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Saarland Informatics Campus

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Topics in Algorithmic Game Theory and Economics, Exercise Sheet 4

https://www.mpi-inf.mpg.de/departments/algorithms-complexity/teaching/winter20/game-theory

Total Points: 3 + 30 = 33 Due: Thursday, **Febr. 4, 23:59 (CET)**, 2021

*You are allowed to collaborate on the exercise sheets, but you have to write down a solution on your own, using your own words. **Always explain your answers**. Please indicate the names of your collaborators for each exercise you solve. Further, cite all external sources that you use (books, websites, research papers, etc.). You need to collect at least 50% of the total points on exercise sheets to be admitted to the exam. Send your solutions in **PDF** format directly to Golnoosh (**gshahkar@mpi-inf.mpg.de**). You get the first 3 points if you hand in typed solutions.*

Exercise 1 — 5 points

Consider the algorithm of Reiffenhäuser (2019) for selling multiple items online.

Input: Bipartite graph $B = (Z \cup Y, E)$ and bids $b : E \to \mathbb{R}_{\geq 0}$. Deterministic algorithm $\mathcal{A}$ for max. weight bipartite matching.

For ordering $\sigma = (z_1, \dots, z_m)$*, do the following:*
Set $M = \emptyset$ (online matching/allocation) and $J = Y$ (set of unallocated items).
Phase I (Observation). For $i = 1, \dots, \lfloor m/e \rfloor$:

- Do not offer bidder $z_i \in Z$ any item.

Phase II (Allocation). For $i = \lfloor m/e \rfloor + 1, \dots, m$:

- For every $y_j \in J$ compute VCG-price

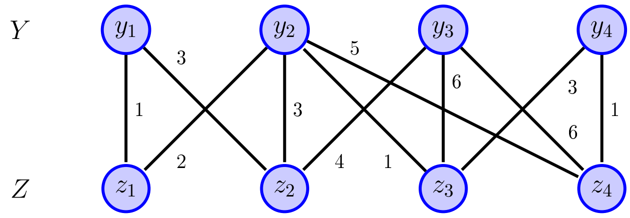
$$p_j(i) = \mathrm{OPT}(\{z_1, \dots, z_{i-1}\}, J) - \mathrm{OPT}(\{z_1, \dots, z_{i-1}\}, J \setminus \{y_j\}).$$

- If there exists $y_j \in J$ with $b_{ij} \geq p_j(i)$:
  - Assign an item $y_j^* = \mathrm{argmax}\{b_{ij} - p_j(i) : y_j \in J\}$ to z_i.
  - Set $M \leftarrow M \cup \{z_i, y_j^*\}$ and $J \leftarrow J \setminus \{y_j^*\}$

Output: Matching M and prices $p_j(i)$ of allocated items in $Y \setminus J$.

Remember that for an induced bipartite graph $B' = (Z' \cup Y', E')$ *of* B*, it holds that* $OPT(Z', Y')$ *is the max. weight (offline) matching on* B' *computed by* $\mathcal{A}$.

Run the algorithm on the following instance.



Exercise 2 — 10 + 5 points

Consider the following threshold based algorithm $\mathcal{A}$ for online bipartite matching in the graph $B = (Y \cup Z, E)$ with weight function $w : E \to \mathbb{R}_{\geq 0}$, where nodes from Z arrive online in a uniform random order.

> *For ordering $\sigma = (z_1, \dots, z_m)$, with m even, do the following:*
> Set $M = \emptyset$.
> **Phase I (Observation):**
> - For $i = 1, \dots, \frac{m}{2}$: Do not select z_i.
>
> **Phase II (Selection):**
> - For every $y \in Y$ set threshold
>
> $$t(y) = \begin{cases} 0 & \text{if there is no } z_i \text{ in Phase I with } \{z_i, y\} \in E, \\ \max_{i=1,\dots,m/2}\{w_{z_i y} : \{z_i, y\} \in E\} & \text{otherwise.} \end{cases}$$
>
> - For $i = \frac{m}{2} + 1, \dots, m$:
>   - Let $A_i = \{y : \{z_i, y\} \in E \text{ and } w_{z_i y} \geq t(y)\}$.
>   - If $A_i \neq \emptyset$, then match up z_i with any (unmatched in M) $y^* \in A_i$, i.e., set $M = M \cup \{z_i, y^*\}$.

(a) Show that for instances with uniform edge weights (Slide 9 of Lecture 9) and in which every node in $Z \cup Y$ has degree at most $\Delta = O(1)$, the above algorithm is a constant-factor approximation. That is,

$$\mathbb{E}_\sigma(w(\mathcal{A}(\sigma))) \geq \Omega\left(\frac{1}{\Delta}\right) \text{OPT},$$

where $w(\mathcal{A}(\sigma))$ is the weight of the matching selected by $\mathcal{A}$, and OPT the offline max. weight bipartite matching.

(b) Turn the algorithm into a strategyproof online mechanism for selling multiple items online (by introducing an appropriate pricing rule and adjustment of the last line of Phase II).

Exercise 3 — 10 points

Give a $\frac{1}{2e}$-approximation for the graphic matroid secretary problem (by filling in the details of the description given on Slides 27-30 of Lecture 10).

You may use, as a blackbox, the fact that there is a $\frac{1}{e}$-approximation, instead of $(\frac{1}{e} - \frac{1}{m})$-approximation, for the (weight-maximization) secretary problem: The problem of selecting the max. weight element from a set of m elements that arrive online in a uniform random arrival order.