Topics in Algorithmic Game Theory and Economics

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Max Planck Institute for Informatics (D1)
Saarland Informatics Campus

January 20, 2020

Lecture 9
Online Bipartite Matching

Offline bipartite matching

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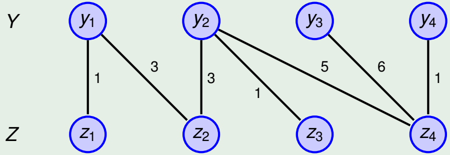

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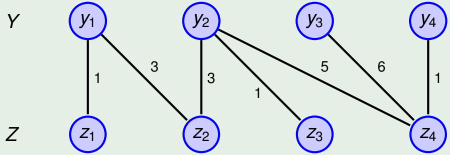

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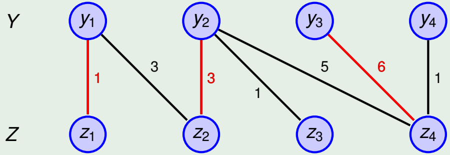

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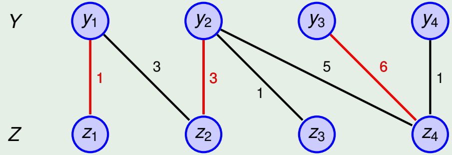

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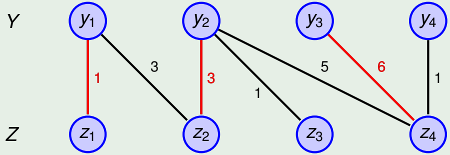

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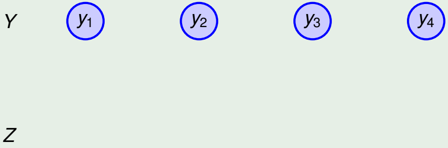

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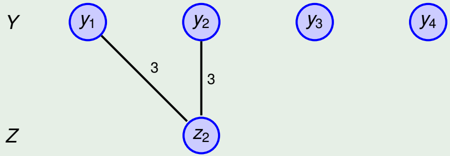

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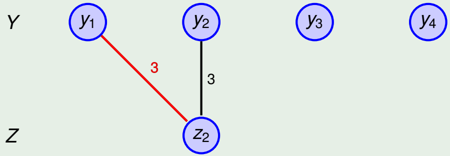

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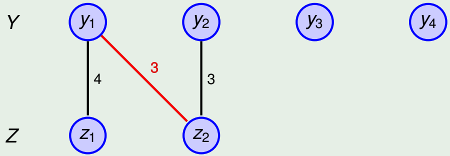

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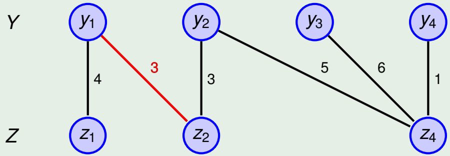

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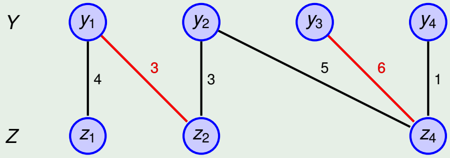

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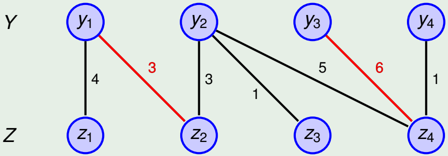

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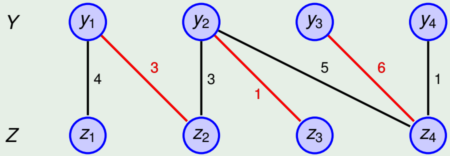

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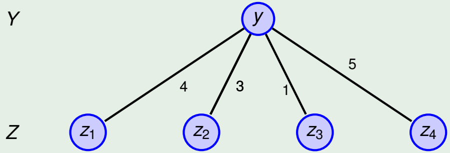

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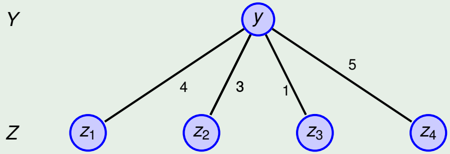

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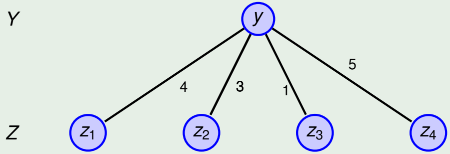

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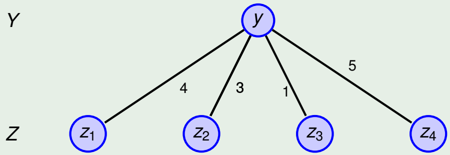

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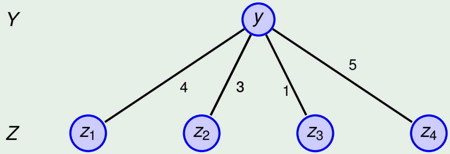

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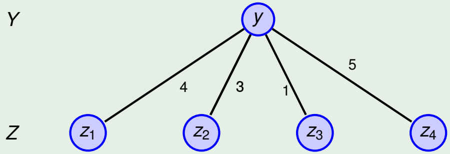

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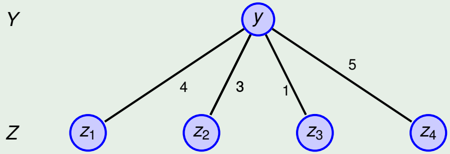

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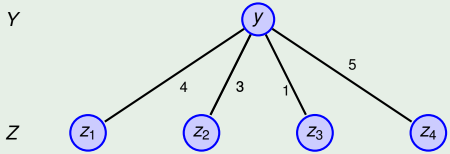

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- [Reiffenhäuser, 2019].
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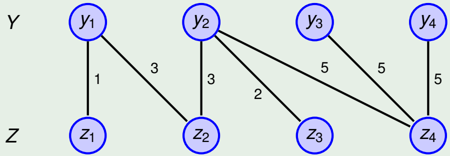

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- Generalization of (weight-maximization) secretary problem.

