Consider the following algorithm $A$ for the secretary problem (assume that $m$ is even).

For ordering $\sigma = (\sigma(1), \ldots, \sigma(m))$, do the following.

**Phase I (Observation):**
- For $i = 1, \ldots, \frac{m}{2}$: Do not select $\sigma(i)$.

**Phase II (Selection):**
- Set threshold $t = \max_{j=1,\ldots,\frac{m}{2}} w_{\sigma(j)}$.
- For $i = \frac{m}{2} + 1, \ldots, m$: If $w_{\sigma(i)} \geq t$, select $\sigma(i)$ and STOP.

Show that this algorithm gives a $\frac{1}{4}$-approximation for the secretary problem (with uniform random arrival order). That is,

$$\mathbb{E}_\sigma(w(A(\sigma))) \geq \frac{1}{4} \max_i w_i,$$

where $w(A(\sigma))$ is the weight of the element selected by the algorithm.

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**Exercise 2**

Consider the setting where we want to select one element from $\{e_1, \ldots, e_m\}$ online (with unknown weights $w_1, \ldots, w_m$), under a **worst-case arrival order** $\sigma$.

(a) Show that there is an online (deterministic or randomized) algorithm $A$, for which

$$\min_{\sigma} w(A(\sigma)) \geq \frac{1}{m} w^*,$$

where $w^* = \max_e w_e$ and $w(A(\sigma))$ the (expected) weight of the element selected by $A$. *(That is, in case $A$ is randomized, $w(A(\sigma))$ denotes the expected weight of the selected element. The expectation here is taken w.r.t. the random choices of the algorithm.)*

(b) Next, consider the case $m = 2$. Show that there is no (deterministic or randomized) online algorithm so that for every choice of weights $w_1$ and $w_2$

$$\min_{\sigma} w(A(\sigma)) \geq \alpha \cdot w^*.$$

for any constant $\alpha > \frac{1}{2}$. 
Consider the following (offline) auction setting. We have $n$ bidders and items $\{1, \ldots, k\}$ with $k < n$. Each item can be assigned to at most one bidder. Each bidder has a private valuation $v_i$ for receiving one of the items. (This means the items are identical, i.e., for every bidder $i$, $v_{i\ell} = v_{i\ell'}$ for every two items $\ell$ and $\ell'$.) Bidder $i$ declares a single bid $b_i \geq 0$ for receiving one of the $k$ (identical) items.

Give a deterministic, strategy proof (meaning bidding truthfully is optimal), individually rational, welfare optimizing mechanism $(x, p)$ that runs in time $O(n \log(n))$. The input of the mechanism is the vector $b = (b_1, \ldots, b_n)$. The utilities of the players are given by

$$u_i(b) = \begin{cases} v_i - p_j(b) & \text{if } i \text{ receives item } j, \\ 0 & \text{if } i \text{ does not get an item,} \end{cases}$$

for bid vector $b = (b_1, \ldots, b_n)$.

Consider the setting of selling one item online under a uniform random arrival order of bidders in $\{1, \ldots, n\}$, where bidder $i$ has (true) valuation $v_i \geq 0$ for the item. Let $v^* = \max_i v_i$.

Give a strategyproof online mechanism $M = (x, p)$, for which

$$E_\sigma[v(M(\sigma))] \geq \left(\frac{1}{e} - \frac{1}{n}\right) \cdot v^*.$$

where $\sigma$ is the arrival order and $v(M(\sigma))$ the value of the bidder that receives the item (if any).