable in polynomial time.
Vickrey-Clarke-Groves (VCG) mechanism

VCG mechanism works in more general settings than unit-demand.

Notation:
- Bipartite graph $B = (X \cup Y, E)$ with edge-weights $w : E \to \mathbb{R}_{\geq 0}$.
- $OPT(X', Y')$ is sum of edge weights of max. weight bipartite matching on induced subgraph $B' = (X' \cup Y', E)$ where $X' \subseteq X$, $Y' \subseteq Y$.

VCG mechanism

- Collect bid vectors $b_1, \ldots, b_n$ from bidders.
- Compute maximum weight bipartite matching $L^*$ (the allocation $x$)
- If bidder $i$ gets item $j$, i.e., $\{i, j\} \in L^*(N, M)$, then charge her
  \[
  p_{ij}(b) = OPT(N \setminus \{i\}, M) - OPT(N \setminus \{i\}, M \setminus \{j\}),
  \]
  and otherwise nothing.

$OPT(N \setminus \{i\}, M) - OPT(N \setminus \{i\}, M \setminus \{j\})$ is welfare loss for other players by assigning $j$ to $i$. 
We use shorthand notation $a_i b_j$ for edge $\{a_i, b_j\}$.

**Example**

Suppose these are reported bids (non-existing edges have $b_{ij} = 0$)

<table>
<thead>
<tr>
<th>Items</th>
<th>Bidders</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>$b_1$</td>
</tr>
<tr>
<td>$a_2$</td>
<td>$b_2$</td>
</tr>
<tr>
<td>$a_3$</td>
<td>$b_3$</td>
</tr>
<tr>
<td>$a_4$</td>
<td>$b_4$</td>
</tr>
</tbody>
</table>

- $L^* (N, M) = \{a_1 b_2, a_2 b_3, a_3 b_4\}$.
  - $\text{OPT}(N, M) = 3 + 4 + 6 = 13$.
- $L^* (N \setminus \{b_4\}, M) = \{a_1 b_1, a_2 b_2, a_3 b_3\}$.
  - $\text{OPT}(N \setminus \{b_4\}, M) = 2 + 2 + 5 = 9$.
- $L^* (N \setminus \{b_4\}, M \setminus \{a_3\}) = \{a_1 b_2, a_2 b_3\}$.
  - $\text{OPT}(N \setminus \{b_4\}, M \setminus \{a_3\}) = 3 + 4 = 7$.
- Price charged to bidder $b_4$ for item $a_3$ is $p_{43}(b) = 9 - 7 = 2$. 

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VCG mechanism satisfies all desired properties:

- Strategyproofness (bidding truthfully is optimal).
  - Exercise: Prove this.

- Individually rational.
  - Bidding truthfully gives non-negative utility.

- Social welfare maximizer.
  - It computes max. weight bipartite matching (where the weights are the true valuations).

- Computationally tractable.
  - Computing max. weight bipartite matching solvable in poly-time.
Online mechanism design

Selling one item
Selling one item online

Setting:
- Bidders have private valuation $v_i \geq 0$ for item.
- Whenever bidder arrives online, it submits bid $b_i$.

_Bidders arrive one by one in unknown order $\sigma = (\sigma(1), \ldots, \sigma(n))$."

**Online mechanism (informal)**

For $k = 1, \ldots, n$, upon arrival of bidder $\sigma(k)$:
- Bid $b_k$ is revealed.
- Decide (irrevocably) whether to allocate item to $\sigma(k)$.
  - If yes, charge price $p(b_{\sigma(1)}, \ldots, b_{\sigma(k)})$ and STOP.

**Goal:** Allocate item to bidder with highest valuation $v^* = \max_i v_i$.

Utility of bidder $i$, when $\sigma(k) = i$, is given by

$$u_{i,k}(b_{\sigma(1)}, \ldots, b_{\sigma(k)}) = \begin{cases} v_i - p(b_{\sigma(1)}, \ldots, b_{\sigma(k)}) & \text{if } i \text{ gets item,} \\ 0 & \text{otherwise.} \end{cases}$$
Requirements for (online) deterministic mechanism \((x, p)\):
Takes as input deterministic ordering \((y_1, \ldots, y_n)\) and bids \(b_1, \ldots, b_n\) for the item.

- Specifies for every \(k = 1, \ldots, n\) whether to allocate to \(y_k\).
- This \(\{0, 1\}\)-variable \(x_k\) (and price \(p\)) for \(k\) is function of:
  - Total number of bidders \(n\).
  - Bidders \(y_1, \ldots, y_k\).
  - Bids \(b_1, \ldots, b_k\).
  - The order \((y_1, \ldots, y_k)\).
    - Last aspect is usually irrelevant.

The variables \(x_i\) induce the allocation rule \(x\).
Desired properties

Bidding truthfully should be dominant strategy for every arrival order $\sigma$ and every arrival time of bidder $i$.

**Definition (Strategyproof)**

Consider arbitrary $i$, ordering $(\sigma(1), \ldots, \sigma(n))$, and $k$ with $i = \sigma(k)$. We say an online mechanism $\mathcal{M} = (x, p)$ is strategyproof if for every alternative bid $b'_i$ and every $b_{\sigma(1)}, \ldots, b_{\sigma(k-1)}$, it holds that

$$u_{i,k}(b_{\sigma(1)}, \ldots, b_{\sigma(k-1)}, v_i) \geq u_{i,k}(b_{\sigma(1)}, \ldots, b_{\sigma(k-1)}, b'_i).$$

- **Individually rational:** Non-negative utility for bidder $i$ when bidding truthful.
- **Constant factor $\alpha$-approximation for welfare maximization**
  - For uniform random arrival model:
    $$\mathbb{E}_{\sigma}[\nu(\mathcal{M}(\sigma))] \geq \alpha \cdot \max_i v_i$$
    - With $\nu(\mathcal{M}(\sigma))$ valuation of bidder that gets item.
- **Computationally tractable:** Decision on who to allocate item to, and computation of charged price, in poly-time.