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There exists a $\left(\frac{1}{e} - \frac{1}{m}\right)$-approximation for the online bipartite matching problem where nodes of one side of the bipartition arrive online in uniform random order.

- Generalization of (weight-maximization) secretary problem.

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Notation:

- Assume arrival order is written as $\sigma = (z_1, \ldots, z_m)$.
- Bipartite graph $B = (Z \cup Y, E)$ with weights $w : E \to \mathbb{R}_{\geq 0}$.
  - Induced subgraph on $Z' \cup Y'$ is given by bipartite graph $B' = (Z' \cup Y', E')$ with $\{y', z'\} \in E' \;\Leftrightarrow\; y' \in Y', z' \in Z'$ and $\{y', z'\} \in E$.

- $\mathrm{OPT}(Z', Y') := w(M^*(Z', Y'))$ is weight of max. weight matching $M^*(Z', Y')$ on induced subgraph $B' = (Z' \cup Y', E')$.

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- Do not match up z_i.

- $\text{OPT}(Z', Y') := w(M^*(Z', Y'))$ is weight of max. weight matching $M^*(Z', Y')$ on induced subgraph $B' = (Z' \cup Y', E')$.

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Phase I (Observation): For $i = 1, \ldots, \lfloor \frac{m}{e} \rfloor$:

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  - z_i is matched up in **offline matching** M^* to some $y \in Y$ and
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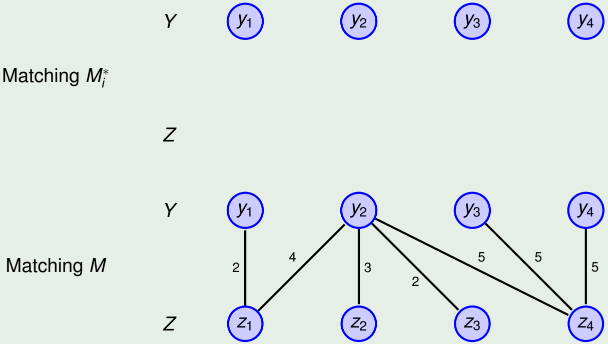
  then set $M = M \cup \{z_i, y\}$.

ALGORITHM 1: KRTV-algorithm for online bipartite matching

Input : Bipartite graph $B = (Z \cup Y, E)$ and weights $w : E \to \mathbb{R}_{\geq 0}$.
Deterministic algorithm $\mathcal{A}$ for max. weight bipartite matching.

Set $M = \emptyset$.
for $i = 1, \dots, \lfloor m/e \rfloor$ **do**
  Do nothing
end
for $i = \lfloor m/e \rfloor + 1, \dots, m$ **do**
  Compute optimal matching $M_i^* = M^*(\{z_1, \dots, z_i\}, Y)$ using $\mathcal{A}$
  if $\{z_i, y\} \in M_i^*$ for some $y \in Y$ **then**
    Set $M \leftarrow M \cup \{z_i, y\}$ if y is unmatched in M.
  end
end
return M

Example (of running Phase II for $i = 1, \dots, m$)



ALGORITHM 2: KRTV-algorithm for online bipartite matching

Input : Bipartite graph $B = (Z \cup Y, E)$ and weights $w : E \to \mathbb{R}_{\geq 0}$.
Deterministic algorithm $\mathcal{A}$ for max. weight bipartite matching.

Set $M = \emptyset$.
for $i = 1, \ldots, \lfloor m/e \rfloor$ **do**
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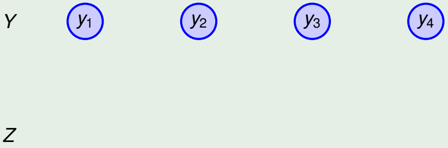
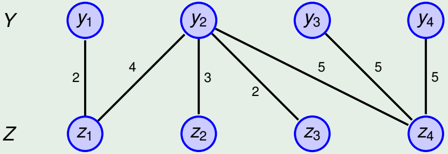
Example (of running Phase II for $i = 1, \ldots, m$)

Matching M_i^*



Matching M



ALGORITHM 3: KRTV-algorithm for online bipartite matching

Input : Bipartite graph $B = (Z \cup Y, E)$ and weights $w : E \to \mathbb{R}_{\geq 0}$.
Deterministic algorithm $\mathcal{A}$ for max. weight bipartite matching.

Set $M = \emptyset$.
for $i = 1, \ldots, \lfloor m/e \rfloor$ **do**
    Do nothing
end
for $i = \lfloor m/e \rfloor + 1, \ldots, m$ **do**
    Compute optimal matching $M_i^* = M^*(\{z_1, \ldots, z_i\}, Y)$ using $\mathcal{A}$
    if $\{z_i, y\} \in M_i^*$ for some $y \in Y$ **then**
        Set $M \leftarrow M \cup \{z_i, y\}$ if y is unmatched in M.
    end
end
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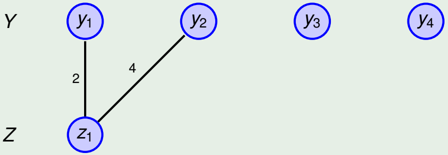
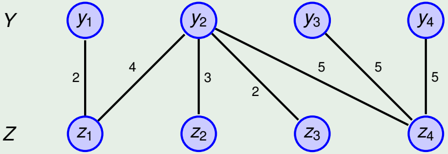
Example (of running Phase II for $i = 1, \ldots, m$)

Matching M_i^*



Matching M



ALGORITHM 4: KRTV-algorithm for online bipartite matching

Input : Bipartite graph $B = (Z \cup Y, E)$ and weights $w : E \to \mathbb{R}_{\geq 0}$.
Deterministic algorithm $\mathcal{A}$ for max. weight bipartite matching.

Set $M = \emptyset$.
for $i = 1, \ldots, \lfloor m/e \rfloor$ **do**
  Do nothing
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for $i = \lfloor m/e \rfloor + 1, \ldots, m$ **do**
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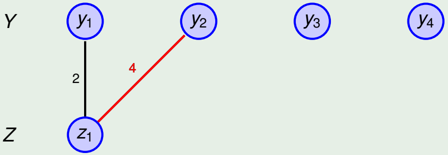
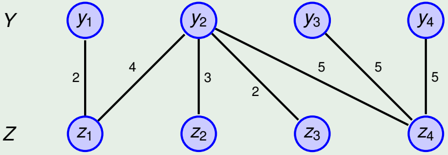
  if $\{z_i, y\} \in M_i^*$ for some $y \in Y$ **then**
    Set $M \leftarrow M \cup \{z_i, y\}$ if y is unmatched in M.
  end
end
return M

Example (of running Phase II for $i = 1, \ldots, m$)

Matching M_i^*



Matching M



ALGORITHM 5: KRTV-algorithm for online bipartite matching

Input : Bipartite graph $B = (Z \cup Y, E)$ and weights $w : E \to \mathbb{R}_{\geq 0}$.
Deterministic algorithm $\mathcal{A}$ for max. weight bipartite matching.

Set $M = \emptyset$.
for $i = 1, \ldots, \lfloor m/e \rfloor$ **do**
  Do nothing
end
for $i = \lfloor m/e \rfloor + 1, \ldots, m$ **do**
  Compute optimal matching $M_i^* = M^*(\{z_1, \ldots, z_i\}, Y)$ using $\mathcal{A}$
  if $\{z_i, y\} \in M_i^*$ for some $y \in Y$ **then**
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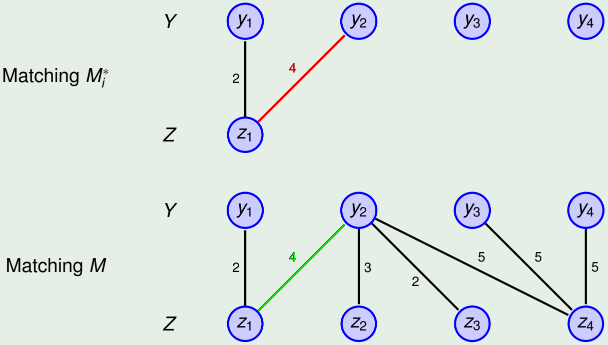
  end
end
return M

Example (of running Phase II for $i = 1, \ldots, m$)

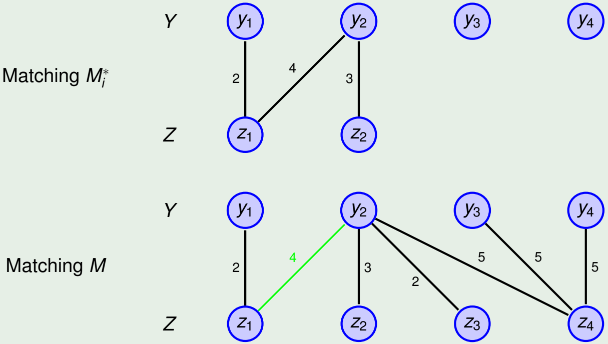


ALGORITHM 6: KRTV-algorithm for online bipartite matching

Input : Bipartite graph $B = (Z \cup Y, E)$ and weights $w : E \to \mathbb{R}_{\geq 0}$.
Deterministic algorithm $\mathcal{A}$ for max. weight bipartite matching.

Set $M = \emptyset$.
for $i = 1, \ldots, \lfloor m/e \rfloor$ **do**
  Do nothing
end
for $i = \lfloor m/e \rfloor + 1, \ldots, m$ **do**
  Compute optimal matching $M_i^* = M^*(\{z_1, \ldots, z_i\}, Y)$ using $\mathcal{A}$
  if $\{z_i, y\} \in M_i^*$ for some $y \in Y$ **then**
    Set $M \leftarrow M \cup \{z_i, y\}$ if y is unmatched in M.
  end
end
return M

Example (of running Phase II for $i = 1, \ldots, m$)

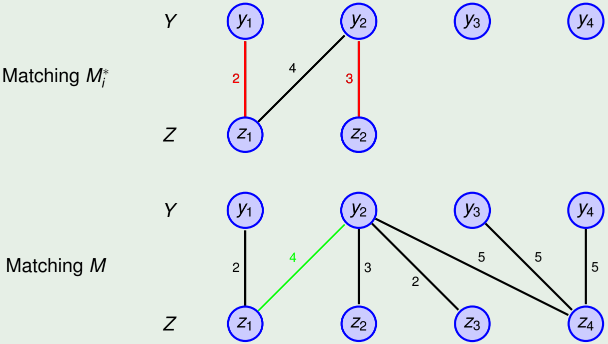


ALGORITHM 7: KRTV-algorithm for online bipartite matching

Input : Bipartite graph $B = (Z \cup Y, E)$ and weights $w : E \to \mathbb{R}_{\geq 0}$.
Deterministic algorithm $\mathcal{A}$ for max. weight bipartite matching.

Set $M = \emptyset$.
for $i = 1, \ldots, \lfloor m/e \rfloor$ **do**
  Do nothing
end
for $i = \lfloor m/e \rfloor + 1, \ldots, m$ **do**
  Compute optimal matching $M_i^* = M^*(\{z_1, \ldots, z_i\}, Y)$ using $\mathcal{A}$
  if $\{z_i, y\} \in M_i^*$ for some $y \in Y$ **then**
    Set $M \leftarrow M \cup \{z_i, y\}$ if y is unmatched in M.
  end
end
return M

Example (of running Phase II for $i = 1, \ldots, m$)



ALGORITHM 8: KRTV-algorithm for online bipartite matching

Input : Bipartite graph $B = (Z \cup Y, E)$ and weights $w : E \to \mathbb{R}_{\geq 0}$.
Deterministic algorithm $\mathcal{A}$ for max. weight bipartite matching.

Set $M = \emptyset$.
for $i = 1, \ldots, \lfloor m/e \rfloor$ **do**
  Do nothing
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for $i = \lfloor m/e \rfloor + 1, \ldots, m$ **do**
  Compute optimal matching $M_i^* = M^*(\{z_1, \ldots, z_i\}, Y)$ using $\mathcal{A}$
  if $\{z_i, y\} \in M_i^*$ for some $y \in Y$ **then**
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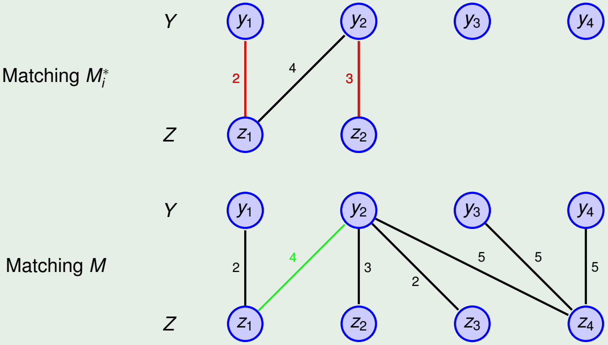
return M

Example (of running Phase II for $i = 1, \ldots, m$)



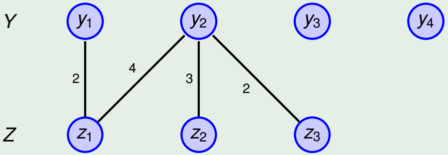
ALGORITHM 9: KRTV-algorithm for online bipartite matching

Input : Bipartite graph $B = (Z \cup Y, E)$ and weights $w : E \to \mathbb{R}_{\geq 0}$.
Deterministic algorithm $\mathcal{A}$ for max. weight bipartite matching.

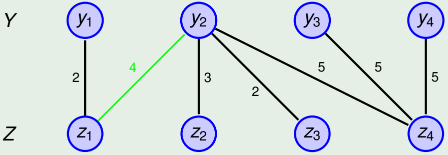
Set $M = \emptyset$.
for $i = 1, \ldots, \lfloor m/e \rfloor$ **do**
  Do nothing
end
for $i = \lfloor m/e \rfloor + 1, \ldots, m$ **do**
  Compute optimal matching $M_i^* = M^*(\{z_1, \ldots, z_i\}, Y)$ using $\mathcal{A}$
  if $\{z_i, y\} \in M_i^*$ for some $y \in Y$ **then**
    Set $M \leftarrow M \cup \{z_i, y\}$ if y is unmatched in M.
  end
end
return M

Example (of running Phase II for $i = 1, \ldots, m$)

Matching M_i^*



Matching M



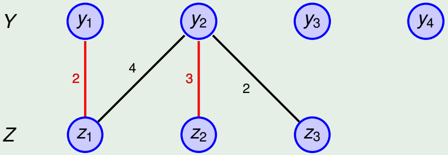
ALGORITHM 10: KRTV-algorithm for online bipartite matching

Input : Bipartite graph $B = (Z \cup Y, E)$ and weights $w : E \to \mathbb{R}_{\geq 0}$.
Deterministic algorithm $\mathcal{A}$ for max. weight bipartite matching.

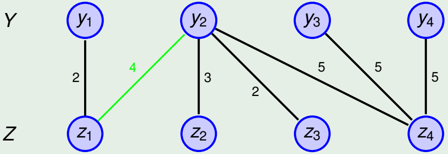
Set $M = \emptyset$.
for $i = 1, \ldots, \lfloor m/e \rfloor$ **do**
  Do nothing
end
for $i = \lfloor m/e \rfloor + 1, \ldots, m$ **do**
  Compute optimal matching $M_i^* = M^*(\{z_1, \ldots, z_i\}, Y)$ using $\mathcal{A}$
  if $\{z_i, y\} \in M_i^*$ for some $y \in Y$ **then**
    Set $M \leftarrow M \cup \{z_i, y\}$ if y is unmatched in M.
  end
end
return M

Example (of running Phase II for $i = 1, \ldots, m$)

Matching M_i^*



Matching M



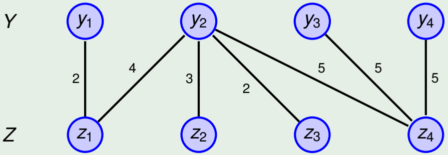
ALGORITHM 11: KRTV-algorithm for online bipartite matching

Input : Bipartite graph $B = (Z \cup Y, E)$ and weights $w : E \to \mathbb{R}_{\geq 0}$.
Deterministic algorithm $\mathcal{A}$ for max. weight bipartite matching.

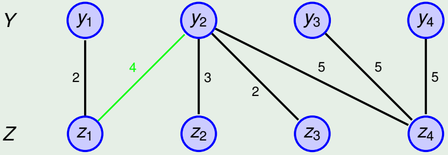
Set $M = \emptyset$.
for $i = 1, \ldots, \lfloor m/e \rfloor$ **do**
  Do nothing
end
for $i = \lfloor m/e \rfloor + 1, \ldots, m$ **do**
  Compute optimal matching $M_i^* = M^*(\{z_1, \ldots, z_i\}, Y)$ using $\mathcal{A}$
  if $\{z_i, y\} \in M_i^*$ for some $y \in Y$ **then**
    Set $M \leftarrow M \cup \{z_i, y\}$ if y is unmatched in M.
  end
end
return M

Example (of running Phase II for $i = 1, \ldots, m$)

Matching M_i^*



Matching M

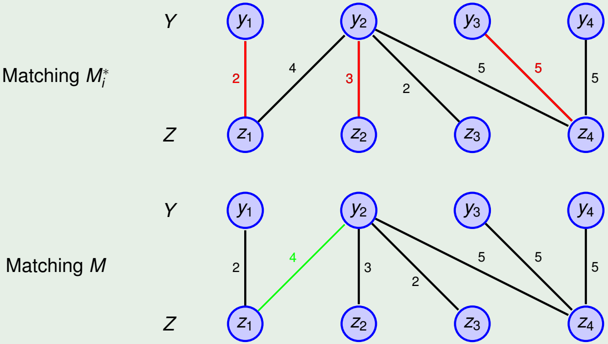


ALGORITHM 12: KRTV-algorithm for online bipartite matching

Input : Bipartite graph $B = (Z \cup Y, E)$ and weights $w : E \to \mathbb{R}_{\geq 0}$.
Deterministic algorithm $\mathcal{A}$ for max. weight bipartite matching.

Set $M = \emptyset$.
for $i = 1, \ldots, \lfloor m/e \rfloor$ **do**
  Do nothing
end
for $i = \lfloor m/e \rfloor + 1, \ldots, m$ **do**
  Compute optimal matching $M_i^* = M^*(\{z_1, \ldots, z_i\}, Y)$ using $\mathcal{A}$
  if $\{z_i, y\} \in M_i^*$ for some $y \in Y$ **then**
    Set $M \leftarrow M \cup \{z_i, y\}$ if y is unmatched in M.
  end
end
return M

Example (of running Phase II for $i = 1, \ldots, m$)



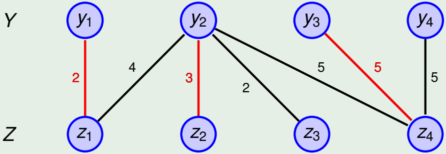
ALGORITHM 13: KRTV-algorithm for online bipartite matching

Input : Bipartite graph $B = (Z \cup Y, E)$ and weights $w : E \to \mathbb{R}_{\geq 0}$.
Deterministic algorithm $\mathcal{A}$ for max. weight bipartite matching.

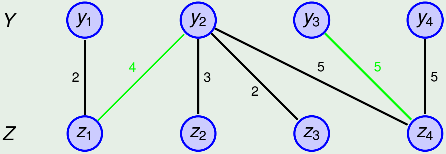
Set $M = \emptyset$.
for $i = 1, \ldots, \lfloor m/e \rfloor$ **do**
  Do nothing
end
for $i = \lfloor m/e \rfloor + 1, \ldots, m$ **do**
  Compute optimal matching $M_i^* = M^*(\{z_1, \ldots, z_i\}, Y)$ using $\mathcal{A}$
  if $\{z_i, y\} \in M_i^*$ for some $y \in Y$ **then**
    Set $M \leftarrow M \cup \{z_i, y\}$ if y is unmatched in M.
  end
end
return M

Example (of running Phase II for $i = 1, \ldots, m$)

Matching M_i^*



Matching M



Online bipartite matching

KRTV-algorithm: Sketch of analysis

Analysis (sketch)

ALGORITHM 14: KRTV-algorithm for online bipartite matching

Input : Bipartite graph $B = (Z \cup Y, E)$ and weights $w : E \to \mathbb{R}_{\geq 0}$.
Deterministic algorithm $\mathcal{A}$ for max. weight bipartite matching.

```
Set M = ∅.
for i = 1, ..., ⌊m/e⌋ do
|   Do nothing
end
for i = ⌊m/e⌋ + 1, ..., m do
|   Compute optimal matching M*_i = M*({z_1, ..., z_i}, Y) using A
|   if {z_i, y} ∈ M*_i for some y ∈ Y then
|   |   Set M ← M ∪ {z_i, y} if y is unmatched in M.
|   end
end
return M
```

Analysis (sketch)

ALGORITHM 15: KRTV-algorithm for online bipartite matching

Input : Bipartite graph $B = (Z \cup Y, E)$ and weights $w : E \to \mathbb{R}_{\geq 0}$.
Deterministic algorithm $\mathcal{A}$ for max. weight bipartite matching.

Set $M = \emptyset$.
for $i = 1, \ldots, \lfloor m/e \rfloor$ **do**
  Do nothing
end
for $i = \lfloor m/e \rfloor + 1, \ldots, m$ **do**
  Compute optimal matching $M_i^* = M^*(\{z_1, \ldots, z_i\}, Y)$ using $\mathcal{A}$
  if $\{z_i, y\} \in M_i^*$ for some $y \in Y$ **then**
    Set $M \leftarrow M \cup \{z_i, y\}$ if y is unmatched in M.
  end
end
return M

We will bound contribution A_i of (random) node i arriving in step $i \geq \lceil \frac{m}{e} \rceil$:

Analysis (sketch)

ALGORITHM 16: KRTV-algorithm for online bipartite matching

Input : Bipartite graph $B = (Z \cup Y, E)$ and weights $w : E \to \mathbb{R}_{\geq 0}$.
Deterministic algorithm $\mathcal{A}$ for max. weight bipartite matching.

```
Set M = ∅.
for i = 1, ..., ⌊m/e⌋ do
  | Do nothing
end
for i = ⌊m/e⌋ + 1, ..., m do
  | Compute optimal matching M*_i = M*({z_1, ..., z_i}, Y) using 𝒜
  | if {z_i, y} ∈ M*_i for some y ∈ Y then
  |   | Set M ← M ∪ {z_i, y} if y is unmatched in M.
  | end
end
return M
```

We will bound contribution A_i of (random) node i arriving in step $i \geq \lceil \frac{m}{e} \rceil$:
(Notation i is used for multiple things to keep everything readable.)

Analysis (sketch)

ALGORITHM 17: KRTV-algorithm for online bipartite matching

Input : Bipartite graph $B = (Z \cup Y, E)$ and weights $w : E \to \mathbb{R}_{\geq 0}$.
Deterministic algorithm $\mathcal{A}$ for max. weight bipartite matching.

```
Set M = ∅.
for i = 1, ..., ⌊m/e⌋ do
  | Do nothing
end
for i = ⌊m/e⌋ + 1, ..., m do
  | Compute optimal matching M*_i = M*({z_1, ..., z_i}, Y) using A
  | if {z_i, y} ∈ M*_i for some y ∈ Y then
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return M
```

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(Notation i is used for multiple things to keep everything readable.)

- For arrival order σ, we have

Analysis (sketch)

ALGORITHM 18: KRTV-algorithm for online bipartite matching

Input : Bipartite graph $B = (Z \cup Y, E)$ and weights $w : E \to \mathbb{R}_{\geq 0}$.
Deterministic algorithm $\mathcal{A}$ for max. weight bipartite matching.

```
Set M = ∅.
for i = 1, ..., ⌊m/e⌋ do
  | Do nothing
end
for i = ⌊m/e⌋ + 1, ..., m do
  | Compute optimal matching M*_i = M*({z_1, ..., z_i}, Y) using A
  | if {z_i, y} ∈ M*_i for some y ∈ Y then
  |   | Set M ← M ∪ {z_i, y} if y is unmatched in M.
  | end
end
return M
```

We will bound contribution A_i of (random) node i arriving in step $i \geq \lceil \frac{m}{e} \rceil$:
(Notation i is used for multiple things to keep everything readable.)

- For arrival order σ, we have

$$A_i = \begin{cases} w_{ir} & \text{if } i \text{ gets matched up with } r \text{ under } \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Analysis (sketch)

ALGORITHM 19: KRTV-algorithm for online bipartite matching

Input : Bipartite graph $B = (Z \cup Y, E)$ and weights $w : E \to \mathbb{R}_{\geq 0}$.
Deterministic algorithm $\mathcal{A}$ for max. weight bipartite matching.

```
Set M = ∅.
for i = 1, ..., ⌊m/e⌋ do
  | Do nothing
end
for i = ⌊m/e⌋ + 1, ..., m do
  | Compute optimal matching M*_i = M*({z_1, ..., z_i}, Y) using A
  | if {z_i, y} ∈ M*_i for some y ∈ Y then
  |   | Set M ← M ∪ {z_i, y} if y is unmatched in M.
  | end
end
return M
```

We will bound contribution A_i of (random) node i arriving in step $i \geq \lceil \frac{m}{e} \rceil$:
(Notation i is used for multiple things to keep everything readable.)

- For arrival order σ, we have

$$A_i = \begin{cases} w_{ir} & \text{if } i \text{ gets matched up with } r \text{ under } \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

- Then

$$\mathbb{E}_\sigma[A_i] = \mathbb{E}_\sigma[\text{Weight of edge } e^{(i)} = \{i, r\} \text{ assigned to } i \text{ in } M_i^*] \times \mathbb{P}_\sigma[\text{Node } i \text{ can be added to the online matching } M].$$

Two claims:

$$\mathbb{E}_\sigma[\text{Weight of edge } e^{(i)} = \{i, r\} \text{ assigned to } i \text{ in } M_i^*] \geq \frac{\text{OPT}}{n}$$

$$\mathbb{P}_\sigma[\text{Node } i \text{ can be added to the online matching } M] \geq \frac{\lfloor n/e \rfloor}{i-1}$$

where OPT is the offline optimum (on the whole instance).

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$$\mathbb{E}_\sigma[w(M)]$$

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The $\left(\frac{1}{e} - \frac{1}{m}\right)$-approximation then follows, because

$$\mathbb{E}_\sigma[w(M)] = \sum_{i=\lfloor m/e \rfloor + 1}^{m} \mathbb{E}_\sigma[A_i]$$

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The $\left(\frac{1}{e} - \frac{1}{m}\right)$-approximation then follows, because

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\mathbb{E}_\sigma[w(M)] &= \sum_{i=\lfloor m/e \rfloor+1}^{m} \mathbb{E}_\sigma[A_i] \geq \sum_{i=\lfloor m/e \rfloor+1}^{m} \frac{\text{OPT}}{m} \frac{\lfloor m/e \rfloor}{i-1} \\
&= \frac{\lfloor m/e \rfloor}{m} \cdot \text{OPT} \cdot \sum_{i=\lfloor m/e \rfloor+1}^{m} \frac{1}{i-1}
\end{aligned}$$

Two claims:

$$\mathbb{E}_\sigma[\text{Weight of edge } e^{(i)} = \{i, r\} \text{ assigned to } i \text{ in } M_i^*] \geq \frac{\text{OPT}}{n}$$

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The $\left(\frac{1}{e} - \frac{1}{m}\right)$-approximation then follows, because

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&= \frac{\lfloor m/e \rfloor}{m} \cdot \text{OPT} \cdot \sum_{i=\lfloor m/e \rfloor+1}^{m} \frac{1}{i-1} \\
&\geq \left(\frac{1}{e} - \frac{1}{m}\right) \cdot \text{OPT} \cdot 1
\end{aligned}$$

Offline mechanism design (recap)

Recap offline setting

Unit-demand setting:

Recap offline setting

Unit-demand setting:

- Set of items $M = \{1, \ldots, m\}$

Recap offline setting

Unit-demand setting:

- Set of items $M = \{1, \ldots, m\}$
- Set of bidders $N = \{1, \ldots, n\}$

Recap offline setting

Unit-demand setting:

- Set of items $M = \{1, \ldots, m\}$
- Set of bidders $N = \{1, \ldots, n\}$
- For every $i \in N$ a private valuation function $v_i : M \to \mathbb{R}_{\geq 0}$.

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- For every $i \in N$ a bid function $b_i : M \to \mathbb{R}_{\geq 0}$.

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The goal is to assign (at most) one item to every bidder.

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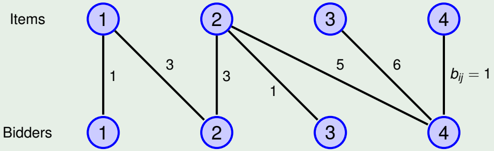
Example

Non-existing edges have $b_{ij} = 0$.



Definition (Mechanism)

An (offline) mechanism (x, p) is given by an allocation rule

$$x : \mathbb{R}_{\geq 0}^{n \times m} \to \{0, 1\}^{n \times m},$$

with $\sum_i x_{ij} \leq 1$ and $\sum_j x_{ij} \leq 1$, and pricing rule $p : \mathbb{R}_{\geq 0}^{n \times m} \to \mathbb{R}_{\geq 0}^{m}$.

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- Utility of bidder i is

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$$u_i(b_{-i}, v_i) \geq u_i(b_{-i}, b_i')$$

for all $b_{-i} = (b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)$ and other bid vector b_i'.

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    for all $b_{-i} = (b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n)$ and other bid vector b_i'.
- Also would like to have *individual rationality*, *welfare maximization*, and *computational tractability*.

Vickrey-Clarke-Groves (VCG) mechanism

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VCG mechanism

- Collect bid vectors $b_1, \ldots, b_n$ from bidders.
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- If bidder i gets item j, i.e., $\{i, j\} \in L^*(N, M)$,

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  - OPT(X', Y') is sum of edge weights of max. weight bipartite matching on induced subgraph $B' = (X' \cup Y', E)$ where $X' \subseteq X, Y' \subseteq Y$.

VCG mechanism

- Collect bid vectors $b_1, \ldots, b_n$ from bidders.
- Compute maximum weight bipartite matching L^* (the allocation x)
- If bidder i gets item j, i.e., $\{i, j\} \in L^*(N, M)$, then charge her

$$p_{ij}(b) = \text{OPT}(N \setminus \{i\}, M) - \text{OPT}(N \setminus \{i\}, M \setminus \{j\}),$$

Vickrey-Clarke-Groves (VCG) mechanism

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$$p_{ij}(b) = \text{OPT}(N \setminus \{i\}, M) - \text{OPT}(N \setminus \{i\}, M \setminus \{j\}),$$

  and otherwise nothing.

$\text{OPT}(N \setminus \{i\}, M) - \text{OPT}(N \setminus \{i\}, M \setminus \{j\})$ is welfare loss for other players by assigning j to i.

Online bipartite matching

Strategyproof online mechanism

Selling multiple items online

Selling multiple items online

Setting:

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- Bidder has valuation vector v_i for items in M.

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For $k = 1, \ldots, n$, upon arrival of bidder $\sigma(k)$:

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For $k = 1, \ldots, n$, upon arrival of bidder $\sigma(k)$:

- Bid vector b_k is revealed.

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Utility of bidder i, when $\sigma(k) = i$, is given by

$$u_{i,k}(b_{\sigma(1)}, \ldots, b_{\sigma(k)}) = \begin{cases} v_{ij} - p(b_{\sigma(1)}, \ldots, b_{\sigma(k)}) & \text{if } i \text{ gets item } j, \\ 0 & \text{otherwise.} \end{cases}$$

Requirements for (online) deterministic mechanism (x, p):

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Takes as input deterministic ordering $(y_1, \ldots, y_n)$ and bid vectors $b_1, \ldots, b_n$ for the item.

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As before, $\sum_k x_{k\ell} \leq 1$ and $\sum_\ell x_{k\ell} \leq 1$.

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> Mechanism is truthful, if, upon arrival, reporting truthful bids is optimal (assuming bidders have full knowledge about (x, p) and bidders arrived so far), for every possible arrival order σ.

An observation regarding the KRTV-algorithm

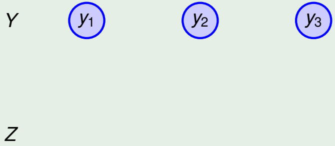
ALGORITHM 20: KRTV-algorithm for online bipartite matching

Input : Bipartite graph $B = (Z \cup Y, E)$ and weights $w : E \to \mathbb{R}_{\geq 0}$.
Deterministic algorithm $\mathcal{A}$ for max. weight bipartite matching.

Set $M = \emptyset$.
for $i = 1, \dots, \lfloor m/e \rfloor$ **do**
  Do nothing
end
for $i = \lfloor m/e \rfloor + 1, \dots, m$ **do**
  Compute optimal matching $M_i^* = M^*(\{z_1, \dots, z_i\}, Y)$ using $\mathcal{A}$
  if $\{z_i, y\} \in M_i^*$ for some $y \in Y$ **then**
    Set $M \leftarrow M \cup \{z_i, y\}$ if y is unmatched in M.
  end
end
return M

Example (of running Phase II for $i = 1, \dots, m$)

Online process



An observation regarding the KRTV-algorithm

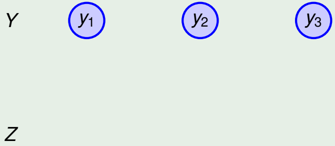
ALGORITHM 21: KRTV-algorithm for online bipartite matching

Input : Bipartite graph $B = (Z \cup Y, E)$ and weights $w : E \to \mathbb{R}_{\geq 0}$.
Deterministic algorithm $\mathcal{A}$ for max. weight bipartite matching.

```
Set M = ∅.
for i = 1, ..., ⌊m/e⌋ do
|   Do nothing
end
for i = ⌊m/e⌋ + 1, ..., m do
|   Compute optimal matching M_i^* = M^*({z_1, ..., z_i}, Y) using A
|   if {z_i, y} ∈ M_i^* for some y ∈ Y then
|   |   Set M ← M ∪ {z_i, y} if y is unmatched in M.
|   end
end
return M
```

Example (of running Phase II for $i = 1, \dots, m$)

Online process



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ALGORITHM 22: KRTV-algorithm for online bipartite matching

Input : Bipartite graph $B = (Z \cup Y, E)$ and weights $w : E \rightarrow \mathbb{R}_{\geq 0}$.
Deterministic algorithm $\mathcal{A}$ for max. weight bipartite matching.

Set $M = \emptyset$.
for $i = 1, \ldots, \lfloor m/e \rfloor$ **do**
  Do nothing
end
for $i = \lfloor m/e \rfloor + 1, \ldots, m$ **do**
  Compute optimal matching $M_i^* = M^*(\{z_1, \ldots, z_i\}, Y)$ using $\mathcal{A}$
  if $\{z_i, y\} \in M_i^*$ for some $y \in Y$ **then**
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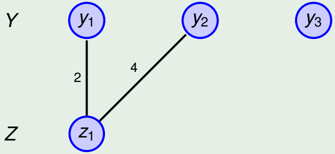
  end
end
return M

Example (of running Phase II for $i = 1, \ldots, m$)

Online process



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ALGORITHM 23: KRTV-algorithm for online bipartite matching

Input : Bipartite graph $B = (Z \cup Y, E)$ and weights $w : E \to \mathbb{R}_{\geq 0}$.
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for $i = 1, \dots, \lfloor m/e \rfloor$ **do**
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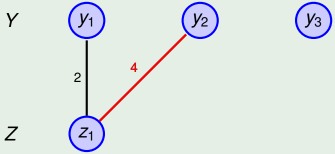
  end
end
return M

Example (of running Phase II for $i = 1, \dots, m$)

Online process



An observation regarding the KRTV-algorithm

ALGORITHM 24: KRTV-algorithm for online bipartite matching

Input : Bipartite graph $B = (Z \cup Y, E)$ and weights $w : E \to \mathbb{R}_{\geq 0}$.
Deterministic algorithm $\mathcal{A}$ for max. weight bipartite matching.

Set $M = \emptyset$.
for $i = 1, \ldots, \lfloor m/e \rfloor$ **do**
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end
for $i = \lfloor m/e \rfloor + 1, \ldots, m$ **do**
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  if $\{z_i, y\} \in M_i^*$ for some $y \in Y$ **then**
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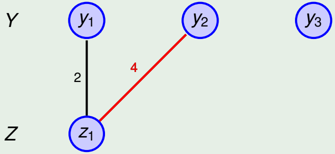
  end
end
return M

Example (of running Phase II for $i = 1, \ldots, m$)

Online process



An observation regarding the KRTV-algorithm

ALGORITHM 25: KRTV-algorithm for online bipartite matching

Input : Bipartite graph $B = (Z \cup Y, E)$ and weights $w : E \to \mathbb{R}_{\geq 0}$.
Deterministic algorithm $\mathcal{A}$ for max. weight bipartite matching.

Set $M = \emptyset$.
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  end
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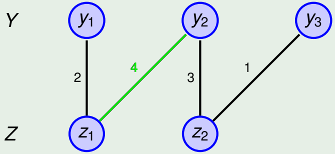
Example (of running Phase II for $i = 1, \ldots, m$)

Online process



An observation regarding the KRTV-algorithm

ALGORITHM 26: KRTV-algorithm for online bipartite matching

Input : Bipartite graph $B = (Z \cup Y, E)$ and weights $w : E \to \mathbb{R}_{\geq 0}$.
Deterministic algorithm $\mathcal{A}$ for max. weight bipartite matching.

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  end
end
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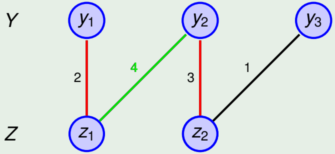
Example (of running Phase II for $i = 1, \ldots, m$)

Online process



An observation regarding the KRTV-algorithm

ALGORITHM 27: KRTV-algorithm for online bipartite matching

Input : Bipartite graph $B = (Z \cup Y, E)$ and weights $w : E \to \mathbb{R}_{\geq 0}$.
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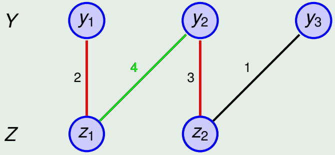
  end
end
return M

Example (of running Phase II for $i = 1, \dots, m$)

Online process



- Bidder might have incentive to misreport true valuations, as, in the offline matching M_i^* she is matched up with item already assigned to an earlier bidder.

Strategyproof online mechanism

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Theorem (Reiffenhäuser, 2019)

There exists a strategyproof $\frac{1}{e}$-approximation for the online bipartite matching problem with uniform random arrivals of the bidders.

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- Upon arrival of bidder z_i, it computes VCG-price for every unallocated item in J:

$$p_j(k) = \mathrm{OPT}(\{z_1, \ldots, z_{i-1}\}, J) - \mathrm{OPT}(\{z_1, \ldots, z_{i-1}\}, J \setminus \{j\}).$$

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- If there exists at least one item $j \in J$ for which $b_{ij} \geq p_j(k)$, then we assign an item

$$j^* = \text{argmax}\{b_{ij} - p_j(k) : j \in J\}$$

to bidder i, and set $J = J \setminus \{j^*\}$.

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- We charge price $p_{j^*}(k)$ to bidder i.

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- Although the algorithm is still relatively simple to describe, analysis is much harder.